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Exceptional Set for Sums of Unlike Powers of Primes

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Abstract. Let N be a sufficiently large integer. In this paper, it is proved that with at most $O(N^{13/16+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4$, where p_1, p_2, p_3, p_4, p_5, p_6 are prime numbers.

1. Introduction and main result

Let $N, k_1, k_2, \ldots, k_s$ be natural numbers such that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_s$, $N > s$. Waring's problem of mixed powers concerns the representation of N as the form

$$N = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}.$$

Not very much is known about results of this kind. For historical literature the reader should consult section P12 of LeVeque's *Reviews in number theory* and the bibliography of Vaughan [15].

In 1970, Vaughan [14] obtained the asymptotic formula for the number of representations of a number as the sum of two squares, two cubes and two fourth powers. He proved that, for any sufficiently large integer N, there holds

$$\sum_{x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4=N} 1 = \frac{\Gamma^2(3/2)\Gamma^2(4/3)\Gamma^2(5/4)}{\Gamma(13/6)} \mathfrak{S}_{2,3,4}(N) N^{7/6} + O(N^{7/6-1/96+\varepsilon}),$$

where the singular series is

$$\mathfrak{S}_{2,3,4}(N) = \sum_{q=1}^{\infty} \frac{1}{q^6} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{3} \left(\sum_{x_i=1}^{q} e\left(\frac{a x_i^{i+1}}{q} \right) \right)^2 e\left(-\frac{aN}{q} \right).$$

In view of Vaughan's result, it is reasonable to conjecture that, for every sufficiently large even integer N, the following Diophantine equation

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 \tag{1.1}$$

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is solvable, where and below the letter p, with or without subscript, always stands for a prime number. But this conjecture is perhaps out of reach at present. However, many authors approach this conjecture in different ways. For instance, in 2015, Lü [9] proved that, for every sufficiently large even integer N, the equation

$$N = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4$$

is solvable with x being an almost-prime $\mathcal{P}_6$ and the p_j $(j = 2, 3, 4, 5, 6)$ primes. On the other hand, in 2017, Liu [8] proved that, every sufficiently large even integer N can be represented as two squares of primes, two cubes of primes, two fourth powers of primes and 41 powers of 2, i.e.,

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{41}}.$$

In this paper, we shall consider the exceptional set of the problem (1.1) and establish the following result.

Theorem 1.1. *Let $E(N)$ denote the number of positive even integers n up to N, which can not be represented as* (1.1). *Then, for any $\varepsilon > 0$, we have*

$$E(N) \ll N^{1-3/16+\varepsilon}.$$

We will establish Theorem 1.1 by the Hardy-Littlewood circle method. In the treatment of the integrals over major arcs, we will apply the iterative method in Liu [7] and the mean-value estimate for Dirichlet polynomials in Choi and Kumchev [1] to construct the asymptotic formula for the number of solutions to the problem. For the treatment of the integrals on the minor arcs, we will employ the methods which is developed by Zhao in [17]. The full details will be explained in the following revelent sections.

Notations. Throughout this paper, let p, with or without subscripts, always denote a prime number; ε and A always denote positive constants which are arbitrary small and sufficiently large, respectively, which may not be the same at different occurrences; $r \sim R$ means $R < r \le 2R$. As usual, we use $\varphi(n)$, $\Lambda(n)$ and $d(n)$ to denote the Euler's function, von Mangoldt's function and Dirichlet's divisor function, respectively. Also, we use $\chi \bmod q$ to denote a Dirichlet character modulo q, and $\chi^0 \bmod q$ the principal character. Especially, we use $\sum^*$ to denote sums over all primitive characters. $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. N is a sufficiently large integer and $n \asymp N$, and thus we use L to denote both $\log N$ and $\log n$. The letter c, with or without subscripts or superscripts, always denote a positive constant.

2. Outline of the proof of Theorem 1.1

Let N be a sufficiently large positive integer. For $k = 2, 3, 4$, we define

$$f_k(\alpha) = \sum_{X_k < p \leq 2X_k} (\log p) e(p^k \alpha),$$

where $X_k = (N/16)^{1/k}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n=p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4 \\ X_2<p_1,p_2\leq 2X_2 \\ X_3<p_3,p_4\leq 2X_3 \\ X_4<p_5,p_6\leq 2X_4}} (\log p_1)(\log p_2)\cdots(\log p_6).$$

Then for any $Q > 0$, we have

$$\mathscr{R}(n) = \int_0^1 \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\,\mathrm{d}\alpha = \int_{1/Q}^{1+1/Q} \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\,\mathrm{d}\alpha.$$

In order to apply the circle method, we set

$$P = N^{9/80-2\varepsilon}, \quad Q = N^{71/80+\varepsilon}. \tag{2.1}$$

By Dirichlet's lemma on rational approximation (for instance, see Lemma 12 on page 104 of Pan and Pan [10]), each $\alpha \in [1/Q, 1+1/Q]$ can be written as the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{qQ}$$

for some integers a, q with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. Then we define the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{q\leq P} \bigcup_{\substack{1\leq a\leq q \\ (a,q)=1}} \mathfrak{M}(q,a), \quad \mathfrak{m} = [1/Q, 1+1/Q] \setminus \mathfrak{M}, \tag{2.2}$$

where

$$\mathfrak{M}(q,a) = [a/q - 1/(qQ), a/q + 1/(qQ)].$$

Then one has

$$\mathscr{R}(n) = \left\{\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right\} \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\,\mathrm{d}\alpha.$$

In order to proof Theorem 1.1, we need the two following propositions, whose proofs will be given in Sections 3 and 6, respectively.

Proposition 2.1. *Let the major arcs* $\mathfrak{M}$ *be defined as in* (2.2) *with* P *and* Q *defined in* (2.1)*. Then, for* $n \in [N/2, N]$ *and any* $A > 0$*, there holds*

$$\int_{\mathfrak{M}} \left(\prod_{k=2}^{4} f_k^2(\alpha) \right) e(-n\alpha)\, \mathrm{d}\alpha = \frac{1}{576} \mathfrak{S}(n) \mathfrak{J}(n) + O(N^{7/6} L^{-A}),$$

where $\mathfrak{S}(n)$ *is the singular series defined in* (3.1)*, which is absolutely convergent and satisfies*

$$0 < c^* \le \mathfrak{S}(n) \ll d(n) \tag{2.3}$$

for any even integer n *and some fixed constant* c^**; while* $\mathfrak{J}(n)$ *is defined by* (3.9) *and satisfies*

$$\mathfrak{J}(n) \asymp N^{7/6}.$$

For the properties (2.3) of singular series, we shall give the proof in Section 5.

Proposition 2.2. *Let the minor arcs* $\mathfrak{m}$ *be defined as in* (2.2) *with* P *and* Q *defined in* (2.1)*. Then we have*

$$\int_{\mathfrak{m}} \left| f_2^4(\alpha) f_3^4(\alpha) f_4^4(\alpha) \right| \mathrm{d}\alpha \ll N^{7/3+1-3/16+\varepsilon}.$$

The remaining part of this section is devoted to establishing Theorem 1.1 by using Propositions 2.1 and 2.2.

Proof of Theorem 1.1. Let $\mathscr{E}(N)$ denote the set of positive integers $n \in [N/2, N]$ such that

$$\left| \int_{\mathfrak{m}} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-n\alpha)\, \mathrm{d}\alpha \right| \gg N^{7/6} L^{-A}.$$

Then we have

$$\begin{aligned} N^{7/3} L^{-2A} |\mathscr{E}(N)| &\ll \sum_{n \in \mathscr{E}(N)} \left| \int_{\mathfrak{m}} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-n\alpha)\, \mathrm{d}\alpha \right|^2 \\ &\ll \sum_{N/2 < n \le N} \left| \int_{\mathfrak{m}} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-n\alpha)\, \mathrm{d}\alpha \right|^2. \end{aligned} \tag{2.4}$$

By Bessel's inequality, we have

$$\sum_{N/2 < n \le N} \left| \int_{\mathfrak{m}} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-n\alpha)\, \mathrm{d}\alpha \right|^2 \le \int_{\mathfrak{m}} \left| f_2^4(\alpha) f_3^4(\alpha) f_4^4(\alpha) \right| \mathrm{d}\alpha. \tag{2.5}$$

