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A Class of α-Carleson Measures

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Abstract. In the present paper, we introduce a class of α-Carleson measures $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$, which is called by the vanishing α-Carleson measures. We prove that $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ is just a predual of the tent space $\widetilde{T}^p_\infty$ $(0 < p < 1)$. Furthermore, we construct the α-Carleson measures and the vanishing α-Carleson measures by the Campanato functions and its a subclass, respectively. Moreover, a characterization of the vanishing α-Carleson measure by the compactness of Poisson integral is given in this paper. Finally, as some applications, we give the $(L^{2/\alpha}, L^2)$ boundedness and compactness for some paraproduct operators.

1. Introduction

It is well known that the Carleson measure has been a very important tool in harmonic analysis, complex analysis and PDE since which was introduced by L. Carleson in [7, 8] (see [3, 11, 13, 24, 35] for some applications of the Carleson measure).

Definition 1.1 (Carleson measure)**.** Suppose that μ is a positive measure on $\mathbb{R}^{n+1}_+$. For any cube $Q \subset \mathbb{R}^n$ and $a > 0$, let

$$\mathcal{N}(\mu, Q) = \frac{\mu(\widehat{Q})}{|Q|} \quad \text{and} \quad \mathcal{N}_a(\mu) = \sup_{|Q|=a} \mathcal{N}(\mu, Q).$$

A positive measure μ on $\mathbb{R}^{n+1}_+$ is called a *Carleson measure* written by $\mu \in \mathcal{C}(\mathbb{R}^{n+1}_+)$ if there exists a constant $C > 0$ such that

$$\|\mu\|_{\mathcal{C}} := \sup_{a>0} \mathcal{N}_a(\mu) \le C,$$

where $\|\mu\|_{\mathcal{C}}$ is called the Carleson constant of μ. A well-known fact is that $\|\cdot\|_{\mathcal{C}}$ is a norm and $\mathcal{C}(\mathbb{R}^{n+1}_+)$ is a Banach space in the norm $\|\cdot\|_{\mathcal{C}}$.

An important result is the duality between the Carleson measure space $\mathcal{C}(\mathbb{R}^{n+1}_+)$ and the tent spaces given by Coifman, Meyer and Stein [14] in 1985. Before stating this result, let us recall the definitions of the tent spaces.

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Definition 1.2 (tent space)**.** For $0 < q \le \infty$ and a measurable function f on $\mathbb{R}^{n+1}_+$, let

$$A_q(f)(x) = \begin{cases} \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}}\right)^{1/q} & \text{if } 0 < q < \infty, \\ \sup_{(y,t)\in\Gamma(x)} |f(y,t)| & \text{if } q = \infty, \end{cases}$$

where and in the sequel, $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$. For $0 < p < \infty$ and $0 < q < \infty$, the tent space T^p_q is defined by

$$T^p_q = \{f : \|f\|_{T^p_q} = \|A_q(f)\|_{L^p} < \infty\}.$$

For $q = \infty$ and $0 < p < \infty$, the tent space T^p_∞ is defined by

$$T^p_\infty = \left\{f \in C(\mathbb{R}^{n+1}_+) : \|f\|_{T^p_\infty} = \|A_\infty(f)\|_{L^p} < \infty \text{ and } \lim_{\epsilon\to 0} \|f - f_\epsilon\|_{T^p_\infty} = 0\right\},$$

where $C(\mathbb{R}^{n+1}_+)$ denotes all continuous functions on $\mathbb{R}^{n+1}_+$ and $f_\epsilon(x,t) = f(x, t+\epsilon)$.

It was showed in [14] that all tent spaces T^p_q $(1 \le p, q \le \infty)$ are Banach spaces with the norm $\|\cdot\|_{T^p_q}$. See [3–5, 10, 27–29, 31] for some applications of the tent spaces in harmonic analysis and PDE.

Theorem 1.3. [14] *The dual of the tent space T^1_∞ is the Carleson measure space $\mathcal{C}(\mathbb{R}^{n+1}_+)$, that is, $(T^1_\infty)^* = \mathcal{C}(\mathbb{R}^{n+1}_+)$.*

At almost same time, Han and Long [26] proved that the generalized α-Carleson measure space is the dual of the tent space T^p_∞ $(0 < p < 1)$. In 1987, Alvarez and Milman [2] gave the same result independently.

Definition 1.4 (α-Carleson measure)**.** Suppose $\alpha > 0$. A positive measure μ on $\mathbb{R}^{n+1}_+$ is said to be an α-Carleson measure, if there exists a constant $C > 0$ such that

$$\|\mu\|_{\mathcal{C}_\alpha} := \sup_{a>0} \mathcal{N}_a(\mu, \alpha) \le C,$$

where for any ball B in $\mathbb{R}^n$,

$$\mathcal{N}_a(\mu, \alpha) = \sup_{|B|=a} \mathcal{N}(\mu, B, \alpha) \quad \text{and} \quad \mathcal{N}(\mu, B, \alpha) = \frac{\mu(\widehat{B})}{|B|^\alpha}.$$

The set of all α-Carleson measures on $\mathbb{R}^{n+1}_+$ is denoted by $\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$.

Remark 1.5. Obviously, the Carleson measure space $\mathcal{C}(\mathbb{R}^{n+1}_+)$ is just 1-Carleson measure space $\mathcal{C}_1(\mathbb{R}^{n+1}_+)$.

The α-Carleson measure was studied intensively from various point of view by many authors, see [16, 23, 26, 30] and references therein for the properties and applications of the α-Carleson measure.

Theorem 1.6. [2, 26] *For $0 < p < 1$, the dual of the tent space T^p_∞ is the $1/p$-Carleson measure space $\mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$, that is, $(T^p_\infty)^* = \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$.*

Recently, in [19] we proved that the predual of the tent space T^1_∞ is the vanishing Carleson measure space $\mathcal{C}_v(\mathbb{R}^{n+1}_+)$ (see its definition in Remark 2.2 below). Thus, a **natural problem** arises: What is the predual of the tent space T^p_∞ for $0 < p < 1$? In 1988, Wang [36] discussed this problem. He pointed out that it is unsuitable to consider the predual of T^p_∞ $(0 < p < 1)$. So, in [36], a tent space $\widetilde{T}^p_\infty$ $(0 < p < 1)$ related to T^p_∞ was introduced (see Definition 2.4), which is the completion of T^p_∞. The author of [36] proved that the dual of $\widetilde{T}^p_\infty$ is also $\mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$. Further, Wang [36] introduced a subclass of $\mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$: for $0 < p < 1$,

$$\mathrm{VCM}_p(\mathbb{R}^{n+1}_+) = \left\{ \mu \in \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+) : \lim_{a \to 0} \mathcal{N}_a \left(\mu, \frac{1}{p} \right) = 0 \right\}.$$

In [36], the author pointed out that the dual of $\mathrm{VCM}_p(\mathbb{R}^{n+1}_+)$ is $\widetilde{T}^p_\infty$ $(0 < p < 1)$ by using the idea of [15]. Note that the author of [36] did not give the proof of his conclusion and even did not introduce a dense subset of $\mathrm{VCM}_p(\mathbb{R}^{n+1}_+)$, which is the key using the Coifman-Weiss's method in [15], so we do not know whether this conclusion in [36] is true or not.

In the present paper, we introduce two subclasses of $\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$, one is the vanishing α-Carleson measure space $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ (see Definition 2.1), another is $\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ (see Subsection 2.1 for its definition). Then we show that for $\alpha > 1$, $\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ is dense in $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ in the norm $\| \cdot \|_{\mathcal{C}_\alpha}$ (see Lemma 2.8), that is,

$$\overline{\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)}^{\| \cdot \|_{\mathcal{C}_\alpha}} = \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+).$$

Further, we prove that $(\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+))^* = \widetilde{T}^p_\infty$ for $0 < p < 1$ (see Theorem 2.5). An obvious fact is that $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+) \subsetneq \mathrm{VCM}_p(\mathbb{R}^{n+1}_+)$ by their definitions.

The another aim of this paper is to give the construction and characterization of the vanishing α-Carleson measure $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$. First, we construct an α-Carleson measure and a vanishing α-Carleson measure by the Campanato functions in Section 3. In Section 4, we give a characterization of the vanishing α-Carleson measure via the compactness of the convolution operator $\mathcal{L}_\varphi$ defined in (4.2). As the consequence of above result, we see that the vanishing α-Carleson measure can be characterized via the compactness of Poisson integral (see Corollary 4.5 below). In the final section, we will apply our results to give the boundedness and compactness of some paraproducts. To be precise, we introduce a paraproduct π_b via Campanato functions and establish its $(L^{2/\alpha}, L^2)$ boundedness and compactness. We also study the $(L^{2/\alpha}, L^2)$ boundedness and compactness of a kind of paraproduct $B_b(f)$ introduced by Coifman and Meyer [12].

In this paper, C will denote a positive constant that may change its value on each statement without the special instruction.

2. Predual of tent spaces $\widetilde{T}^p_\infty$ with $0 < p < 1$

In this section, we first introduce a subclass of $\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$, the vanishing α-Carleson measure space $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ and prove that for $0 < p < 1$, the dual of $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ is the tent space $\widetilde{T}^p_\infty$, which was introduced by Wang in [36].

Definition 2.1 (vanishing α-Carleson measure)**.** Let $\alpha > 1$. An α-Carleson measure μ on $\mathbb{R}^{n+1}_+$ is called a vanishing α-Carleson measure denoted by $\mu \in \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ if μ satisfies

(i) $\lim_{a\to 0} \mathcal{N}_a(\mu, \alpha) = 0$;

(ii) $\lim_{a\to \infty} \mathcal{N}_a(\mu, \alpha) = 0$;

(iii) $\lim_{|x|\to \infty} \mathcal{N}(\mu, B + x, \alpha) = 0$ for each ball B in $\mathbb{R}^n$.