Combining (2.4), (2.5) and Proposition 2.2, we have

$$|\mathscr{E}(N)| \ll N^{1-3/16+\varepsilon}.$$

Therefore, with at most $O(N^{13/16+\varepsilon})$ exceptions, all even integers $n \in [N/2, N]$ satisfies

$$\left| \int_{\mathfrak{m}} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-n\alpha)\, \mathrm{d}\alpha \right| \ll N^{7/6} L^{-A},$$

from which and Proposition 2.1, we deduce that, with at most $O(N^{13/16+\varepsilon})$ exceptions, all even positive integers $n \in [N/2, N]$ can be represented in the form $p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4$, where p_1, p_2, p_3, p_4, p_5, p_6 are prime numbers. By a splitting argument, we obtain

$$E(N) \ll N^{1-3/16+\varepsilon}.$$

This completes the proof of Theorem 1.1. □

3. Proof of Proposition 2.1

In this section, we shall concentrate on proving Proposition 2.1. We first introduce some notations. For a Dirichlet character χ mod q and $2 \le k \le 4$, we define

$$C_k(\chi, a) = \sum_{h=1}^{q} \overline{\chi(h)} e\left(\frac{ah^k}{q}\right), \quad C_k(q,a) = C_k(\chi^0, a),$$

where χ^0 is the principal character modulo q, and $C_k(q,a)$ is the Ramanujan sum. Let $\chi_2^{(1)}$, $\chi_2^{(2)}$, $\chi_3^{(1)}$, $\chi_3^{(2)}$, $\chi_4^{(1)}$, $\chi_4^{(2)}$ be Dirichlet characters modulo q. Define

$$B\left(n, q, \chi_2^{(1)}, \chi_2^{(2)}, \chi_3^{(1)}, \chi_3^{(2)}, \chi_4^{(1)}, \chi_4^{(2)}\right) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{k=2}^{4} \prod_{i=1}^{2} C_k\left(\chi_k^{(i)}, a\right) e\left(-\frac{an}{q}\right),$$

$$B(n,q) = B\left(n, q, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0\right),$$

and write

$$A(n,q) = \frac{B(n,q)}{\varphi^6(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q). \tag{3.1}$$

Lemma 3.1. *For $(a,q)=1$ and any Dirichlet character χ* mod q*, there holds*

$$|C_k(\chi, a)| \le 2q^{1/2} d^{\beta_k}(q)$$

with $\beta_k = (\log k)/\log 2$.

Proof. See the Problem 14 of Chapter VI of Vinogradov [16]. □

Lemma 3.2. *The singular series $\mathfrak{S}(n)$ satisfies* (2.3)*.*

The proof of Lemma 3.2 will be given in Section 5.

Lemma 3.3. *Let $f(x)$ be a real differentiable function in the interval $[a,b]$. If $f'(x)$ is monotonic and satisfies $|f'(x)|\le\theta<1$. Then we have*

$$\sum_{a<n\le b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,\mathrm{d}x + O(1).$$

Proof. See Lemma 4.8 of Titchmarsh [13]. □

Lemma 3.4. *Let $\chi_k^{(i)}$ mod $r_k^{(i)}$ with $k=2,3,4$ and $i=1,2$ be primitive characters, $r_0=\left[r_2^{(1)},r_2^{(2)},r_3^{(1)},r_3^{(2)},r_4^{(1)},r_4^{(2)}\right]$, and χ^0 the principal character modulo q. Then there holds*

$$\sum_{\substack{q\le x\\ r_0|q}} \frac{1}{\varphi^6(q)}\left|B\left(n,q,\chi_2^{(1)}\chi^0,\chi_2^{(2)}\chi^0,\chi_3^{(1)}\chi^0,\chi_3^{(2)}\chi^0,\chi_4^{(1)}\chi^0,\chi_4^{(2)}\chi^0\right)\right| \ll r_0^{-2+\varepsilon}\log^c x. \tag{3.2}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned}&\left|B\left(n,q,\chi_2^{(1)}\chi^0,\chi_2^{(2)}\chi^0,\chi_3^{(1)}\chi^0,\chi_3^{(2)}\chi^0,\chi_4^{(1)}\chi^0,\chi_4^{(2)}\chi^0\right)\right|\\ &\ll \sum_{\substack{a=1\\(a,q)=1}}^{q}\prod_{k=2}^{4}\prod_{i=1}^{2}\left|C_k\left(\chi_k^{(i)}\chi^0,a\right)\right| \ll q^3\varphi(q)d^{10}(q).\end{aligned}$$

Therefore, the left-hand side of (3.2) is

$$\ll \sum_{\substack{q\le x\\ r_0|q}}\frac{q^3\varphi(q)d^{10}(q)}{\varphi^6(q)} = \sum_{t\le x/r_0}\frac{r_0^3t^3d^{10}(r_0t)}{\varphi^5(r_0t)} \ll r_0^{-2+\varepsilon}(\log x)\sum_{t\le x}\frac{d^{10}(t)}{t^2} \ll r_0^{-2+\varepsilon}\log^c x.$$

This completes the proof of Lemma 3.4. □

Write

$$V_k(\lambda) = \sum_{X_k<m\le 2X_k} e(m^k\lambda),$$

$$W_k(\chi,\lambda) = \sum_{X_k<p\le 2X_k}(\log p)\chi(p)e(p^k\lambda) - \delta_\chi\sum_{X_k<m\le 2X_k} e(m^k\lambda), \tag{3.3}$$

where $\delta_\chi=1$ or 0 according to χ is principal or not. Then by the orthogonality of Dirichlet characters, for $(a,q)=1$, we have

$$f_k\left(\frac{a}{q}+\lambda\right) = \frac{C_k(q,a)}{\varphi(q)}V_k(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C_k(\chi,a)W_k(\chi,\lambda).$$

For $j = 1, 2, \ldots, 27$, we define the sets $\mathscr{S}_j$ as follows:

$$\mathscr{S}_j = \begin{cases} \{2,2,3,3,4,4\} & \text{if } j=1, \quad \{2,2,3,3,4\} \quad \text{if } j=2, \quad \{2,2,3,4,4\} \quad \text{if } j=3, \\ \{2,3,3,4,4\} & \text{if } j=4, \quad \{2,2,3,3\} \quad \text{if } j=5, \quad \{2,2,4,4\} \quad \text{if } j=6, \\ \{3,3,4,4\} & \text{if } j=7, \quad \{2,2,3,4\} \quad \text{if } j=8, \quad \{2,3,3,4\} \quad \text{if } j=9, \\ \{2,3,4,4\} & \text{if } j=10, \quad \{2,2,3\} \quad \text{if } j=11, \quad \{2,2,4\} \quad \text{if } j=12, \\ \{2,3,3\} & \text{if } j=13, \quad \{2,4,4\} \quad \text{if } j=14, \quad \{3,3,4\} \quad \text{if } j=15, \\ \{3,4,4\} & \text{if } j=16, \quad \{2,3,4\} \quad \text{if } j=17, \quad \{2,2\} \quad \text{if } j=18, \\ \{3,3\} & \text{if } j=19, \quad \{4,4\} \quad \text{if } j=20, \quad \{2,3\} \quad \text{if } j=21, \\ \{2,4\} & \text{if } j=22, \quad \{3,4\} \quad \text{if } j=23, \quad \{2\} \quad \text{if } j=24, \\ \{3\} & \text{if } j=25, \quad \{4\} \quad \text{if } j=26, \quad \emptyset \quad \text{if } j=27. \end{cases}$$

Also, we write $\overline{\mathscr{S}_j} = \{2,2,3,3,4,4\} \setminus \mathscr{S}_j$. Then we have

$$\begin{aligned} &\int_{\mathfrak{M}} \left(\prod_{k=2}^{4} f_k^2(\alpha) \right) e(-n\alpha)\,\mathrm{d}\alpha \\ &= I_1 + 2I_2 + 2I_3 + 2I_4 + I_5 + I_6 + I_7 + 4I_8 + 4I_9 + 4I_{10} \\ &\quad + 2I_{11} + 2I_{12} + 2I_{13} + 2I_{14} + 2I_{15} + 2I_{16} + 8I_{17} + I_{18} + I_{19} + I_{20} \\ &\quad + 4I_{21} + 4I_{22} + 4I_{23} + 2I_{24} + 2I_{25} + 2I_{26} + I_{27}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} I_j = {} & \sum_{q \le P} \frac{1}{\varphi^6(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\prod_{k \in \mathscr{S}_j} C_k(q,a) \right) e\left(-\frac{an}{q} \right) \\ & \times \int_{-1/(qQ)}^{1/(qQ)} \left(\prod_{k \in \mathscr{S}_j} V_k(\lambda) \right) \left(\prod_{k \in \overline{\mathscr{S}_j}} \sum_{\chi \bmod q} C_k(\chi,a) W_k(\chi,\lambda) \right) e(-n\lambda)\,\mathrm{d}\lambda. \end{aligned}$$

In the following content of this section, we shall prove that I_1 produces the main term, while the others contribute the error term.

For $k = 2, 3, 4$, applying Lemma 3.3 to $V_k(\lambda)$, we have

$$\begin{aligned} V_k(\lambda) &= \int_{X_k}^{2X_k} e(u^k\lambda)\,\mathrm{d}u + O(1) = \frac{1}{k} \int_{X_k^k}^{(2X_k)^k} e(v\lambda) v^{1/k-1}\,\mathrm{d}v + O(1) \\ &= \frac{1}{k} \sum_{X_k^k < m \le (2X_k)^k} m^{1/k-1} e(m\lambda) + O(1). \end{aligned} \tag{3.5}$$

Putting (3.5) into I_1, we see that

$$\begin{aligned} I_1 = {} & \frac{1}{576}\sum_{q\le P}\frac{B(n,q)}{\varphi^6(q)}\int_{-1/(qQ)}^{1/(qQ)}\prod_{k=2}^{4}\Bigg(\sum_{X_k^k<m\le(2X_k)^k} m^{1/k-1}e(m\lambda)\Bigg)^2 e(-n\lambda)\,\mathrm{d}\lambda \\ & + O\Bigg(\sum_{q\le P}\frac{|B(n,q)|}{\varphi^6(q)}\int_{-1/(qQ)}^{1/(qQ)}\Bigg|\sum_{X_k^k<m\le(2X_k)^k} m^{-3/4}e(m\lambda)\Bigg| \\ & \qquad\times\prod_{k=2}^{3}\Bigg|\sum_{X_k^k<m\le(2X_k)^k} m^{1/k-1}e(m\lambda)\Bigg|^2\mathrm{d}\lambda\Bigg). \end{aligned} \tag{3.6}$$

By using the elementary estimate

$$\sum_{X_k^k<m\le(2X_k)^k} m^{1/k-1}e(m\lambda)\ll N^{1/k-1}\min\left(N,\frac{1}{|\lambda|}\right), \tag{3.7}$$

and Lemma 3.4 with $r_0=1$, the O-term in (3.6) can be estimated as

$$\ll\sum_{q\le P}\frac{|B(n,q)|}{\varphi^6(q)}\left(\int_0^{1/N}N^{23/12}\,\mathrm{d}\lambda+\int_{1/N}^{\infty}N^{-37/12}\cdot\frac{1}{\lambda^5}\,\mathrm{d}\lambda\right)\ll N^{11/12}L^c\ll N^{7/6}L^{-A}.$$

If we extend the interval of the integral in the main term of (3.6) to $[-1/2,1/2]$, then from (2.1) we can see that the resulting error is

$$\ll L^c\int_{1/(qQ)}^{1/2}N^{-23/6}\cdot\frac{\mathrm{d}\lambda}{\lambda^6}\ll N^{-23/6}q^5Q^5L^c\ll N^{-23/6}(PQ)^5L^c\ll N^{7/6-\varpi}$$

for some $\varpi>0$. Therefore, by Lemma 3.2, (3.6) becomes

$$I_1=\frac{1}{576}\mathfrak{S}(n)\mathfrak{J}(n)+O(N^{7/6}L^{-A}), \tag{3.8}$$

where

$$\mathfrak{J}(n):=\sum_{\substack{m_1+m_2+\cdots+m_6=n\\ X_2^2<m_1,m_2\le(2X_2)^2\\ X_3^3<m_3,m_4\le(2X_3)^3\\ X_4^4<m_5,m_6\le(2X_4)^4}}(m_1m_2)^{-1/2}(m_3m_4)^{-2/3}(m_5m_6)^{-3/4}\asymp N^{7/6}. \tag{3.9}$$

In order to estimate the contribution of I_j for $j=2,3,\ldots,27$, we shall need the following three preliminary lemmas, i.e., Lemmas 3.5–3.7, which will be proved in Section 4. In view of this, for $k=2,3,4$, we recall the definition of $W_k(\chi,\lambda)$ in (3.3) and write

$$J_k(g)=\sum_{r\le P}[g,r]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\max_{|\lambda|\le1/(rQ)}|W_k(\chi,\lambda)|$$

and

$$K_k(g) = \sum_{r\le P}[g,r]^{-2+\varepsilon} \sum_{\chi \bmod r}^{*} \left(\int_{-1/(rQ)}^{1/(rQ)} |W_k(\chi,\lambda)|^2 \,\mathrm{d}\lambda \right)^{1/2}.$$

Here and below, $\sum^*$ indicates that the summation is taken over all primitive characters.

Lemma 3.5. *Let P, Q be defined as in* (2.1)*. For $k = 2, 3$, we have*

$$J_k(g) \ll g^{-2+\varepsilon} N^{1/k} L^c.$$

Lemma 3.6. *Let P, Q be defined as in* (2.1)*. For $k = 2$ and $g = 1$, Lemma* 3.5 *can be improved to*

$$J_2(1) \ll N^{1/2} L^{-A}.$$

Lemma 3.7. *Let P, Q be defined as in* (2.1)*. For $k = 4$, we have*

$$K_4(g) \ll g^{-2+\varepsilon} N^{-1/4} L^c.$$

Now, we concentrate on estimating the terms I_j for $j = 2, 3, \ldots, 27$. We begin with the term I_{27}, which is the most complicated one. Reducing the Dirichlet characters in I_{27} into primitive characters, we have

$$\begin{aligned}
|I_{27}| &= \left| \sum_{q\le P} \frac{1}{\varphi^6(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-1/(qQ)}^{1/(qQ)} \prod_{k=2}^{4} \left(\sum_{\chi_k \bmod q} C_k(\chi_k,a) W_k(\chi_k,\lambda) \right)^2 e(-n\lambda)\,\mathrm{d}\lambda \right| \\
&= \Bigg| \sum_{q\le P} \sum_{\chi_2^{(1)}} \sum_{\chi_2^{(2)}} \sum_{\chi_3^{(1)}} \sum_{\chi_3^{(2)}} \sum_{\chi_4^{(1)}} \sum_{\chi_4^{(2)}} \frac{1}{\varphi^6(q)} \cdot B\left(n, q, \chi_2^{(1)}, \chi_2^{(2)}, \chi_3^{(1)}, \chi_3^{(2)}, \chi_4^{(1)}, \chi_4^{(2)}\right) \\
&\qquad \times \int_{-1/(qQ)}^{1/(qQ)} W_2(\chi_2^{(1)},\lambda) W_2(\chi_2^{(2)},\lambda) \cdots W_4(\chi_4^{(2)},\lambda) e(-n\lambda)\,\mathrm{d}\lambda \Bigg| \\
&\le \sum_{r_2^{(1)}\le P} \cdots \sum_{r_4^{(2)}\le P} \sum_{\chi_2^{(1)} \bmod r_2^{(1)}}^{*} \cdots \sum_{\chi_4^{(2)} \bmod r_4^{(2)}}^{*} \sum_{\substack{q\le P\\ r_0|q}} \frac{\left| B(n, q, \chi_2^{(1)}\chi^0, \ldots, \chi_4^{(2)}\chi^0) \right|}{\varphi^6(q)} \\
&\qquad \times \int_{-1/(qQ)}^{1/(qQ)} \left| W_2(\chi_2^{(1)}\chi^0,\lambda) \right| \cdots \left| W_4(\chi_4^{(2)}\chi^0,\lambda) \right| \mathrm{d}\lambda,
\end{aligned}$$

where χ^0 is the principal character modulo q and $r_0 = \left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}, r_4^{(2)}\right]$. For $q \le P$ and $X_k < p \le 2X_k$ with $k = 2, 3, 4$, we have $(q, p) = 1$. From this and the definition of $W_k(\chi, \lambda)$, we obtain $W_k(\chi_k^{(i)}\chi^0, \lambda) = W_k(\chi_k^{(i)}, \lambda)$ for primitive characters $\chi_k^{(i)}$ above with

$k = 2, 3, 4$ and $i = 1, 2$. Therefore, by Lemma 3.4, we obtain

$$
\begin{aligned}
|I_{27}| \leq & \sum_{r_2^{(1)} \leq P} \cdots \sum_{r_4^{(2)} \leq P} \sideset{}{^*}\sum_{\chi_2^{(1)} \bmod r_2^{(1)}} \cdots \sideset{}{^*}\sum_{\chi_4^{(2)} \bmod r_4^{(2)}} \int_{-1/(r_0 Q)}^{1/(r_0 Q)} \left| W_2(\chi_2^{(1)}, \lambda) \right| \cdots \left| W_4(\chi_4^{(2)}, \lambda) \right| \mathrm{d}\lambda \\
& \times \sum_{\substack{q \leq P \\ r_0 | q}} \frac{\left| B\left(n, q, \chi_2^{(1)}\chi^0, \chi_2^{(2)}\chi^0, \chi_3^{(1)}\chi^0, \chi_3^{(2)}\chi^0, \chi_4^{(1)}\chi^0, \chi_4^{(2)}\chi^0\right) \right|}{\varphi^6(q)} \\
& \ll L^c \cdot \sum_{r_2^{(1)} \leq P} \cdots \sum_{r_4^{(2)} \leq P} r_0^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi_2^{(1)} \bmod r_2^{(1)}} \cdots \sideset{}{^*}\sum_{\chi_4^{(2)} \bmod r_4^{(2)}} \\
& \times \int_{-1/(r_0 Q)}^{1/(r_0 Q)} \left| W_2(\chi_2^{(1)}, \lambda) \right| \cdots \left| W_4(\chi_4^{(2)}, \lambda) \right| \mathrm{d}\lambda.
\end{aligned}
$$