See Definition 1.4 for the definition of $\mathcal{N}_a(\mu, \alpha)$.

Remark 2.2. When $\alpha = 1$, the vanishing 1-Carleson measure is called by the vanishing Carleson measure, the collection of all vanishing Carleson measures is denoted by $\mathcal{C}_v(\mathbb{R}^{n+1}_+)$.

In [14,26], the authors gave the atom decomposition of the tent space T^p_∞ for $0 < p < 1$. A function a on $\mathbb{R}^{n+1}_+$ is said to be a T^p_∞ *atom* if

(i) $\operatorname{supp} a \subset \widehat{Q}$ for some cube $Q \subset \mathbb{R}^n$;

(ii) $\|a\|_{L^\infty(\mathbb{R}^{n+1}_+)} \le |Q|^{-1/p}$.

Lemma 2.3. [14, 26] *Suppose $f \in T^p_\infty$ ($0 < p < 1$). Then $f = \sum_{i=1}^\infty \lambda_i a_i$, where each a_i is a T^p_∞ atom, $\lambda_i \in \mathbb{C}$, and $\sum_{i=1}^\infty |\lambda_i|^p \le C\|f\|^p_{T^p_\infty}$, the constant C is independent of $\{\lambda_i\}$ and f.*

In [36], Wang introduced a tent space $\widetilde{T}^p_\infty$ ($0 < p < 1$).

Definition 2.4. For $0 < p < 1$, the tent space $\widetilde{T}^p_\infty$ is defined by

$$\widetilde{T}^p_\infty = \left\{ f = \sum_i \lambda_i a_i : a_i \text{ is a } T^p_\infty \text{ atom and } \sum_i |\lambda_i| < \infty \right\},$$

and the $\widetilde{T}^p_\infty$ norm is defined by $\|f\|_{\widetilde{T}^p_\infty} = \inf \big\{ \sum_i |\lambda_i| : f = \sum_i \lambda_i a_i \big\}$.

It was showed in [36] that $\widetilde{T}^p_\infty$ is the completion of T^p_∞ for $0 < p < 1$ and $\widetilde{T}^p_\infty$ and T^p_∞ have the same dual. That is, $(\widetilde{T}^p_\infty)^* = \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$.

The first main result in this section is to prove that the tent space $\widetilde{T}^p_\infty$ is the dual space of $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$.

Theorem 2.5. *For $0 < p < 1$, the dual of $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ is the tent space $\widetilde{T}^p_\infty$, that is, $(\mathcal{C}_{1/p,v})^* = \widetilde{T}^p_\infty$. More precisely, the pairing $\langle f, d\mu\rangle = \int_{\mathbb{R}^{n+1}_+} f(x,t)\, d\mu(x,t)$ realizes the duality of $\mathcal{C}_{1/p,v}$ with $\widetilde{T}^p_\infty$.*

To prove Theorem 2.5, we need some elementary properties of $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ and $\widetilde{T}^p_\infty$ which are given in Subsections 2.1 and 2.2, respectively.

2.1. A dense subset of $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$

In this subsection, we introduce a subclass $\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ of $\mathcal{C}_{\alpha}(\mathbb{R}^{n+1}_+)$ and prove that $\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ is a dense subset of $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathcal{C}_\alpha}$. We first give two lemmas. The first one is obvious, and we omit its proof.

Lemma 2.6. (i)

$$\lim_{a\to 0} \mathcal{N}_a(\mu,\alpha) = 0 \quad \Longleftrightarrow \quad \lim_{a\to 0} \sup_{|Q|\le a} \mathcal{N}(\mu,Q,\alpha) = 0,$$

$$\lim_{a\to \infty} \mathcal{N}_a(\mu,\alpha) = 0 \quad \Longleftrightarrow \quad \lim_{a\to \infty} \sup_{|Q|\ge a} \mathcal{N}(\mu,Q,\alpha) = 0.$$

(ii)

$$\lim_{|x|\to \infty} \mathcal{N}(\mu,Q+x,\alpha) = 0 \quad \Longleftrightarrow \quad \lim_{a\to \infty} \sup_{|x|> a} \mathcal{N}(\mu,Q+x,\alpha) = 0,$$

where Q is any cube in $\mathbb{R}^n$.

Lemma 2.7. *$\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{C}_\alpha}$.*

Proof. Suppose $\{\mu_k\}$ is a Cauchy sequence in $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$, then $\{\mu_k\}$ is also the Cauchy sequence in $\mathcal{C}_{\alpha}(\mathbb{R}^{n+1}_+)$. Applying the completeness of $\mathcal{C}_{\alpha}(\mathbb{R}^{n+1}_+)$, there exists a measure $\mu \in \mathcal{C}_{\alpha}(\mathbb{R}^{n+1}_+)$, such that $\|\mu - \mu_k\|_{\mathcal{C}_\alpha} \to 0$ as $k\to\infty$. It remains to show that $\mu \in \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$.

For any $a > 0$, note that

$$\mathcal{N}_a(\mu,\alpha) \le \mathcal{N}_a(\mu_k,\alpha) + \mathcal{N}_a(\mu-\mu_k,\alpha) \le \mathcal{N}_a(\mu_k,\alpha) + \|\mu-\mu_k\|_{\mathcal{C}_\alpha}.$$

Thus, for both cases $s = 0$ and $s = \infty$, we have

$$\lim_{a\to s} \mathcal{N}_a(\mu,\alpha) \le \|\mu-\mu_k\|_{\mathcal{C}_\alpha} \to 0 \quad \text{as } k\to\infty.$$

We also notice that for any cube Q in $\mathbb{R}^n$,

$$\mathcal{N}(\mu, Q, \alpha) \leq \mathcal{N}(\mu_k, Q, \alpha) + \mathcal{N}(\mu - \mu_k, Q, \alpha) \leq \mathcal{N}(\mu_k, Q, \alpha) + \|\mu - \mu_k\|_{\mathcal{C}_\alpha}.$$

So

$$\lim_{|x|\to\infty} \mathcal{N}(\mu, Q + x, \alpha) \leq \|\mu - \mu_k\|_{\mathcal{C}_\alpha} \to 0 \quad \text{as } k \to \infty.$$

That is, $\mu \in \mathcal{C}_{\alpha,v}$ and $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ is a Banach space. □

Now we introduce a subclass of $\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$ as follows.

$$\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+) = \{\mu \in \mathcal{C}_\alpha : \text{there exists a compact set } K \text{ in } \mathbb{R}^{n+1}_+, \text{ such that for any } \mu\text{-measurable set } E \text{ in } \mathbb{R}^{n+1}_+,\ \mu(E) = \mu(E \cap K)\}.$$

We claim that $\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ is dense in $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathcal{C}_\alpha}$.

Lemma 2.8. *$\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)$ is dense in $\mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$ in the norm $\|\cdot\|_{\mathcal{C}_\alpha}(\mathbb{R}^{n+1}_+)$. That is,*

$$\overline{\mathcal{C}_{\alpha,c}(\mathbb{R}^{n+1}_+)}^{\|\cdot\|_{\mathcal{C}_\alpha}} = \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+).$$

Proof. We first prove $\mathcal{C}_{\alpha,c} \subset \mathcal{C}_{\alpha,v}$. In fact, for any $\mu \in \mathcal{C}_{\alpha,c}$, there exists a compact set $K \subset \mathbb{R}^{n+1}_+$ such that for any μ-measurable set $E \subset \mathbb{R}^{n+1}_+$, $\mu(E) = \mu(E \cap K)$. For $a > 0$ and a cube $Q \subset \mathbb{R}^n$ with $|Q| = a$, then we have the following facts:

(i) If a is small enough, then $\widehat{Q} \cap K = \emptyset$, so $\mu(\widehat{Q})/|Q|^\alpha = \mu(\widehat{Q} \cap K)/|Q|^\alpha = 0$.

(ii) If a is large enough, such that $\widehat{Q} \cap K \neq \emptyset$, then $\mu(\widehat{Q})/|Q|^\alpha \leq a^{-\alpha}\mu(K)$. Thus, $\mathcal{N}_a(\mu, \alpha) \leq a^{-\alpha}\mu(K)$ and $\lim_{a\to\infty} \mathcal{N}_a(\mu, \alpha) = 0$.

(iii) If $|x| \to \infty$, then $\widehat{Q + x} \cap K = \emptyset$. Thus, $\mu(\widehat{Q + x}) = 0$. So $\lim_{|x|\to\infty} \mu(\widehat{Q + x})/|Q|^\alpha = 0$. Hence, $\mathcal{C}_{\alpha,c} \subset \mathcal{C}_{\alpha,v}$ and $\overline{\mathcal{C}_{\alpha,c}}^{\|\cdot\|_{\mathcal{C}_\alpha}} \subset \mathcal{C}_{\alpha,v}$ by Lemma 2.7.