In the last integral, we pick out $\left|W_2(\chi_2^{(1)}, \lambda)\right|$, $\left|W_2(\chi_2^{(2)}, \lambda)\right|$, $\left|W_3(\chi_3^{(1)}, \lambda)\right|$ and $\left|W_3(\chi_3^{(2)}, \lambda)\right|$, and then use Cauchy's inequality to derive that

$$
\begin{aligned}
|I_{27}| \ll L^c & \prod_{k=2}^{3} \prod_{i=1}^{2} \left(\sum_{r_k^{(i)} \leq P} \sideset{}{^*}\sum_{\chi_k^{(i)} \bmod r_k^{(i)}} \max_{|\lambda| \leq 1/(r_k^{(i)} Q)} \left| W_k(\chi_k^{(i)}, \lambda) \right| \right) \\
& \times \sum_{r_4^{(1)} \leq P} \sideset{}{^*}\sum_{\chi_4^{(1)} \bmod r_4^{(1)}} \left(\int_{-1/(r_4^{(1)} Q)}^{1/(r_4^{(1)} Q)} \left| W_4(\chi_4^{(1)}, \lambda) \right|^2 \mathrm{d}\lambda \right)^{1/2} \\
& \times \sum_{r_4^{(2)} \leq P} r_0^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi_4^{(2)} \bmod r_4^{(2)}} \left(\int_{-1/(r_4^{(2)} Q)}^{1/(r_4^{(2)} Q)} \left| W_4(\chi_4^{(2)}, \lambda) \right|^2 \mathrm{d}\lambda \right)^{1/2}.
\end{aligned} \tag{3.10}
$$

Now we introduce the iterative procedure to bound the sums over $r_4^{(2)}, \ldots, r_2^{(1)}$ consecutively. We first estimate the above sum over $r_4^{(2)}$ in (3.10) via Lemma 3.7. Since

$$
r_0 = \left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}, r_4^{(2)}\right] = \left[\left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}\right], r_4^{(2)}\right],
$$

the sum over $r_4^{(2)}$ is

$$
\begin{aligned}
& \sum_{r_4^{(2)} \leq P} \left[\left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}\right], r_4^{(2)}\right]^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi_4^{(2)} \bmod r_4^{(2)}} \left(\int_{-1/(r_4^{(2)} Q)}^{1/(r_4^{(2)} Q)} \left| W_4(\chi_4^{(2)}, \lambda) \right|^2 \mathrm{d}\lambda \right)^{1/2} \\
& = K_4\left(\left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}\right]\right) \ll \left[r_2^{(1)}, r_2^{(2)}, r_3^{(1)}, r_3^{(2)}, r_4^{(1)}\right]^{-2+\varepsilon} N^{-1/4} L^c.
\end{aligned} \tag{3.11}
$$

By Lemma 3.7 again, the contribution of the quantity on the right-hand side of (3.11) to the sum over $r_4^{(1)}$ in (3.10) is

(3.12)

$$\begin{aligned}&\ll N^{-1/4}L^c\cdot\sum_{r_4^{(1)}\le P}\big[\big[r_2^{(1)},r_2^{(2)},r_3^{(1)},r_3^{(2)}\big],r_4^{(1)}\big]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi_4^{(1)}\bmod r_4^{(1)}}\bigg(\int_{-1/(r_4^{(1)}Q)}^{1/(r_4^{(1)}Q)}\Big|W_4(\chi_4^{(1)},\lambda)\Big|^2\,\mathrm{d}\lambda\bigg)^{1/2}\\&=N^{-1/4}L^c\cdot K_4\big(\big[r_2^{(1)},r_2^{(2)},r_3^{(1)},r_3^{(2)}\big]\big)\ll\big[r_2^{(1)},r_2^{(2)},r_3^{(1)},r_3^{(2)}\big]^{-2+\varepsilon}N^{-1/2}L^c.\end{aligned}$$

By Lemma 3.5, the contribution of the quantity on the right-hand side of (3.12) to the sum over $r_3^{(2)}$ in (3.10) is

$$\begin{aligned}&\ll N^{-1/2}L^c\cdot\sum_{r_3^{(2)}\le P}\big[\big[r_2^{(1)},r_2^{(2)},r_3^{(1)}\big],r_3^{(2)}\big]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi_3^{(2)}\bmod r_3^{(2)}}\max_{|\lambda|\le 1/(r_3^{(2)}Q)}\Big|W_3(\chi_3^{(2)},\lambda)\Big|\\&=N^{-1/2}L^c\cdot J_3\big(\big[r_2^{(1)},r_2^{(2)},r_3^{(1)}\big]\big)\ll\big[r_2^{(1)},r_2^{(2)},r_3^{(1)}\big]^{-2+\varepsilon}N^{-1/6}L^c.\end{aligned}\tag{3.13}$$

The contribution of the quantity on the right-hand side of (3.13) to the sum over $r_3^{(1)}$ in (3.10) is

$$\begin{aligned}&\ll N^{-1/6}L^c\cdot\sum_{r_3^{(1)}\le P}\big[\big[r_2^{(1)},r_2^{(2)}\big],r_3^{(1)}\big]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi_3^{(1)}\bmod r_3^{(1)}}\max_{|\lambda|\le 1/(r_3^{(1)}Q)}\Big|W_3(\chi_3^{(1)},\lambda)\Big|\\&=N^{-1/6}L^c\cdot J_3\big(\big[r_2^{(1)},r_2^{(2)}\big]\big)\ll\big[r_2^{(1)},r_2^{(2)}\big]^{-2+\varepsilon}N^{1/6}L^c.\end{aligned}\tag{3.14}$$

The contribution of the quantity on the right-hand side of (3.14) to the sum over $r_2^{(2)}$ in (3.10) is

$$\begin{aligned}&\ll N^{1/6}L^c\cdot\sum_{r_2^{(2)}\le P}\big[r_2^{(1)},r_2^{(2)}\big]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi_2^{(2)}\bmod r_2^{(2)}}\max_{|\lambda|\le 1/(r_2^{(2)}Q)}\Big|W_2(\chi_2^{(2)},\lambda)\Big|\\&=N^{1/6}L^c\cdot J_2\big(r_2^{(1)}\big)\ll\big(r_2^{(1)}\big)^{-2+\varepsilon}N^{2/3}L^c.\end{aligned}\tag{3.15}$$

At last, from Lemma 3.6, inserting the bound on the right-hand side of (3.15) to the sum over $r_2^{(1)}$ in (3.10), we get

$$\begin{aligned}|I_{27}|&\ll N^{2/3}L^c\cdot\sum_{r_2^{(1)}\le P}\big[1,r_2^{(1)}\big]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi_2^{(1)}\bmod r_2^{(1)}}\max_{|\lambda|\le 1/(r_2^{(1)}Q)}\Big|W_2(\chi_2^{(1)},\lambda)\Big|\\&=N^{2/3}L^c\cdot J_2(1)\ll N^{7/6}L^{-A}.\end{aligned}\tag{3.16}$$

For the estimation of the terms $I_2,\dots,I_{26}$, by noting (3.5) and (3.7), we obtain

$$\begin{aligned}\bigg(\int_{-1/Q}^{1/Q}|V_k(\lambda)|^2\,\mathrm{d}\lambda\bigg)^{1/2}&\ll\bigg(\int_{-1/Q}^{1/Q}N^{2/k-2}\min\Big(N,\frac{1}{|\lambda|}\Big)^2\,\mathrm{d}\lambda+\frac{1}{Q}\bigg)^{1/2}\\&\ll N^{1/k-1}\bigg(\int_0^{1/N}N^2\,\mathrm{d}\lambda+\int_{1/N}^{1/Q}\frac{\mathrm{d}\lambda}{\lambda^2}\bigg)^{1/2}+\frac{1}{Q^{1/2}}\ll N^{1/k-1/2}.\end{aligned}$$

Using this estimate and the upper bound of $V_k(\lambda)$, which derives from (3.5) and (3.7), that $V_k(\lambda) \ll N^{1/k}$, we can argue similarly to the treatment of I_{27} and obtain

$$\sum_{j=2}^{26} I_j \ll N^{7/6} L^{-A}. \tag{3.17}$$

Combining (3.4), (3.8), (3.16) and (3.17), we can get the conclusion of Proposition 2.1.

4. Estimation of $J_k(g)$, $J_k(1)$ and $K_k(g)$

In this section, we will establish Lemmas 3.5–3.7. We shall need the following lemmas.