Below we verify $\mathcal{C}_{\alpha,v} \subset \overline{\mathcal{C}_{\alpha,c}}^{\|\cdot\|_{\mathcal{C}_\alpha}}$. Let $E_k = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y| \leq k, 1/k \leq t \leq k\}$ for any $k \in \mathbb{N}$. For $\mu \in \mathcal{C}_{\alpha,v}$, denote $\mu_k(E) = \mu(E \cap E_k)$ for any μ-measurable set E in $\mathbb{R}^{n+1}_+$, then it is easy to see $\mu_k \in \mathcal{C}_{\alpha,c}$. So, to finish the proof of Lemma 2.8 it remains to show that

$$\lim_{k\to\infty} \|\mu - \mu_k\|_{\mathcal{C}_\alpha} = 0. \tag{2.1}$$

Let

$$F^1_k = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : t > k\right\},$$
$$F^2_k = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : 0 < t < \frac{1}{k}\right\},$$
$$F^3_k = \left\{(y,t) \in \mathbb{R}^{n+1}_+ : |y| > k, \frac{1}{k} \leq t \leq k\right\}.$$

Then it is easy to see that $\mathbb{R}^{n+1}_+ = E_k \cup F_k^1 \cup F_k^2 \cup F_k^3$ for any $k \in \mathbb{N}$, and for any cube Q in $\mathbb{R}^n$ with center x_Q and side length $\ell(Q)$,

$$\frac{(\mu-\mu_k)(\widehat{Q})}{|Q|^\alpha} \leq \frac{\mu(\widehat{Q}\cap F_k^1)}{|Q|^\alpha} + \frac{\mu(\widehat{Q}\cap F_k^2)}{|Q|^\alpha} + \frac{\mu(\widehat{Q}\cap F_k^3)}{|Q|^\alpha} =: I_1 + I_2 + I_3.$$

Thus, to get (2.1) we only need to show that

$$\lim_{k\to\infty} \sup_{Q\subset\mathbb{R}^n} I_i = 0 \quad \text{for } i = 1,2,3. \tag{2.2}$$

Case $i = 1$. If $\ell(Q) \leq k/2$, we have $\widehat{Q} \cap F_k^1 = \emptyset$. Then we have $\sup_{|Q|\leq(k/2)^n} I_1 = 0$. If $\ell(Q) > k/2$, then $|Q| \geq (k/2)^n \to \infty$ as $k \to \infty$. Thus, $\lim_{k\to\infty} \sup_{|Q|\geq(k/2)^n} I_1 = 0$ by $\mu \in \mathcal{C}_{\alpha,v}$ and Lemma 2.6. Hence (2.2) holds for $i = 1$.

Case $i = 2$. If $\ell(Q) \leq 2/k$, since $\mu \in \mathcal{C}_{\alpha,v}$ and applying Lemma 2.6, we have

$$\sup_{\ell(Q)\leq 2/k} I_2 \leq \sup_{\ell(Q)\leq 2/k} \frac{\mu(\widehat{Q})}{|Q|^\alpha} \to 0 \quad \text{as } k\to\infty.$$

If $\ell(Q) > 2/k$, applying Besicovitch covering lemma (see [25, p. 39]), then there exists a sequence of cubes $\{Q_j\}$ and c_n only depending on the dimension n such that:

(i) $\ell(Q_j) \in (1/k, 2/k)$;

(ii) $Q \subset \bigcup_j Q_j$;

(iii) $\sum_j \chi_{Q_j}(x) \leq c_n$ for each $x \in \mathbb{R}^n$.

Then it is easy to see that $(\widehat{Q} \cap F_k^2) \subset \bigcup_j \widehat{Q}_j$. Notice that $\alpha > 1$, we have

$$\begin{aligned} I_2 &\leq \sum_j \frac{\mu(\widehat{Q}_j)}{|Q|^\alpha} \leq \sum_j \frac{|Q_j|^\alpha}{|Q|^\alpha} \sup_{\ell(Q)\leq 2/k} \frac{\mu(\widehat{Q})}{|Q|^\alpha} \leq \left(\sum_j \frac{|Q_j|}{|Q|}\right)^\alpha \sup_{\ell(Q)\leq 2/k} \frac{\mu(\widehat{Q})}{|Q|^\alpha} \\ &\leq c_n^\alpha \sup_{\ell(Q)\leq 2/k} \frac{\mu(\widehat{Q})}{|Q|^\alpha} \to 0 \quad \text{as } k\to\infty. \end{aligned}$$

From this we see that (2.2) holds for $i = 2$.

Case $i = 3$. Obviously, we need only to consider the limit of $\sup_{\widehat{Q}\cap F_k^3\neq\emptyset} I_3$ as $k \to \infty$. By $\mu \in \mathcal{C}_{\alpha,v}$ and Lemma 2.6, we have

$$\begin{aligned} \sup_{\widehat{Q}\cap F_k^3\neq\emptyset} I_3 &\leq \sup_{|x_Q|\geq k} I_3 + \sup_{\substack{|x_Q|<k \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 \\ &\leq \sup_{|x_Q|\geq k} I_3 + \sup_{\substack{|x_Q|<k/2 \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 + \sup_{\substack{k/2\leq|x_Q|<k \\ \frac{1}{2}\ell(Q)>k-|x_Q|}} I_3 \end{aligned}$$

$$\le \sup_{|x_Q|\ge k} I_3 + \sup_{\ell(Q)>k} I_3 + \sup_{k/2\le|x_Q|<k} I_3$$
$$\le 2\sup_{|x_Q|\ge k/2}\frac{\mu(\widehat{Q})}{|Q|^\alpha} + \sup_{\ell(Q)>k}\frac{\mu(\widehat{Q})}{|Q|^\alpha} \to 0 \quad \text{as } k\to\infty.$$

Thus, (2.2) still holds in this case. We therefore show that $\mu \in \overline{\mathcal{C}_{\alpha,c}}^{\|\cdot\|_{\mathcal{C}_\alpha}}$ and complete the proof of Lemma 2.8. □

2.2. Some facts on tent spaces $\widetilde{T}^p_\infty$ $(0<p<1)$

In this subsection, we give some facts on the tent space $\widetilde{T}^p_\infty$ $(0<p<1)$, which will be used in the proof of Theorem 2.5.

Lemma 2.9. *For $0<p<1$, the norm of $\widetilde{T}^p_\infty$ can be characterized via $\mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+)$. More precisely,*

$$\|f\|_{\widetilde{T}^p_\infty} = \sup_{\substack{\mu\in\mathcal{C}_{1/p,c}\\ \|\mu\|_{\mathcal{C}_{1/p}}\le 1}} \left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t)\right|.$$

Proof. Since $(\widetilde{T}^p_\infty)^* = \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$ and $\mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+)\subset\mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$, we have

$$\|f\|_{\widetilde{T}^p_\infty} = \sup_{\substack{\mu\in\mathcal{C}_{1/p}\\ \|\mu\|_{\mathcal{C}_{1/p}}\le 1}} \left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t)\right| \ge \sup_{\substack{\mu\in\mathcal{C}_{1/p,c}\\ \|\mu\|_{\mathcal{C}_{1/p}}\le 1}} \left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t)\right|. \tag{2.3}$$

It remains to prove that for any $\epsilon>0$, there exists $\mu_0\in\mathcal{C}_{1/p,c}$ with $\|\mu_0\|_{\mathcal{C}_{1/p}}\le 1$ such that

$$\left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu_0(y,t)\right| \ge \|f\|_{\widetilde{T}^p_\infty} - \epsilon.$$

In fact, from (2.3), there exists a measure $\mu\in\mathcal{C}_{1/p}$ with $\|\mu\|_{\mathcal{C}_{1/p}}\le 1$ such that

$$\left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t)\right| \ge \|f\|_{\widetilde{T}^p_\infty} - \frac{\epsilon}{2}.$$

For $k\in\mathbb{N}$, let $\chi_k = \chi_{\{(y,t)\in\mathbb{R}^{n+1}_+ : |y|\le k, 1/k\le t\le k\}}$. Note that

$$\left|\int_{\mathbb{R}^{n+1}_+} f(y,t)(1-\chi_k(y,t))\,d\mu(y,t)\right| \to 0 \quad \text{as } k\to\infty.$$

Thus, there exists $k_0>0$, such that

$$\left|\int_{\mathbb{R}^{n+1}_+} f(y,t)(1-\chi_{k_0}(y,t))\,d\mu(y,t)\right| < \frac{\epsilon}{2}.$$

Hence,

$$\left|\int_{\mathbb{R}^{n+1}_+} f(y,t)\chi_{k_0}(y,t)\,d\mu(y,t)\right| \geq \|f\|_{\widetilde{T}^p_\infty} - \epsilon.$$

Denote $d\mu_0 := \chi_{k_0}\,d\mu$, then it is easy to see $\mu_0 \in \mathcal{C}_{1/p,c}$ and $\|\mu_0\|_{\mathcal{C}_{1/p}} \leq 1$. We prove Lemma 2.9. □

Notice that from Definition 2.4, we know that the space $\widetilde{T}^p_\infty$ $(0 < p < 1)$ has an atom decomposition. But the relationship of the support set of each atom is not clear. In [36], Wang gave a more delicate atom decomposition of $\widetilde{T}^p_\infty$ $(0 < p < 1)$.

Lemma 2.10. [36] *For every fixed $k \in \mathbb{Z}$, there is a sequence $\{Q_{jk}\}_j$ of cubes in $\mathbb{R}^n$ which satisfies*

- (i) $|Q_{jk}| = \beta^k$, $\beta = 3^n$ *for any* $j = 1, 2, \ldots$;
- (ii) $\bigcup_{j=1}^\infty Q_{jk} = \mathbb{R}^n$;
- (iii) $\sum_{j=1}^\infty \chi_{Q_{jk}}(x) \leq \beta$ *for any* $x \in \mathbb{R}^n$;
- (iv) *for each* $f \in \widetilde{T}^p_\infty$ $(0 < p < 1)$, *we have* $f = \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty \lambda_{jk}a_{jk}$, *where* a_{jk} *is the* T^p_∞ *atom supported in* $\widehat{Q}_{jk}$ *and* $\sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{jk}| \leq C\|f\|_{\widetilde{T}^p_\infty}$, *the constant* C *is independent of the sequence* $\{\lambda_{jk}\}$ *and* f.