Lemma 4.1. *Let $R \geq 1$, $\mathcal{X} \geq 2$, $T \geq 2$. Then we have*

$$\sum_{\substack{r\sim R \\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_T^{2T} \left| \sum_{\mathcal{X}<n\leq 2\mathcal{X}} \Lambda(n)\chi(n)n^{-it} \right| \mathrm{d}t \ll \left(\frac{R^2T}{d}\mathcal{X}^{11/20} + \mathcal{X} \right) \log^c(RT\mathcal{X}).$$

Proof. See Theorem 1.1 of Choi and Kumchev [1]. □

Lemma 4.2. *Suppose that*

$$S(t) = \sum_{\nu} c(\nu)e(\nu t)$$

is an absolutely convergent exponential sum. Here the frequencies ν run over an arbitrary sequence of real numbers and the coefficients are complex. Let $\delta = \theta/T$ with $0 < \theta < 1$. Then

$$\int_{-T}^{T} |S(t)|^2 \,\mathrm{d}t \ll_\theta \int_{-\infty}^{+\infty} \left| \delta^{-1} \sum_{x<\nu\leq x+\delta} c(\nu) \right|^2 \mathrm{d}x.$$

Proof. See Lemma 1 of Gallagher [3]. □

Lemma 4.3. *Let $G(x)$, $F(x)$ be real functions and twice differentiable on $[a,b]$, and $G(x)/F'(x)$ be monotonic.*

(i) *If $|F'(x)/G(x)| \geq V_1 > 0$ on $[a,b]$, then*

$$\int_a^b G(x)e(F(x))\,\mathrm{d}x \ll \frac{1}{V_1}.$$

(ii) *If $|F''(x)| \geq V_2 > 0$ on $[a,b]$ and $|G(x)| \leq M$, then*

$$\int_a^b G(x)e(F(x))\,\mathrm{d}x \ll \frac{M}{\sqrt{V_2}}.$$

Proof. For the proofs of (i) and (ii), one can see Lemmas 4.3 and 4.5 of Titchmarsh [13], respectively. □

Lemma 4.4. *Let $\chi(m)$ be a Dirichlet character modulo q, then we have the explicit formula*

$$\sum_{m\le u}\Lambda(m)\chi(m)=\delta_\chi u-\sum_{|\gamma|\le T}\frac{u^\rho}{\rho}+O\left(\left(\frac{u}{T}+1\right)\log^2(quT)\right), \tag{4.1}$$

where $\delta_\chi=1$ or 0 according to χ is principal or not, $\rho=\beta+i\gamma$ runs over non-trivial zeros of the function $L(s,\chi)$, and $2\le T\le u$ is a parameter.

Proof. See p. 109 and p. 120 of Davenport [2]. □

Lemma 4.5. *For $T\ge 2$, let $\mathscr{N}^*(\alpha,q,T)$ denote the number of zeros of all the Dirichlet L-functions $L(s,\chi)$ with primitive characters $\chi \bmod q$ in the region $\Re s\ge\alpha$, $|\Im s|\le T$. Then we have*

$$\sum_{q\le\mathscr{Z}}\mathscr{N}^*(\alpha,q,T)\ll(\mathscr{Z}^2T)^{\frac{12}{5}(1-\alpha)}\log^{c_1}(\mathscr{Z}T),$$

where c_1 is an absolute constant.

Proof. See p. 164 of Huxley [5] or pp. 75–76 of Pan and Pan [10]. □

Lemma 4.6. *Let $T\ge 2$. There exists an absolute constant $c_2>0$, such that the product*

$$\prod_{\chi \bmod q}L(s,\chi)\ne 0$$

in the region

$$\Re s\ge 1-\frac{c_2}{\max\left\{\log q,\log^{4/5}T\right\}},\quad |\Im s|\le T,$$

except for the possible Siegel zero.

Proof. See Satz VIII.6.2 of Prachar [11]. □

4.1. Estimation of $K_4(g)$

We approximate the $W_4(\chi,\lambda)$ in (3.3) by

$$\widehat{W}_4(\chi,\lambda)=\sum_{X_4<m\le 2X_4}(\Lambda(m)\chi(m)-\delta_\chi)e(m^4\lambda).$$

Then the error is

$$W_4(\chi,\lambda)-\widehat{W}_4(\chi,\lambda)\ll\sum_{\substack{X_4<p^t\le 2X_4\\ t\ge 2}}\log p\ll\sum_{p\le X_4^{1/2}}\log p\ll N^{1/8}. \tag{4.2}$$

Therefore, we have

$$\begin{aligned}
&\sum_{r\le P}[g,r]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\left(\int_{-1/(rQ)}^{1/(rQ)}\left|W_4(\chi,\lambda)-\widehat{W}_4(\chi,\lambda)\right|^2\,\mathrm{d}\lambda\right)^{1/2}\\
&\ll N^{1/8}\sum_{r\le P}[g,r]^{-2+\varepsilon}\left(\frac{r}{Q}\right)^{1/2}=N^{1/8}\sum_{r\le P}\left(\frac{gr}{(g,r)}\right)^{-2+\varepsilon}\left(\frac{r}{Q}\right)^{1/2}\\
&=g^{-2+\varepsilon}N^{1/8}Q^{-1/2}\sum_{\substack{d|g\\ d\le P}}\sum_{\substack{r\le P\\ d|r}}\left(\frac{r}{d}\right)^{-2+\varepsilon}r^{1/2}\\
&\ll g^{-2+\varepsilon}N^{1/8}Q^{-1/2}P^{1/2+\varepsilon}\ll g^{-2+\varepsilon}N^{-1/4}L^c.
\end{aligned}$$

Thus, it is sufficient to show that

$$\sum_{r\sim R}[g,r]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi \bmod r}\left(\int_{-1/(rQ)}^{1/(rQ)}\left|\widehat{W}_4(\chi,\lambda)\right|^2\,\mathrm{d}\lambda\right)^{1/2}\ll g^{-2+\varepsilon}N^{-1/4}L^c, \tag{4.3}$$

where $R\ll P$.

By Lemma 4.2, we have

$$\begin{aligned}
\int_{-1/(rQ)}^{1/(rQ)}\left|\widehat{W}_4(\chi,\lambda)\right|^2\,\mathrm{d}\lambda &\ll\left(\frac{1}{RQ}\right)^2\int_{-\infty}^{+\infty}\Bigg|\sum_{\substack{\nu<m^4\le\nu+rQ\\ X_4<m\le 2X_4}}(\Lambda(m)\chi(m)-\delta_\chi)\Bigg|^2\,\mathrm{d}\nu\\
&\ll\left(\frac{1}{RQ}\right)^2\int_{X_4^4-rQ}^{(2X_4)^4}\Bigg|\sum_{\substack{\nu<m^4\le\nu+rQ\\ X_4<m\le 2X_4}}(\Lambda(m)\chi(m)-\delta_\chi)\Bigg|^2\,\mathrm{d}\nu.
\end{aligned}\tag{4.4}$$

Let x, y be two parameters, which satisfy

$$X_4\le y<x\le 2X_4$$

and

$$x-y\ll(\nu+rQ)^{1/4}-\nu^{1/4}\ll\nu^{1/4}\left(\left(1+\frac{rQ}{\nu}\right)^{1/4}-1\right)\ll rQ\nu^{-3/4}\ll rQN^{-3/4}.$$

Then the last sum in (4.4) can be written as

$$\sum_{y<m\le x}(\Lambda(m)\chi(m)-\delta_\chi). \tag{4.5}$$

If $R<1$, the quantity in (4.4) is

$$\ll L(x-y)\ll LQN^{-3/4}.$$

The above contribution to (4.3) is

$$g^{-2+\varepsilon}\left(\frac{1}{Q^2}\left((2X_4)^4-X_4^4+Q\right)L^2Q^2N^{-3/2}\right)^{1/2}\ll g^{-2+\varepsilon}N^{-1/4}L,$$

which is acceptable. If $R\geq 1$, we have $\chi\neq\chi^0$ and hence $\delta_\chi=0$. Applying Perron's formula, (4.5) can be written as

$$\sum_{y<m\leq x}\Lambda(m)\chi(m)=\frac{1}{2\pi i}\int_{-iT}^{iT}F(s,\chi)\cdot\frac{x^s-y^s}{s}\,\mathrm{d}s+O\left(\frac{N^{1/4}L^2}{T}\right),$$

where $T=N^{1/4}$ and

$$F(s,\chi)=\sum_{y<m\leq x}\frac{\Lambda(m)\chi(m)}{m^s}.$$

For the factor $(x^s-y^s)/s$, on one hand, we have

$$\begin{aligned}\left|\frac{x^s-y^s}{s}\right|&=\left|\int_y^x u^{s-1}\,\mathrm{d}u\right|\leq\int_y^x u^{-1}\,\mathrm{d}u\\&\ll N^{-1/4}(x-y)\ll N^{-1/4}RQN^{-3/4}\ll RQN^{-1}=T_0^{-1},\end{aligned}\tag{4.6}$$

say. On the other hand, we get

$$\left|\frac{x^s-y^s}{s}\right|\ll\frac{|x^s|+|y^s|}{|s|}\ll\frac{1}{|t|}.\tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\left|\frac{x^s-y^s}{s}\right|\ll\min\left(\frac{1}{T_0},\frac{1}{|t|}\right).\tag{4.8}$$