In order to prove Theorem 2.5, we also need the following lemma which is given by Coifman and Weiss in [15].

Lemma 2.11. [15] *Suppose* $\lambda_{jk} \geq 0$, $j, k = 1, 2, \ldots$, *satisfies* $\sum_{j=1}^\infty \lambda_{jk} \leq 1$ *for each* $k = 1, 2, \ldots$, *then there exists an increasing sequence of natural numbers* $k_1 < k_2 < \cdots < k_l < \cdots$ *such that* $\lim_{l\to\infty} \lambda_{jk_l} = \lambda_j$ *for each* j *and* $\sum_{j=1}^\infty \lambda_j \leq 1$.

The following result plays a key important role in the proof of Theorem 2.5.

Lemma 2.12. *Suppose* $0 < p < 1$, $\{f_l\}_{l\in\mathbb{N}} \subset \widetilde{T}^p_\infty$ *with* $\|f_l\|_{\widetilde{T}^p_\infty} \leq M$, *where* $M > 0$ *is independent of* $l = 1, 2, \ldots$. *Then there exist a function* $f \in \widetilde{T}^p_\infty$ *and a subsequence* $\{f_{l_s}\}_s$ *such that*

$$\lim_{s\to\infty}\int_{\mathbb{R}^{n+1}_+} f_{l_s}(y,t)\,d\mu(y,t) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t) \quad \text{for any } \mu \in \mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+). \tag{2.4}$$

Proof. Applying Lemma 2.10, $f_l = \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty \lambda_{l,jk}a_{l,jk}$ with $\sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{l,jk}| \leq C\|f_l\|_{\widetilde{T}^p_\infty}$, where C is independent of $\{\lambda_{l,jk}\}$ and f_l. For any fixed $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, by Lemma 2.11, there exist a subsequence $\{\lambda_{l_s,jk}\}_s$ and λ_{jk} such that

$$\lim_{s\to\infty}|\lambda_{l_s,jk}| = |\lambda_{jk}| \quad \text{and} \quad \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty |\lambda_{jk}| \leq C\|f_l\|_{\widetilde{T}^p_\infty} \leq CM.$$

Notice that all $a_{l,jk}$ are T^p_∞ atoms supported in $\widehat{Q}_{jk}$ with Q_{jk} satisfying (i), (ii), (iii) in Lemma 2.10. From the proof of lemma 2.10, it is easy to see that the cube sequence $\{Q_{jk}\}$ is independent of l_s. And for (j,k) fixed,

$$\|a_{l_s,jk}\|_{L^\infty} \le |Q_{jk}|^{-1/p} = \beta^{-k/p} < \infty \tag{2.5}$$

holds uniformly with the bound independent of l_s. Hence there exist a subsequence, which is still denoted by $\{a_{l_s,jk}\}_s$, and a function $a_{jk} \in L^\infty$ supported in $\widehat{Q}_{jk}$ such that

$$\lim_{s\to\infty} \int_{\widehat{Q}_{jk}} a_{l_s,jk}(y,t)g(y,t)\,\frac{dydt}{t} = \int_{\widehat{Q}_{jk}} a_{jk}(y,t)g(y,t)\,\frac{dydt}{t} \quad \text{for all } g \in L^1\left(\widehat{Q}_{jk};\frac{dydt}{t}\right). \tag{2.6}$$

Thus, by (2.6) it is easy to see that

$$\begin{aligned}
\|a_{jk}\|_{L^\infty} &= \sup_{\|g\|_{L^1\left(\widehat{Q}_{jk};\frac{dydt}{t}\right)}\le 1} \left|\int_{\widehat{Q}_{jk}} a_{jk}(y,t)g(y,t)\,\frac{dydt}{t}\right| \\
&= \sup_{\|g\|_{L^1\left(\widehat{Q}_{jk};\frac{dydt}{t}\right)}\le 1} \lim_{s\to\infty} \left|\int_{\widehat{Q}_{jk}} a_{l_s,jk}(y,t)g(y,t)\,\frac{dydt}{t}\right| \\
&\le \sup_{\|g\|_{L^1\left(\widehat{Q}_{jk};\frac{dydt}{t}\right)}\le 1} |Q_{jk}|^{-1/p}\|g\|_{L^1\left(\widehat{Q}_{jk};\frac{dydt}{t}\right)} \\
&\le |Q_{jk}|^{-1/p}.
\end{aligned}$$

Thus, a_{jk} is a T^p_∞ atom. Let $f = \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty \lambda_{jk}a_{jk}$, then $f \in \widetilde{T}^p_\infty$. Below we prove that f satisfies (2.4).

Assume that $\mu \in \mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+)$, then there exists a compact set $K \subset \mathbb{R}^{n+1}_+$ such that for any μ-measurable set $E \subset \mathbb{R}^{n+1}_+$, $\mu(E) = \mu(E\cap K)$, then

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} f_{l_s}(y,t)\,d\mu(y,t) &= \int_{\mathbb{R}^{n+1}_+} \left(\sum_{-N\le k\le N} + \sum_{k<-N} + \sum_{k>N}\right)\sum_{j=1}^\infty \lambda_{l_s,jk}a_{l_s,jk}(y,t)\,d\mu(y,t) \\
&=: II_1 + II_2 + II_3.
\end{aligned}$$

Since K is a compact set in $\mathbb{R}^{n+1}_+$, so there is a $t_0 > 0$ such that $K \subset \{(x,t) \in \mathbb{R}^{n+1}_+ : t > t_0\}$. We now take N large enough such that $\beta^{-N/n} < t_0$.

Estimate of II_2. Note that $\ell(Q_{jk}) = \beta^{k/n} < \beta^{-N/n} < t_0$ for all $j \in \mathbb{N}$. Thus,

$$\left(\bigcup_{k<-N}\bigcup_{j=1}^\infty \widehat{Q}_{jk}\right)\bigcap K = \emptyset.$$

Therefore, $II_2 = 0$.

Estimate of II_3. By (2.5), it is easy to see that

$$\begin{aligned}|II_3| &\le \sum_{k>N}\sum_{j=1}^{\infty}|\lambda_{l_s,jk}|\|a_{l_s,jk}\|_{L^\infty}\mu(K)\\ &\le C_K\|\mu\|_{\mathcal{C}_{1/p}}\sum_{k>N}\sum_{j=1}^{\infty}|\lambda_{l_s,jk}||Q_{jk}|^{-1/p}\\ &\le CMC_K\|\mu\|_{\mathcal{C}_{1/p}}\beta^{-N/p}\to 0 \quad \text{as } N\to\infty.\end{aligned}$$

Estimate of II_1. Notice that for any $k\in[-N,N]$ fixed, the set $\{j\in\mathbb{N}:\widehat{Q}_{jk}\cap K\neq\emptyset\}$ is a finite set. That is, there exists a integer $m>0$ such that

$$\begin{aligned}II_1 &= \int_{\mathbb{R}^{n+1}_+}\sum_{-N\le k\le N}\sum_{j=1}^{m}\lambda_{l_s,jk}a_{l_s,jk}(y,t)\,d\mu(y,t)\\ &= \sum_{-N\le k\le N}\sum_{j=1}^{m}\lambda_{l_s,jk}\int_{\mathbb{R}^{n+1}_+}a_{l_s,jk}(y,t)\,d\mu(y,t).\end{aligned}$$

Notice that $t\,d\mu\in L^1(\widehat{Q}_{jk};\frac{dydt}{t})$, then from (2.6),

$$\lim_{s\to\infty}II_1 = \sum_{-N\le k\le N}\sum_{j=1}^{m}\lambda_{jk}\int_{\mathbb{R}^{n+1}_+}a_{jk}(y,t)\,d\mu(y,t).$$

Thus,

$$\begin{aligned}\lim_{s\to\infty}\lim_{N\to\infty}II_1 &= \lim_{N\to\infty}\sum_{-N\le k\le N}\sum_{j=1}^{m}\lambda_{jk}\int_{\mathbb{R}^{n+1}_+}a_{jk}(y,t)\,d\mu(y,t)\\ &= \lim_{N\to\infty}\int_{\mathbb{R}^{n+1}_+}\sum_{-N\le k\le N}\sum_{j=1}^{\infty}\lambda_{jk}a_{jk}(y,t)\,d\mu(y,t)\\ &= \int_{\mathbb{R}^{n+1}_+}f(y,t)\,d\mu(y,t).\end{aligned}$$

We therefore complete the proof of Lemma 2.12. □

2.3. Proof of Theorem 2.5

The proof of Theorem 2.5 needs to use a general result in functional analysis. Let us give the definition of the total set, which can be found in [21, p. 58].

Definition 2.13 (total set)**.** A set W of maps which map a vector space X into another vector space Y is called a total set if $x=0$ is the only vector for which $\phi(x)=0$ for all $\phi\in W$.

Lemma 2.14. [21, p. 439] *Let X be a locally convex linear topological space and W be a linear subspace of X^*. Then W is X-dense in X^* if and only if W is a total set of functionals on X.*

Proof of Theorem 2.5. Note that $(\widetilde{T}^p_\infty)^* = \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+) \supset \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$, so $\widetilde{T}^p_\infty \subset (\mathcal{C}_{1/p,v})^*$.