According to (4.8), we get

$$\sum_{y<m\leq x}\Lambda(m)\chi(m)\ll\frac{1}{T_0}\int_{|t|\leq T_0}|F(it,\chi)|\,\mathrm{d}t+\int_{T_0<|t|\leq T}|F(it,\chi)|\frac{\mathrm{d}t}{|t|}+O(L^2).\tag{4.9}$$

From (4.4) and (4.9), we can see that the left-hand side of (4.3) is

$$\begin{aligned}&\ll N^{-1/2}\sum_{r\sim R}[g,r]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{|t|\leq T_0}|F(it,\chi)|\,\mathrm{d}t\\&\quad+N^{1/2}(RQ)^{-1}\sum_{r\sim R}[g,r]^{-2+\varepsilon}\sideset{}{^*}\sum_{\chi\bmod r}\int_{T_0<|t|\leq T}|F(it,\chi)|\frac{\mathrm{d}t}{|t|}\\&\quad+g^{-2+\varepsilon}L^2N^{1/2+\varepsilon}Q^{-1}.\end{aligned}$$

Trivially, the third term in the above estimate is acceptable. Therefore, it follows that (4.3) is a consequence of the following two estimates:

(i) For $R \ll P$ and $0 < T_1 \ll T_0$, we have

$$\sum_{r\sim R} [g,r]^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |F(it,\chi)| \, \mathrm{d}t \ll g^{-2+\varepsilon} N^{1/4} L^c. \tag{4.10}$$

(ii) For $R \ll P$ and $T_0 \ll T_2 \ll T$, we have

$$\sum_{r\sim R} [g,r]^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} |F(it,\chi)| \, \mathrm{d}t \ll g^{-2+\varepsilon} RQN^{-3/4} T_2 L^c. \tag{4.11}$$

In order to prove (4.10), we use the identity that $[g,r](g,r) = gr$. By Lemma 4.1, then the left-hand side of (4.10) is

$$\begin{aligned}
&\sum_{r\sim R} \left(\frac{gr}{(g,r)}\right)^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |F(it,\chi)| \, \mathrm{d}t \\
&\ll g^{-2+\varepsilon} \sum_{\substack{d|g \\ d\leq 2R}} \left(\frac{R}{d}\right)^{-2+\varepsilon} \sum_{\substack{r\sim R \\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |F(it,\chi)| \, \mathrm{d}t \\
&\ll g^{-2+\varepsilon} \sum_{\substack{d|g \\ d\leq 2R}} \left(\frac{R}{d}\right)^{-2+\varepsilon} \left(\frac{R^2 T_1}{d} N^{11/80} + N^{1/4}\right) L^c \\
&\ll g^{-2+\varepsilon} L^c R^{\varepsilon} Q^{-1} N^{91/80} + g^{-2+\varepsilon} N^{1/4} L^c \ll g^{-2+\varepsilon} N^{1/4} L^c.
\end{aligned}$$

Similarly, the left-hand side of (4.11) is

$$\begin{aligned}
&\sum_{r\sim R} \left(\frac{gr}{(g,r)}\right)^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} |F(it,\chi)| \, \mathrm{d}t \\
&\ll g^{-2+\varepsilon} \sum_{\substack{d|g \\ d\leq 2R}} \left(\frac{R}{d}\right)^{-2+\varepsilon} \sum_{\substack{r\sim R \\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} |F(it,\chi)| \, \mathrm{d}t \\
&\ll g^{-2+\varepsilon} \sum_{\substack{d|g \\ d\leq 2R}} \left(\frac{R}{d}\right)^{-2+\varepsilon} \left(\frac{R^2 T_2}{d} N^{11/80} + N^{1/4}\right) L^c \\
&\ll g^{-2+\varepsilon} L^c R^{1+\varepsilon} T_2 N^{11/80} + g^{-2+\varepsilon} N^{1/4} L^c \ll g^{-2+\varepsilon} RQN^{-3/4} T_2 L^c.
\end{aligned}$$

This completes the proof of Lemma 3.7.

4.2. Estimation of $J_k(g)$ with $k = 2, 3$

As is shown in the estimation of $K_4(g)$, we approximate $W_k(\chi,\lambda)$ by $\widehat{W}_k(\chi,\lambda)$. Similar to (4.2), we know that the error $N^{1/(2k)}$ contributes to $J_k(g)$ is

$$\ll N^{1/(2k)} \sum_{r\le P} [g,r]^{-2+\varepsilon} r = g^{-2+\varepsilon} N^{1/(2k)} \sum_{r\le P} \frac{r^{-1+\varepsilon}}{(g,r)^{-2+\varepsilon}} = g^{-2+\varepsilon} N^{1/(2k)} \sum_{\substack{d|g\\ d\le P}} \sum_{\substack{r\le P\\ d|r}} \frac{r^{-1+\varepsilon}}{d^{-2+\varepsilon}}$$

$$\ll g^{-2+\varepsilon} N^{1/(2k)} \sum_{\substack{d|g\\ d\le P}} d\cdot \left(\frac{P}{d}\right)^{\varepsilon} \ll g^{-2+\varepsilon} N^{1/(2k)} P^{1+\varepsilon} \ll g^{-2+\varepsilon} N^{1/k} L^c.$$

Therefore, Lemma 3.5 is a consequence of the following estimate

$$\sum_{r\sim R} [g,r]^{-2+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} \left|\widehat{W}_k(\chi,\lambda)\right| \ll g^{-2+\varepsilon} N^{1/k} L^c, \tag{4.12}$$

where $R \ll P$.

If $R < 1$, then we have

$$\left|\widehat{W}_k(\chi,\lambda)\right| \ll \sum_{X_k<m\le 2X_k} (\Lambda(m)+1) \ll \sum_{X_k<m\le 2X_k} \log m \ll N^{1/k} L.$$

The above contribution to (4.12) is $\ll g^{-2+\varepsilon} N^{1/k} L$, which is acceptable. If $R \ge 1$, we have $\delta_\chi = 0$ for all χ mod r in the definition of $\left|\widehat{W}_k(\chi,\lambda)\right|$, and thus

$$\widehat{W}_k(\chi,\lambda) = \sum_{X_k<m\le 2X_k} \Lambda(m)\chi(m) e(m^k\lambda).$$

By partial summation, we obtain

$$\widehat{W}_k(\chi,\lambda) = \int_{X_k}^{2X_k} e(u^k\lambda)\,\mathrm{d}\left(\sum_{X_k<m\le u} \Lambda(m)\chi(m)\right). \tag{4.13}$$

For the inner sum in (4.13), from Perron's formula, we deduce that

$$\begin{aligned} \sum_{X_k<m\le u} \Lambda(m)\chi(m) &= \frac{1}{2\pi i}\int_{-iT}^{iT} H(s,\chi)\cdot \frac{u^s - X_k^s}{s}\,\mathrm{d}s + O\left(\frac{N^{1/k}L^2}{T}\right) \\ &= \frac{1}{2\pi}\int_{-T}^{T} H(it,\chi)\cdot \frac{u^{it} - X_k^{it}}{it}\,\mathrm{d}t + O(L^2), \end{aligned} \tag{4.14}$$

where $T = N^{1/k}$ and

$$H(s,\chi) = \sum_{X_k<m\le u} \frac{\Lambda(m)\chi(m)}{m^s}.$$

Putting (4.14) into (4.13) and changing the order of the integration, we derive that

$$\widehat{W}_k(\chi,\lambda)=\frac{1}{2\pi}\int_{-T}^{T}H(it,\chi)\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u\,\mathrm{d}t+O\big((1+|\lambda|N)L^2\big). \tag{4.15}$$

For the inner integral in (4.15), we have the trivial estimate

$$\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u\ll\int_{X_k}^{2X_k}u^{-1}\,\mathrm{d}u\ll 1. \tag{4.16}$$

On the other hand, we change the variable by setting $u^k=v$ and obtain

$$\begin{aligned}\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u&=\int_{X_k}^{2X_k}u^{-1}e\left(u^k\lambda+\frac{t\log u}{2\pi}\right)\mathrm{d}u\\&=\frac{1}{k}\int_{X_k^k}^{(2X_k)^k}v^{-1}e\left(v\lambda+\frac{t\log v}{2k\pi}\right)\mathrm{d}v.\end{aligned} \tag{4.17}$$