On the other hand, if there exists a $\mu \in \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$, such that

$$\int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t) = 0 \quad \text{for all } f \in \widetilde{T}^p_\infty, \tag{2.7}$$

then by $(\widetilde{T}^p_\infty)^* = \mathcal{C}_{1/p}(\mathbb{R}^{n+1}_+)$, we see that

$$\|\mu\|_{\mathcal{C}_{1/p}} = \sup_{\|f\|_{\widetilde{T}^p_\infty} \le 1} \left| \int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t) \right| = 0.$$

Thus, $\mu = 0$. In particular, since $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ is a Banach space, so it is obvious that μ is the only measure in $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ such that (2.7) holds. Thus, $\widetilde{T}^p_\infty$ is a total set on $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ by Definition 2.13. Clearly, $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ is locally convex linear topological space by Lemma 2.7. Applying Lemma 2.14, $\widetilde{T}^p_\infty$ is weak* dense in $(\mathcal{C}_{1/p,v})^*$. So, for any $\ell \in (\mathcal{C}_{1/p,v})^*$, there exists a sequence of functions $\{f_k\} \subset \widetilde{T}^p_\infty$ such that

$$\ell(\mu) = \lim_{k\to\infty} \langle f_k, \mu\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\,d\mu(y,t) \quad \text{for all } \mu \in \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+).$$

From this, we see that for each fixed $\mu \in \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$,

$$\sup_k \left| \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\,d\mu(y,t) \right| < \infty. \tag{2.8}$$

The uniform boundedness principle (Banach-Steinhaus Theorem) and (2.8) imply that

$$\sup_k \left| \int_{\mathbb{R}^{n+1}_+} f_k(y,t)\,d\mu(y,t) \right| \le C\|\mu\|_{\mathcal{C}_{1/p}} \quad \text{for any } \mu \in \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+).$$

Thus, by Lemma 2.9, we get $\|f_k\|_{\widetilde{T}^p_\infty} \le C$ where C is independent of k. Now, applying Lemma 2.12, we can obtain a subsequence $\{f_{k_j}\}_j$ and a function $f \in \widetilde{T}^p_\infty$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^{n+1}_+} f_{k_j}(y,t)\,d\mu(y,t) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t) \quad \text{for any } \mu \in \mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+).$$

Therefore,

$$\ell(\mu) = \int_{\mathbb{R}^{n+1}_+} f(y,t)\,d\mu(y,t) \quad \text{for any } \mu \in \mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+). \tag{2.9}$$

Finally, by the density of $\mathcal{C}_{1/p,c}(\mathbb{R}^{n+1}_+)$ in $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ (i.e., Lemma 2.8), we can check that (2.9) still holds for all $\mu \in \mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$, which shows that the linear functional ℓ on $\mathcal{C}_{1/p,v}(\mathbb{R}^{n+1}_+)$ can be represented by a function f in $\widetilde{T}^p_\infty$. Therefore, we prove indeed that $(\mathcal{C}_{1/p,v})^* \subset \widetilde{T}^p_\infty$ and complete the proof of Theorem 2.5. □

3. α-Carleson measure and Campanato functions

It is known that one may construct a Carleson measure on $\mathbb{R}^{n+1}_+$ via a BMO($\mathbb{R}^n$) function. More precisely, if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class) with $\int \psi = 0$, then $|\psi_t * b(x)|^2 \frac{dxdt}{t} \in \mathcal{C}(\mathbb{R}^{n+1}_+)$ (see [35], for example). In 2007, J. Xiao constructed an α-Carleson measure via a Campanato function (see Lemma 2.1 in [37]). First let's recall the definition of the Campanato space.

Definition 3.1. [32] Suppose $1 \le p < \infty$, $\lambda \ge 0$. For any ball $B = B(x,r) \subset \mathbb{R}^n$, $a > 0$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\mathcal{M}(f,B,p,\lambda) = \left(\frac{1}{r^\lambda}\int_B |f(y) - f_B|^p\,dy\right)^{1/p},$$

and

$$\mathcal{M}_a(f,p,\lambda) = \sup_{|B|=a} \mathcal{M}(f,B,p,\lambda),$$

where $f_B = \frac{1}{|B|}\int_B f(y)\,dy$. Then the Campanato space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is defined as

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{p,\lambda} < \infty\},$$

where

$$\|f\|_{p,\lambda} := \sup_{a>0} \mathcal{M}_a(f,p,\lambda) < \infty.$$

Lemma 3.2. [37, p. 233] *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. Suppose $b \in \mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ with $\alpha > 1$. Let $d\mu(x,t) := |b * \psi_t(x)|^2 \frac{dxdt}{t}$, then μ is an α-Carleson measure on $\mathbb{R}^{n+1}_+$, and $\|\mu\|_{\mathcal{C}_\alpha} \le C\|b\|^2_{2,n\alpha}$.*

3.1. A vanishing α-Carleson measure constructed via a vanishing Campanato function

Recently, in [18], we give a construction of vanishing Carleson measures via CMO space, a subspace of BMO space. In this subsection, we can use a similar way to construct a vanishing α-Carleson measure via a subclass of the Campanato space, which we define as the vanishing Campanato space.

Definition 3.3 (vanishing Campanato functions)**.** Suppose $1 \le p < \infty$ and $\lambda \ge 0$. If $f \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ which satisfies:

(i) $\lim_{a\to 0} \mathcal{M}_a(f,p,\lambda) = 0$;

(ii) $\lim_{a\to\infty} \mathcal{M}_a(f,p,\lambda) = 0$;

(iii) $\lim_{|x|\to\infty} \mathcal{M}(f,B+x,p,\lambda) = 0$ for any ball B in $\mathbb{R}^n$.

Then f is called a vanishing Campanato function. The set of all vanishing Campanato functions is defined by $\mathcal{L}_v^{p,\lambda}(\mathbb{R}^n)$.

Theorem 3.4. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$. Suppose $b \in \mathcal{L}_v^{2,n\alpha}(\mathbb{R}^n)$ with $\alpha > 1$. Let $d\mu(x,t) := |b * \psi_t(x)|^2 \frac{dxdt}{t}$, then μ is a vanishing α-Carleson measure on $\mathbb{R}_+^{n+1}$.*

Proof. Fix $a > 0$, for any ball $B \subset \mathbb{R}^n$ with $|B| = a$, let $B^* = 2B$. From the proof of Lemma 3.2, we know

$$\frac{1}{|B|^\alpha} \int_{\widehat{B}} |b * \psi_t(x)|^2 \frac{dxdt}{t} \le C \frac{1}{|B^*|^\alpha} \int_{B^*} |b(y) - b_{B^*}|^2\,dy + C \left[\sum_{k=1}^{\infty} \beta_k g_k(b,B)\right]^2$$

holds. Since $b \in \mathcal{L}_v^{2,n\alpha}(\mathbb{R}^n)$, then

$$\lim_{a\to 0} \sup_{|B|=a} g_k(b,B) = \lim_{a\to\infty} \sup_{|B|=a} g_k(b,B) = \lim_{|x|\to\infty} g_k(b, B+x) = 0.$$

Combining with Lemma 3.2 in [18] and the fact

$$\sup_k \sup_{a>0} \sup_{|B|=a} |g_k(b,B)| \le C,$$

we have, for both cases $s = 0$ and $s = \infty$,

$$\lim_{a\to s} \sup_{|B|=a} \sum_{k=1}^{\infty} \beta_k g_k(b,B) = 0$$

and

$$\lim_{|x|\to\infty} \sum_{k=1}^{\infty} \beta_k g_k(b, B+x) = 0,$$

which implies that $\mu \in \mathcal{C}_{\alpha,v}$. Thus, we prove Theorem 3.4. □

3.2. The weakened results

Note that in Lemma 3.2 and Theorem 3.4, we assumed the smoothness condition on the function ψ. In this subsection, we show that the smoothness condition assumed on ψ can be removed. Before stating our result, let us recall some known results. Suppose that the function Ω satisfies the following conditions:

$$\Omega(\lambda x) = \Omega(x) \quad \text{for any } \lambda > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}, \tag{3.1}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\,d\sigma(x') = 0, \tag{3.2}$$

$$\Omega \in L^1(\mathbb{S}^{n-1}).$$

Let $\psi(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|<1\}}(x)$, then the Littlewood-Paley operator g_Ω (i.e., Marcinkiewicz integral introduced by Stein [34]) associated with ψ is defined by

$$\begin{aligned} g_\psi(f)(x) &= \left(\int_0^\infty |f*\psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \\ &= \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right)^{1/2} =: g_\Omega(f)(x). \end{aligned} \tag{3.3}$$

It is well known that g_Ω is bounded on L^p for $1<p<\infty$ with Ω satisfying (3.1) and (3.2) and $\Omega \in H^1(\mathbb{S}^{n-1})$ or $\Omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$ (see [1,17]).

Remark 3.5. The following containing relation between some function spaces on $\mathbb{S}^{n-1}$ is well known:

$$L^\infty(\mathbb{S}^{n-1}) \subsetneq L^r(\mathbb{S}^{n-1})\ (1<r<\infty) \subsetneq L\log^+ L(\mathbb{S}^{n-1}) \subsetneq H^1(\mathbb{S}^{n-1}) \subsetneq L^1(\mathbb{S}^{n-1}).$$

Moreover, the spaces $H^1(\mathbb{S}^{n-1})$ and $L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$ do not contain each other.

Now we may construct an α-Carleson measure and a vanishing α-Carleson measure under some weaker conditions.

Theorem 3.6. *Let $b \in \mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ with $\alpha > 1$. Suppose that $\psi(x) = \frac{\Omega(x)}{|x|^{n-1}}\chi_{\{|x|<1\}}(x)$ with Ω satisfying* (3.1) *and* (3.2). *Let $d\nu := |b*\psi_t(x)|^2\,\frac{dxdt}{t}$, if $\Omega \in H^1(\mathbb{S}^{n-1})$ or $\Omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$, then $\nu \in \mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$.*

Proof. For any ball $B = B(x_0, r) \subset \mathbb{R}^n$, let $B^* = B(x_0, 2r)$. Decompose b as follows:

$$b = b_{B^*} + (b - b_{B^*})\chi_{B^*} + (b-b_{B^*})\chi_{(B^*)^c} =: b_0 + b_1 + b_2.$$

From (3.2), we have $b_0 * \psi_t = 0$.