It is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{t\log v}{2k\pi}+v\lambda\right)=\frac{t}{2k\pi v}+\lambda \tag{4.18}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}v}\left(\frac{t\log v^2}{2k\pi}+v\lambda\right)=-\frac{t}{2k\pi v^2}. \tag{4.19}$$

By (4.17), (4.18) and Lemma 4.3(i), we get

$$\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u\ll\frac{1}{\min_{X_k^k\le v\le(2X_k)^k}|t+2k\pi\lambda v|}. \tag{4.20}$$

From (4.17), (4.19) and Lemma 4.3(ii), we obtain

$$\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u\ll\frac{N}{\sqrt{|t|}}\cdot N^{-1}\ll\frac{1}{\sqrt{|t|}}. \tag{4.21}$$

Combining (4.16), (4.20) and (4.21), we deduce that

$$\begin{aligned}\int_{X_k}^{2X_k}u^{-1+it}e(u^k\lambda)\,\mathrm{d}u&\ll\min\left(1,\frac{1}{\sqrt{|t|}},\frac{1}{\min_{X_k^k\le v\le(2X_k)^k}|t+2k\pi\lambda v|}\right)\\&\ll\begin{cases}\frac{1}{\sqrt{|t|+1}} & \text{if } |t|\le T^*,\\ \frac{1}{|t|} & \text{if } T^*<|t|\le T,\end{cases}\end{aligned} \tag{4.22}$$

where

$$T^*=\frac{4k\pi(2X_k)^k}{RQ}.$$

Here we illustrate that the choice of T^* is to ensure $|t+2k\pi\lambda v| > |t|/2$ for $|t| > T^*$. Actually, there holds

$$\begin{aligned}|t+2k\pi\lambda v| &\ge |t| - 2k\pi|\lambda v| \ge |t| - \frac{2k\pi|v|}{rQ} \ge |t| - \frac{2k\pi|v|}{RQ}\\ &= \frac{|t|}{2} + \frac{|t|}{2} - \frac{2k\pi|v|}{RQ} > \frac{|t|}{2} + \frac{T^*}{2} - \frac{2k\pi|v|}{RQ} \ge \frac{|t|}{2}.\end{aligned}$$

From (4.15) and (4.22), we obtain

$$\widehat{W}_k(\chi,\lambda) = \int_{|t|\le T^*} \frac{|H(it,\chi)|}{\sqrt{|t|+1}}\,\mathrm{d}t + \int_{T^*<|t|\le T} \frac{|H(it,\chi)|}{|t|}\,\mathrm{d}t + O\big((1+|\lambda|N)L^2\big).$$

The contribution of $(1+|\lambda|N)L^2$ to the left-hand side of (4.12) is

$$\begin{aligned}&\ll \frac{NL^2}{RQ}\sum_{r\sim R}[g,r]^{-2+\varepsilon} r \ll \frac{g^{-2+\varepsilon}NL^2}{RQ}\sum_{r\sim R}\frac{r^{-1+\varepsilon}}{(g,r)^{-2+\varepsilon}}\\ &\ll \frac{g^{-2+\varepsilon}NL^2}{RQ}\sum_{\substack{d\mid g\\ d\le 2R}} d^{2-\varepsilon}\sum_{\substack{r\sim R\\ d\mid r}} r^{-1+\varepsilon} \ll g^{-2+\varepsilon}NL^2R^\varepsilon Q^{-1} \ll g^{-2+\varepsilon}N^{1/k}L^c.\end{aligned}$$

Therefore, (4.12) is a consequence of the following two estimates:

(i) For $R \ll P$ and $0 < T_1 \ll T^*$, there holds

$$\sum_{r\sim R}[g,r]^{-2+\varepsilon}\sum_{\chi \bmod r}^{*}\int_{T_1}^{2T_1}|H(it,\chi)|\,\mathrm{d}t \ll g^{-2+\varepsilon}N^{1/k}(T_1+1)^{1/2}L^c. \tag{4.23}$$

(ii) For $R \ll P$ and $T^* \ll T_2 \ll T$, there holds

$$\sum_{r\sim R}[g,r]^{-2+\varepsilon}\sum_{\chi \bmod r}^{*}\int_{T_2}^{2T_2}|H(it,\chi)|\,\mathrm{d}t \ll g^{-2+\varepsilon}N^{1/k}T_2L^c. \tag{4.24}$$

The estimates (4.23) and (4.24) follow from Lemma 4.1 via the arguments similar to those of the estimates which lead to (4.10) and (4.11). So we omit the details. This completes the proof of Lemma 3.5.

4.3. Estimation of $J_2(1)$

Clearly, Lemma 3.6 is the same as that of Lemma 3.5 except for the saving L^{-A} on its right-hand side. Because of this saving, we have to distinguish two cases according as R small or large.

Also, we approximate $W_2(\chi,\lambda)$ by $\widehat{W}_2(\chi,\lambda)$. Similar to (4.2), we know that the error $N^{1/4}$ contributes to $J_k(1)$ is

$$\ll N^{1/4}\sum_{r\le P} r^{-1+\varepsilon} \ll P^\varepsilon N^{1/4} \ll N^{1/2}L^{-A},$$

which is acceptable. Therefore, Lemma 3.6 is a consequence of the estimate

$$\sum_{r\sim R} r^{-2+\varepsilon} \sum_{\chi \bmod r}^{*} \max_{|\lambda|\le 1/(rQ)} \left|\widehat{W}_2(\chi,\lambda)\right| \ll N^{1/2}L^{-A}, \tag{4.25}$$

where $R \ll P$ and $A > 0$ is arbitrary.

If $L^C \ll R \ll P$, where C is a constant which depends on A, we follow the arguments step by step in the proof of Lemma 3.5 to (4.23) and (4.24) with $g = 1$, and find that (4.25) is a consequence of the following two estimates:

(i) For $R \ll P$ and $0 < T_1 \ll T^*$, there holds

$$\sum_{r\sim R} r^{-2+\varepsilon} \sum_{\chi \bmod r}^{*} \int_{T_1}^{2T_1} |H(it,\chi)|\,\mathrm{d}t \ll N^{1/2}(T_1+1)^{1/2}L^{-A}. \tag{4.26}$$

(ii) For $R \ll P$ and $T^* \ll T_2 \ll T$, there holds

$$\sum_{r\sim R} r^{-2+\varepsilon} \sum_{\chi \bmod r}^{*} \int_{T_2}^{2T_2} |H(it,\chi)|\,\mathrm{d}t \ll N^{1/2}T_2L^{-A}. \tag{4.27}$$

The estimates (4.26) and (4.27) follow from Lemma 4.1 via the arguments similar to those of the estimates leading to (4.10) and (4.11). Especially, we use the condition $R \gg L^C$ to obtain the saving factor L^{-A}. So we omit the details.

Now, we concentrate on the case $R \ll L^C$, where $C > 0$ is arbitrary. Taking $T = N^{5/24-\varepsilon}$ in (4.1) of Lemma 4.4 and inserting (4.1) into $\widehat{W}_2(\chi,\lambda)$, from partial summation, we deduce that

$$\begin{aligned}
\widehat{W}_2(\chi,\lambda) &= \int_{X_2}^{2X_2} e(u^2\lambda)\,\mathrm{d}\left(\sum_{m\le u}(\Lambda(m)\chi(m)-\delta_\chi)\right)\\
&= \int_{X_2}^{2X_2} e(u^2\lambda)\,\mathrm{d}\left(-\sum_{|\gamma|\le T}\frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T}+1\right)\log^2(ruT)\right)\right)\\
&= -\int_{X_2}^{2X_2} e(u^2\lambda)\sum_{|\gamma|\le T} u^{\rho-1}\,\mathrm{d}u + O\left(\frac{N^{1/2}L^2}{T}(1+|\lambda|N)\right)\\
&\ll N^{1/2}\sum_{|\gamma|\le T} N^{(\beta-1)/2} + \frac{N^{1/2}L^2}{T}(1+|\lambda|N).
\end{aligned} \tag{4.28}$$

The contribution of the second term in (4.28) to the left-hand side of (4.25) is

$$\ll \frac{N^{1/2}L^2}{T}\cdot\frac{N}{RQ}\sum_{r\sim R}\varphi(r) \ll \frac{N^{1/2}L^2RN}{TQ} \ll \frac{N^{3/2}L^{C+2}}{TQ} \ll N^{1/2}L^{-A}.$$

Let $\eta(T) = c_3 \log^{-4/5} T$. By Lemma 4.6, $\prod_{\chi \bmod r} L(s,\chi)$ is zero-free in the region $\Re s \geq 1 - \eta(T)$, $|\Im s| \leq T$ except for the possible Siegel zero. On the other hand, by Siegel's theorem (see, for instance, Lemma 9 on page 255 of Pan-Pan [10] or Chapter 21 of Davenport [2]), the Siegel zero does not exist in the present situation, since $r \sim R$ and $R \ll L^C$. Therefore, by partial summation and Lemma 4.5, we have

$$\begin{aligned}\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\leq T} N^{(\beta-1)/2} &= \int_0^{1-\eta(T)} N^{(\alpha-1)/2}\,\mathrm{d}\Bigg(\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{\substack{0<\beta\leq\alpha\\ |\gamma|\leq T}} 1\Bigg)\\ &= \int_0^{1-\eta(T)} N^{(\alpha-1)/2}\,\mathrm{d}\left(\sum_{r\sim R} \mathscr{N}^*(\alpha,r,T)\right)\\ &\ll \exp\big(-c_4 L^{1/5}\big).\end{aligned}$$

Consequently, we deduce that, for any $A > 0$, there holds

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\leq 1/(rQ)} \left|\widehat{W}_2(\chi,\lambda)\right| \ll N^{1/2} L^{-A},$$

which implies (4.25) in the second case. This completes the proof of Lemma 3.6.