Applying Theorem 4.1, one may see that

$$\begin{aligned} \frac{1}{|B|^\alpha}\int_{\widehat{B}} |b_1*\psi_t(x)|^2\,\frac{dxdt}{t} &\le \frac{1}{|B|^\alpha}\|g_\Omega(b_1)\|_2^2 \\ &\le C\frac{1}{|B^*|^\alpha}\int_{B^*}|b(y)-b_{B^*}|^2\,dy \le C\|b\|_{2,n\alpha}^2. \end{aligned} \tag{3.4}$$

For $(x,t) \in \widehat{B}$ and $y \in (B^*)^c$, we see that $|x-y| \ge |y-x_0| - |x-x_0| > 2r - r = r$ and $t < r$. Thus, when $(x,t) \in \widehat{B}$,

$$b_2 * \psi_t(x) = \int_{(B^*)^c} \frac{\Omega(x-y)}{t|x-y|^{n-1}}(b(y)-b_{B^*})\chi_{\{|x-y|<t\}}(y)\,dy = 0.$$

Therefore, $\nu \in \mathcal{C}_\alpha$ and $\|\nu\|_{\mathcal{C}_\alpha} \le C\|b\|_{2,n\alpha}^2$. We finish the proof of Theorem 3.6. $\square$

Similarly, from (3.4), we can obtain a vanishing α-Carleson measure under improved conditions. We omit the proof.

Theorem 3.7. *Let* $b \in \mathcal{L}_v^{2,n\alpha}(\mathbb{R}^n)$ *with* $\alpha > 1$. *Suppose that* $\psi(x) = \frac{\Omega(x)}{|x|^{n-1}} \chi_{\{|x|<1\}}(x)$ *with* Ω *satisfying* (3.1) *and* (3.2). *Let* $d\nu := |b * \psi_t(x)|^2 \frac{dxdt}{t}$, *if* $\Omega \in H^1(\mathbb{S}^{n-1})$ *or* $\Omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$, *then* $\nu \in \mathcal{C}_{\alpha,v}(\mathbb{R}_+^{n+1})$.

4. A characterization of the vanishing α-Carleson measure

It is well known that the Carleson measure can be characterized via the boundedness of Poisson integral (see [7,8]). For $f \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, Poisson integral of f is defined by $u(x,t) := p_t * f(x)$ $(t > 0)$, where $p_t(x) = c_n \frac{t}{(|x|^2+t^2)^{(n+1)/2}}$ with $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$ is the Poisson kernel on $\mathbb{R}_+^{n+1}$.

For the α-Carleson measure, Barker [6] and Johnson [30] showed respectively that if $\alpha > 1$, then the α-Carleson measure μ on $\mathbb{R}_+^{n+1}$ can also be characterized via the boundedness of Poisson integral.

Theorem 4.1. (see [6,30]) *Suppose* $\alpha > 1$, *the following are equivalent:*

(i) *The measure* μ *is an* α*-Carleson measure on* $\mathbb{R}_+^{n+1}$;

(ii) *Poisson integral is bounded from* $L^p(\mathbb{R}^n; dx)$ *to* $L^{p\alpha}(\mathbb{R}_+^{n+1}; d\mu)$ *for every* $1 < p < \infty$;

(iii) *Poisson integral is bounded from* $L^p(\mathbb{R}^n; dx)$ *to* $L^{p\alpha}(\mathbb{R}_+^{n+1}; d\mu)$ *for some* $1 < p < \infty$.

Further, using the method of proving Theorem 7.3.7 in [24], one can give an extension of Theorem 4.1 (see Corollary 4.2 below), we omit its proof here. Suppose that the function φ on $\mathbb{R}^n$ satisfies

$$|\varphi(x)| \le C(1 + |x|)^{-n-\epsilon} \quad \text{for some } C, \epsilon > 0 \text{ and all } x \in \mathbb{R}^n. \tag{4.1}$$

The convolution operator associated with φ is denoted by

$$\mathcal{L}_\varphi \colon f \mapsto \varphi_t * f, \tag{4.2}$$

where and in the sequel, $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$ and $x \in \mathbb{R}^n$.

Corollary 4.2. *Suppose* $\alpha > 1$ *and* φ *satisfies* (4.1).

(i) *If the measure* μ *on* $\mathbb{R}_+^{n+1}$ *is an* α*-Carleson measure, then for every* $1 < p < \infty$, *the operator* $\mathcal{L}_\varphi$ *is bounded from* $L^p(\mathbb{R}^n; dx)$ *to* $L^{p\alpha}(\mathbb{R}_+^{n+1}; d\mu)$ *with* $\|\mathcal{L}_\varphi\|_{L^p(dx)\to L^{p\alpha}(d\mu)} \le C_{n,p}\|\mu\|_{\mathcal{C}_\alpha}^{1/(p\alpha)}$.

(ii) *If $\varphi \geq 0$ and $\int_{|x|\leq 1} \varphi(x)\,dx > 0$ yet, and a measure μ is defined on $\mathbb{R}^{n+1}_+$ such that the operator $\mathcal{L}_\varphi$ is bounded from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$ for some $1 < p < \infty$, then μ is an α-Carleson measure, and $\|\mu\|_{\mathcal{C}_\alpha} \leq \|\mathcal{L}_\varphi\|^{p\alpha}_{L^p(dx)\to L^{p\alpha}(d\mu)}$.*

Remark 4.3. Similarly, using the idea of proving Theorem 9.5 in [22], it is easy to see that Corollary 4.2 still holds if the condition (4.1) assumed on φ is replaced by $\phi \in L^1 \cap L^\infty$ with $\phi(x) := \operatorname{ess\,sup}_{|x|\leq|y|} |\varphi(y)|$.

Now, we want to give a characterization of the vanishing α-Carleson measure via the compactness of the convolution operator $\mathcal{L}_\varphi$.

Theorem 4.4. *Suppose $\alpha > 1$ and φ satisfies* (4.1).

(i) *If the measure μ on $\mathbb{R}^{n+1}_+$ is a vanishing α-Carleson measure, then for every $1 < p < \infty$, the operator $\mathcal{L}_\varphi$ is a compact operator from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$.*

(ii) *If $\varphi \geq 0$ and $\int_{|x|\leq 1} \varphi(x)\,dx > 0$ yet, and a measure μ is defined on $\mathbb{R}^{n+1}_+$ such that the operator $\mathcal{L}_\varphi$ is compact from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$ for some $1 < p < \infty$, then μ is a vanishing α-Carleson measure.*

The following consequence of Theorem 4.4 is obvious.

Corollary 4.5. *Suppose $\alpha > 1$, the following are equivalent:*

(i) *The measure μ is a vanishing α-Carleson measure on $\mathbb{R}^{n+1}_+$;*

(ii) *Poisson integral is a compact operator from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$ for every $1 < p < \infty$;*

(iii) *Poisson integral is a compact operator from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$ for some $1 < p < \infty$.*

Remark 4.6. For the case $\alpha = 1$, the conclusions of Theorem 4.4 and Corollary 4.5 have been given by authors in [18].

Proof of Theorem 4.4. (i) Note that $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ is a reflexive space, by [33, p. 113], to show that $\mathcal{L}_\varphi$ is compact from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$, it suffices to prove that

$$\lim_{k\to\infty} \|\mathcal{L}_\varphi(f_k)\|_{L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)} = 0 \tag{4.3}$$

holds for any sequence $\{f_k\} \subset L^p(\mathbb{R}^n)$ which converges to zero weakly. Since $\mu \in \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$, then for any $\epsilon > 0$, there exist $\delta, M, N > 0$, for any ball $B \subset \mathbb{R}^n$ with the radius r_B and center x_B, then

$$\frac{\mu(\widehat{B})}{|B|^\alpha} < \epsilon \quad \text{if } r_B \leq \delta \text{ or } r_B \geq M \text{ or } |x_B| \geq N. \tag{4.4}$$

Set

$$\begin{aligned}
&E_1 := B(0,\delta)\times(0,\delta) = \widehat{B(0,\delta)},\\
&E_2 := \big(B(0,M+N+1)\setminus B(0,\delta)\big)\times(0,\delta/8),\\
&E_3 := \big(B(0,M+N+1)\setminus B(0,\delta)\big)\times[\delta/8,\delta)\cup B(0,M+N+1)\times[\delta,M+N+1),\\
&E_4 := \big(B(0,M+N+1)\times(0,M+N+1)\big)^c.
\end{aligned}$$

Denote $d\mu_i := \chi_{E_i}\,d\mu$ $(1\le i\le 4)$, then $\mu=\mu_1+\mu_2+\mu_3+\mu_4$. Since $\mu\in\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$, so it is easy to see that $\mu_i\in\mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$ for $1\le i\le 4$. Thus, to prove (4.3) it suffices to show the following four limits hold for any sequence $\{f_k\}$ which converges to zero weakly in $L^p(\mathbb{R}^n)$:

$$\lim_{k\to\infty}\int_{\mathbb{R}^{n+1}_+}|f_k*\varphi_t(x)|^{p\alpha}\,d\mu_i(x,t)=0,\quad 1\le i\le 4. \tag{4.5}$$

Note that the weak convergence of $\{f_k\}$ in $L^p(\mathbb{R}^n)$ together with the uniform boundedness principle implies that $\{f_k\}$ is uniformly bounded in $L^p(\mathbb{R}^n)$. That is, there exists a constant $C_0>0$, independent of k, such that $\|f_k\|_p\le C_0$.

We first give the estimate of (4.5) for $i=3$. Notice that for any $(x,t)\in E_3$, we have

$$|f_k*\varphi_t(x)|\le C_0\left(\frac{\delta}{8}\right)^{-n/p}\|\varphi\|_{p'}$$

and

$$\int_{\mathbb{R}^{n+1}_+}d\mu_3(x,t)\le\mu(B(0,\widehat{M+N}+1))\le C_{n,\alpha,M,N}\|\mu\|_{\mathcal{C}_\alpha}<\infty,$$

where the constant $C_{n,\alpha,M,N}$ only depends on the dimension n, α, M and N. Applying Lebesgue dominated convergence theorem and note that $\{f_k\}$ converges weakly to zero in $L^p(\mathbb{R}^n)$, we get (4.5) for $i=3$.

In order to prove (4.5) for $i=1,2,4$, we only need to give the following estimates

$$\|\mu_i\|_{\mathcal{C}_\alpha}<C_n\epsilon\quad\text{for } i=1,2,4, \tag{4.6}$$

where the constant $C_n>0$ only depends on the dimension n. In fact, by the conclusion (i) of Corollary 4.2 and (4.6), the following inequalities hold uniformly in k:

$$\left(\int_{\mathbb{R}^{n+1}_+}|f_k*\varphi_t(x)|^{p\alpha}\,d\mu_i(x,t)\right)^{1/(p\alpha)}\le C_nC_0\|\mu_i\|_{\mathcal{C}_\alpha}^{1/(p\alpha)}<\widetilde{C}C_0\epsilon^{1/(p\alpha)},\quad i=1,2,4,$$

where C_n and $\widetilde{C}$ are constants depending only on the dimension n.

Hence, to finish the proof of conclusion (i) in Theorem 4.4, it remains to verify (4.6). That is, we need to show that there exists $C_n>0$, depending on the dimension n only, such that (4.6) holds. For any fixed ball $B:=B(x_B,r_B)\subset\mathbb{R}^n$.

Case I: $r_B \le \delta$ or $r_B \ge M$ or $|x_B| \ge N$. By (4.4)

$$\frac{\mu_i(\widehat{B})}{|B|^\alpha} \le \frac{\mu(\widehat{B})}{|B|^\alpha} < \epsilon, \quad i = 1, 2, 4.$$

Case II: $\delta < r_B < M$ and $|x_B| < N$. In this case, by the definition of μ_1 and using (4.4), we get

$$\frac{\mu_1(\widehat{B})}{|B|^\alpha} \le \frac{\mu(\widehat{B(0,\delta)})}{|B(0,\delta)|^\alpha} < \epsilon.$$

Notice that $\widehat{B} \subset E_4^c$. Thus, $\mu_4(\widehat{B}) = 0$. It remains to verify (4.6) for $i = 2$.

Let $F := \{x \in \mathbb{R}^n : \delta \le |x| < M + N + 1\}$, then $\bigcup_{x \in F} B(x, \delta/4) \supset F$. Applying Besicovitch covering lemma (see [25, p. 39]), there exist $\{x_j\}_{1 \le j \le m} \subset F$ and $c_n > 0$ depending only on the dimension n such that

(i) $\bigcup_{1 \le j \le m} B(x_j, \delta/4) \supset F$;

(ii) $\sum_{1 \le j \le m} \chi_{B_j}(x) \le c_n$ for each $x \in \mathbb{R}^n$, where $B_j := B(x_j, \delta/4)$.

Note that $B \cap F \ne \emptyset$, let $\mathcal{H} := \{B_j : B_j \cap (B \cap F) \ne \emptyset, 1 \le j \le m\}$. Then

$$B \cap F \subset \bigcup_{B_j \in \mathcal{H}} B_j \quad \text{and} \quad \widehat{B} \cap E_2 \subset \bigcup_{B_j \in \mathcal{H}} \widehat{B}_j.$$

Moreover, it is clear that $|\bigcup_{B_j \in \mathcal{H}} B_j| \le 4^n |B|$ since $r_B > \delta$. Therefore, notice that $\alpha > 1$,

$$\frac{\mu_2(\widehat{B})}{|B|^\alpha} \le \frac{\sum_{B_j \in \mathcal{H}} \mu(\widehat{B_j})}{|B|^\alpha} \le \epsilon \frac{\sum_{B_j \in \mathcal{H}} |B_j|^\alpha}{|B|^\alpha} \le \epsilon c_n^\alpha \left(\frac{|\bigcup_{B_j \in \mathcal{H}} B_j|}{|B|} \right)^\alpha \le (4^n c_n)^\alpha \epsilon.$$

Thus, we finish the proof of conclusion (i) of Theorem 4.4.

(ii) Since $\mathcal{L}_\varphi$ is a compact operator from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$, so $\mathcal{L}_\varphi$ is also bounded from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$. Thus, $\mu \in \mathcal{C}_\alpha(\mathbb{R}^{n+1}_+)$ by the conclusion (ii) of Corollary 4.2. Below we verify μ satisfies the conditions (i)–(iii) in Definition 2.1.

Note that in the definitions of $\mathcal{C}_\alpha$ (Definition 1.4) and $\mathcal{C}_{\alpha,v}$ (Definition 2.1), we may replace $\widehat{B}$ by $T(B) := \{(x,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset B\}$ for any ball B in $\mathbb{R}^n$. Hence, in order to prove $\mu \in \mathcal{C}_{\alpha,v}(\mathbb{R}^{n+1}_+)$, we only need to verify

$$\lim_{k \to \infty} \frac{\mu(T(B_k))}{|B_k|^\alpha} = 0$$

holds for any sequence of balls $\{B_k\}$ in $\mathbb{R}^n$, which satisfies one of the following three conditions:

(a) $\lim_{k \to \infty} |B_k| = 0$;

(b) $\lim_{k \to \infty} |B_k| = \infty$;

(c) $B_k = B + x_k$ with $\lim_{k\to\infty} |x_k| = \infty$ for each ball B in $\mathbb{R}^n$.

Let $f_k(x) = \frac{1}{|B_k|^{1/p}} \chi_{2B_k}(x)$, then from Lemma 2.3 in [18], $\{f_k\}$ converges to zero weakly in L^p under one of the above cases (a), (b) and (c). Since the linear operator $\mathcal{L}_\varphi$ is compact from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$, so

$$\lim_{k\to\infty} \|\mathcal{L}_\varphi(f_k)\|_{L^{p\alpha}(\mathbb{R}^{n+1}_+;d\mu)} = 0. \tag{4.7}$$

Notice that for any ball $B = B(x_0, r) \subset \mathbb{R}^n$, for any $(x,t) \in T(B)$, we have

$$\chi_{2B} * \varphi_t(x) = \int_{B(x_0,2r)} \varphi_t(x-y)\,dy \geq \int_{B(0,t)} \varphi_t(y)\,dy = \int_{B(0,1)} \varphi(y)\,dy =: A. \tag{4.8}$$

Thus, by (4.7) and (4.8), we have

$$\begin{aligned} \frac{\mu(T(B_k))}{|B_k|^\alpha} &\leq \frac{A^{-p\alpha}}{|B_k|^\alpha} \int_{T(B_k)} |\chi_{2B_k} * \varphi_t(x)|^{p\alpha}\,d\mu(x,t) \\ &\leq A^{-p\alpha} \int_{\mathbb{R}^{n+1}_+} |f_k * \varphi_t(x)|^{p\alpha}\,d\mu(x,t) \to 0 \quad \text{as } k\to\infty. \end{aligned}$$

Hence, we finish the proof of conclusion (ii). □

5. Applications: Boundedness and compactness of some paraproducts

As some applications of our results obtained above, we will give the boundedness and compactness of some paraproducts in this section.

Let us recall the definition of the paraproduct π_b with $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\pi_b(f)(x) = \int_0^\infty \eta_t * ((f*\varphi_t)(b*\psi_t))(x)\,\frac{dt}{t}, \tag{5.1}$$

where φ satisfies (4.1). It is well known that the paraproduct plays an important role in proving $T1$ Theorem (see [9, 22, 24], for example). We know that when η and ψ satisfy some appropriate conditions, the paraproduct π_b is bounded on L^2 if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and compact on L^2 if $b \in \mathrm{CMO}(\mathbb{R}^n)$ (see [18,20,35]), where $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$ norm.

5.1. Paraproduct with the smooth kernel

We first consider the case where both of ψ and η are Schwartz functions.

Theorem 5.1. *Suppose that the paraproduct π_b is defined by* (5.1)*, where*

(i) $\psi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$;

(ii) $\eta \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \eta(x)\,dx = 0$*;*

(iii) φ *satisfies* (4.1) *(or* $\phi \in L^1 \cap L^\infty$ *with* $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|} |\varphi(y)|$*);*

(iv) $b \in \mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ *with* $1 < \alpha < 2$*.*

Then π_b *is bounded from* $L^{2/\alpha}(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$ *with the bound* $C\|b\|_{2,n\alpha}$*.*

Proof. By Lemma 3.2, $|b * \psi_t(x)|^2 \frac{dxdt}{t}$ is an α-Carleson measure. Applying Corollary 4.2 (or Remark 4.3), for any $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, we get

$$\left(\int_{\mathbb{R}^{n+1}_+} |f * \varphi_t(x)|^{p\alpha} |b * \psi_t(x)|^2 \,\frac{dxdt}{t}\right)^{1/(p\alpha)} \le C\|b\|_{2,n\alpha}^{2/(p\alpha)} \|f\|_{L^p}. \tag{5.2}$$

Now, for any $h \in L^2(\mathbb{R}^n)$ with $\|h\|_{L^2} \le 1$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} \pi_b(f)(x)h(x)\,dx\right| \\ &= \left|\int_{\mathbb{R}^n}\int_0^\infty \left[\eta_t * \big((f*\varphi_t)(b*\psi_t)\big)\right](x)h(x)\,\frac{dxdt}{t}\right| \\ &= \left|\int_{\mathbb{R}^n}\int_0^\infty (f*\varphi_t)(x)(b*\psi_t)(x)(h*\widetilde{\eta}_t)(x)\,\frac{dxdt}{t}\right| \\ &\le \left(\int_{\mathbb{R}^{n+1}_+} |h*\widetilde{\eta}_t(x)|^2\,\frac{dxdt}{t}\right)^{1/2} \left(\int_{\mathbb{R}^{n+1}_+} |f*\varphi_t(x)|^2 |b*\psi_t(x)|^2\,\frac{dxdt}{t}\right)^{1/2}, \end{aligned} \tag{5.3}$$

where $\widetilde{\eta}(x) = \eta(-x)$. So, applying the L^2 boundedness of the Littlewood-Paley operator $g_{\widetilde{\eta}}$ defined in (3.3) and (5.2) (for $p\alpha = 2$), we obtain

$$\|\pi_b(f)\|_{L^2} = \sup_{h\in L^2, \|h\|_{L^2}\le 1} \left|\int_{\mathbb{R}^n} \pi_b(f)(x)h(x)\,dx\right| \le C\|f\|_{L^{2/\alpha}} \|b\|_{2,n\alpha}.$$

Thus, we complete the proof of Theorem 5.1. □

Further, if replacing $\mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ by $\mathcal{L}^{2,n\alpha}_v(\mathbb{R}^n)$ in Theorem 5.1, then we get the $(L^{2/\alpha}, L^2)$ compactness of π_b.

Theorem 5.2. *Under the same conditions of Theorem* 5.1 *if replacing* $b \in \mathcal{L}^{2,n\alpha}$ *by* $b \in \mathcal{L}^{2,n\alpha}_v$*, then* π_b *is a compact operator from* $L^{2/\alpha}(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$*.*

Proof. First note that $d\mu(x,t) := |b * \psi_t(x)|^2 \frac{dxdt}{t}$ is a vanishing α-Carleson measure by Theorem 3.4. Applying Theorem 4.4, the operator $\mathcal{L}_\varphi$ is compact from $L^p(\mathbb{R}^n; dx)$ to $L^{p\alpha}(\mathbb{R}^{n+1}_+; d\mu)$ for all $1 < p < \infty$. Hence, for any sequence $\{f_k\}$ in $L^p(\mathbb{R}^n)$, which converges weakly to zero, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^{n+1}_+} |f_k * \varphi_t(x)|^{p\alpha} |b * \psi_t(x)|^2 \,\frac{dxdt}{t} = 0. \tag{5.4}$$

Now, to prove π_b is a compact operator from $L^{2/\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, it suffices to verify that for any sequence $\{f_k\}$ in $L^{2/\alpha}(\mathbb{R}^n)$ which converges weakly to zero, $\{\pi_b(f_k)\}$ converges to zero in L^2 norm. Equivalently, we need to show that

$$\lim_{k\to\infty}\sup_{\|h\|_2\le 1}\left|\int_{\mathbb{R}^n}\pi_b(f_k)(x)h(x)\,dx\right| = 0. \tag{5.5}$$

In fact, for any $h \in L^2(\mathbb{R}^n)$ with $\|h\|_2 \le 1$, by (5.3) with f instead by f_k, then

$$\begin{aligned}
&\sup_{\|h\|_2\le 1}\left|\int_{\mathbb{R}^n}\pi_b(f_k)(x)h(x)\,dx\right| \\
&\le \sup_{\|h\|_2\le 1}\left(\int_{\mathbb{R}^{n+1}_+}|h*\widetilde{\eta}_t(x)|^2\,\frac{dxdt}{t}\right)^{1/2}\left(\int_{\mathbb{R}^{n+1}_+}|f_k*\varphi_t(x)|^2|b*\psi_t(x)|^2\,\frac{dxdt}{t}\right)^{1/2} \\
&= \sup_{\|h\|_2\le 1}\|g_{\widetilde{\eta}}(h)\|_{L^2}\left(\int_{\mathbb{R}^{n+1}_+}|f_k*\varphi_t(x)|^2|b*\psi_t(x)|^2\,\frac{dxdt}{t}\right)^{1/2}.
\end{aligned}$$

Hence (5.5) holds by using again the L^2 boundedness of the operator $g_{\widetilde{\eta}}$ and (5.4) (for $p\alpha = 2$) and we prove Theorem 5.2. □

5.2. Paraproduct with the rough kernel

In this subsection, one will see that the conditions $\psi, \eta \in \mathcal{S}(\mathbb{R}^n)$ in Theorems 5.1 and 5.2 can be weakened.

Theorem 5.3. *Suppose that the paraproduct π_b is defined by (5.1), where*

(i) $\psi(x) = \Omega(x)|x|^{1-n}\chi_{\{|x|<1\}}(x)$ *with Ω satisfying (3.1), (3.2);*

(ii) $\eta(x) = \omega(x)|x|^{1-n}\chi_{\{|x|<1\}}(x)$ *with ω satisfying (3.1), (3.2);*

(iii) φ *satisfies (4.1) (or $\phi \in L^1 \cap L^\infty$ with $\phi(x) := \operatorname{ess\,sup}_{|x|\le|y|}|\varphi(y)|$);*

(iv) $b \in \mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ *with $1 < \alpha < 2$.*

If $\Omega \in H^1(\mathbb{S}^{n-1})$ or $\Omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$ and $\omega \in H^1(\mathbb{S}^{n-1})$ or $\omega \in L(\log^+ L)^{1/2}(\mathbb{S}^{n-1})$, then π_b is bounded from $L^{2/\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with the bound $C\|b\|_{2,n\alpha}$.

Theorem 5.4. *Under the same conditions of Theorem 5.3 if replacing $b \in \mathcal{L}^{2,n\alpha}$ by $b \in \mathcal{L}^{2,n\alpha}_v$, then π_b is compact from $L^{2/\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Applying Theorems 3.6, 3.7, Corollary 4.2, Theorems 4.4 and 4.1, we may get Theorems 5.3 and 5.4, its proof is similar to proving Theorems 5.1 and 5.2, we omit the details here.

5.3. Coifman-Meyer-type paraproduct

As an application of Theorems 5.1 and 5.2, we give the $(L^{2/\alpha}, L^2)$ boundedness and compactness of a kind of paraproduct introduced by Coifman and Meyer in [12], which is defined by

$$B_b(f)(x) = \int_0^\infty (f * \varphi_t)(x)(b * \psi_t)(x) \frac{\theta(t)}{t}\, dt,$$

where $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\theta \in L^\infty(\mathbb{R}^n)$ and

(i) $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$;

(ii) $\widehat{\varphi}$, $\widehat{\psi}$ have compact support;

(iii) $0 \notin \operatorname{supp} \widehat{\psi}$.

There Coifman and Meyer proved that the paraproduct B_b is a bounded operator on $L^2(\mathbb{R}^n)$ with $b \in \mathrm{BMO}(\mathbb{R}^n)$. Recently, in [18], we verified that B_b is a compact operator on $L^2(\mathbb{R}^n)$ with $b \in \mathrm{CMO}(\mathbb{R}^n)$ and φ, ψ satisfying

(iv) $0 \notin \operatorname{supp} \widehat{\varphi} + \operatorname{supp} \widehat{\psi}$.

In this paper, we can give the $(L^{2/\alpha}, L^2)$ boundedness of B_b with $b \in \mathcal{L}^{2,n\alpha}$.

Theorem 5.5. *Assume that $\theta \in L^\infty(\mathbb{R}^n)$, φ, ψ satisfy the above conditions* (i)–(iv) *and $b \in \mathcal{L}^{2,n\alpha}(\mathbb{R}^n)$ with $1 < \alpha < 2$. Then there exists a constant $C > 0$, depending on φ, ψ and θ only, such that for all $f \in L^{2/\alpha}(\mathbb{R}^n)$,*

$$\|B_b(f)\|_2 \le C \|b\|_{2,n\alpha} \|f\|_{2/\alpha}. \tag{5.6}$$

Proof. By the condition (iv), we can choose an $\eta \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\eta} \in C_c^\infty$, $\widehat{\eta} = 1$ on $\operatorname{supp} \widehat{\varphi} + \operatorname{supp} \widehat{\psi}$ and $0 \notin \operatorname{supp} \widehat{\eta}$. Thus, it is easy to see that

$$B_b(f) = \int_0^\infty \eta_t * ((f * \varphi_t)(b * \psi_t)) \frac{\theta(t)}{t}\, dt.$$

By Theorem 5.1 (note that $\theta \in L^\infty(\mathbb{R}^n)$), the paraproduct B_b is bounded from $L^{2/\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and (5.6) holds. □

Similarly, we can obtain the $(L^{2/\alpha}, L^2)$ compactness of B_b with $b \in \mathcal{L}_v^{2,n\alpha}(\mathbb{R}^n)$. From the discussion in the proof of Theorems 5.5 and 5.2, we can easily obtain the following result.

Theorem 5.6. *Assume that $\theta \in L^\infty(\mathbb{R}^n)$, φ, ψ satisfy the above conditions* (i)–(iv) *and $b \in \mathcal{L}_v^{2,n\alpha}(\mathbb{R}^n)$ with $1 < \alpha < 2$. Then B_b is compact from $L^{2/\alpha}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

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