5. The singular series

In this section, we shall investigate the properties of the singular series which appear in Proposition 2.1.

Lemma 5.1. *Let p be a prime and $p^\alpha \| k$. For $(a,p) = 1$, if $\ell \geq \gamma(p)$, we have $C_k(p^\ell, a) = 0$, where*

$$\gamma(p) = \begin{cases} \alpha + 2 & \text{if } p \neq 2 \text{ or } p = 2,\ \alpha = 0, \\ \alpha + 3 & \text{if } p = 2,\ \alpha > 0. \end{cases}$$

Proof. See Lemma 8.3 of Hua [4]. □

For $k \geq 1$, we define

$$S_k(q,a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right).$$

Lemma 5.2. *Suppose that $(p,a) = 1$. Then*

$$S_k(p,a) = \sum_{\chi \in \mathscr{A}_k} \overline{\chi(a)}\tau(\chi),$$

where $\mathscr{A}_k$ denotes the set of non-principal characters χ modulo p for which χ^k is principal, and $\tau(\chi)$ denotes the Gauss sum

$$\sum_{m=1}^{p} \chi(m) e\left(\frac{m}{p}\right).$$

Also, there hold $|\tau(\chi)| = p^{1/2}$ *and* $|\mathscr{A}_k| = (k, p-1) - 1$.

Proof. See Lemma 4.3 of Vaughan [15]. □

Lemma 5.3. *For* $(p, n) = 1$, *we have*

$$\left|\sum_{a=1}^{p-1}\left(\prod_{k=2}^{4}\frac{S_k^2(p,a)}{p^2}\right)e\left(-\frac{an}{p}\right)\right| \leq 36p^{-5/2}. \tag{5.1}$$

Proof. We denote by $\mathcal{S}$ the left-hand side of (5.1). By Lemma 5.2, we have

$$\mathcal{S} = \frac{1}{p^6}\sum_{a=1}^{p-1}\left(\prod_{k=2}^{4}\left(\sum_{\chi_k\in\mathscr{A}_k}\overline{\chi_k(a)}\tau(\chi_k)\right)^2\right)e\left(-\frac{an}{p}\right).$$

If $|\mathscr{A}_k| = 0$ for some $k \in \{2, 3, 4\}$, then $\mathcal{S} = 0$. If this is not the case, then

$$\begin{aligned}\mathcal{S} = \frac{1}{p^6}&\sum_{\chi_2^{(1)}\in\mathscr{A}_2}\sum_{\chi_2^{(2)}\in\mathscr{A}_2}\sum_{\chi_3^{(1)}\in\mathscr{A}_3}\sum_{\chi_3^{(2)}\in\mathscr{A}_3}\sum_{\chi_4^{(1)}\in\mathscr{A}_4}\sum_{\chi_4^{(2)}\in\mathscr{A}_4}\\ &\times\tau(\chi_2^{(1)})\tau(\chi_2^{(2)})\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_4^{(1)})\tau(\chi_4^{(2)})\\ &\times\sum_{a=1}^{p-1}\overline{\chi_2^{(1)}(a)\chi_2^{(2)}(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_4^{(1)}(a)\chi_4^{(2)}(a)}e\left(-\frac{an}{p}\right).\end{aligned}$$

From Lemma 5.2, the sextuple outer sums have not more than $\prod_{k=2}^{4}\big((k,p-1)-1\big)^2 \leq (3!)^2 = 36$ terms. In each of these terms, we have

$$\left|\tau(\chi_2^{(1)})\tau(\chi_2^{(2)})\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_4^{(1)})\tau(\chi_4^{(2)})\right| = p^3.$$

Since in any one of these terms

$$\overline{\chi_2^{(1)}(a)\chi_2^{(2)}(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_4^{(1)}(a)\chi_4^{(2)}(a)}$$

is a Dirichlet character $\chi \pmod p$, the inner sum is

$$\sum_{a=1}^{p-1}\chi(a)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\sum_{a=1}^{p-1}\chi(-an)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\tau(\chi).$$

From the fact that $\tau(\chi^0) = -1$ for principal character χ^0 mod p, we have

$$\left|\overline{\chi(-n)}\tau(\chi)\right| \leq p^{1/2}.$$

By the above arguments, we obtain

$$|\mathcal{S}| \leq \frac{1}{p^6}\cdot 36\cdot p^3\cdot p^{1/2} = 36p^{-5/2}.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let $\mathcal{L}(p,n)$ denote the number of solutions of the congruence*

$$x_1^2+x_2^2+x_3^3+x_4^3+x_5^4+x_6^4\equiv n \pmod p,\quad 1\le x_1,x_2,\ldots,x_6\le p-1.$$

Then, for $n\equiv 0 \pmod 2$, we have $\mathcal{L}(p,n)>0$.

Proof. We have

$$p\cdot\mathcal{L}(p,n)=\sum_{a=1}^{p}C_2^2(p,a)C_3^2(p,a)C_4^2(p,a)e\left(-\frac{an}{p}\right)=(p-1)^6+E_p,$$

where

$$E_p=\sum_{a=1}^{p-1}C_2^2(p,a)C_3^2(p,a)C_4^2(p,a)e\left(-\frac{an}{p}\right).$$

By Lemma 5.2, we obtain

$$|E_p|\le(p-1)(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(3\sqrt{p}+1)^2.$$

It is easy to check that $|E_p|<(p-1)^6$ for $p\ge 13$. Therefore, we obtain $\mathcal{L}(p,n)>0$ for $p\ge 13$. If $p<13$, we can check $\mathcal{L}(p,n)>0$ directly. This completes the proof of Lemma 5.4. □

Lemma 5.5. *$A(n,q)$ is multiplicative in q.*

Proof. By the definition of $A(n,q)$ in (3.1), we only need to show that $B(n,q)$ is multiplicative in q. Suppose $q=q_1q_2$ with $(q_1,q_2)=1$. Then we have

$$\begin{aligned}B(n,q_1q_2)&=\sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2}\left(\prod_{k=2}^{4}C_k^2(q_1q_2,a)\right)e\left(-\frac{an}{q_1q_2}\right)\\&=\sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2}\left(\prod_{k=2}^{4}C_k^2(q_1q_2,a_1q_2+a_2q_1)\right)e\left(-\frac{a_1n}{q_1}\right)e\left(-\frac{a_2n}{q_2}\right).\end{aligned}\tag{5.2}$$

For $(q_1,q_2)=1$, there holds

$$\begin{aligned}C_k(q_1q_2,a_1q_2+a_2q_1)&=\sum_{\substack{m=1\\(m,q_1q_2)=1}}^{q_1q_2}e\left(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\right)\\&=\sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2}e\left(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\right)\\&=\sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}e\left(\frac{a_1(m_1q_2)^k}{q_1}\right)\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2}e\left(\frac{a_2(m_2q_1)^k}{q_2}\right)\\&=C_k(q_1,a_1)C_k(q_2,a_2).\end{aligned}\tag{5.3}$$

Putting (5.3) into (5.2), we deduce that

$$B(n,q_1q_2)=\sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\left(\prod_{k=2}^{4}C_k^2(q_1,a_1)\right)e\left(-\frac{a_1n}{q_1}\right)\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2}\left(\prod_{k=2}^{4}C_k^2(q_2,a_2)\right)e\left(-\frac{a_2n}{q_2}\right)$$
$$=B(n,q_1)B(n,q_2).$$

This completes the proof of Lemma 5.5. □

Lemma 5.6. *Let $A(n,q)$ be as defined in* (3.1). *Then*

(i) *We have*

$$\sum_{q>Z}|A(n,q)|\ll Z^{-3/2+\varepsilon}d(n).$$

Hence $\sum_{q=1}^{\infty}A(n,q)$ is absolutely convergent and satisfies $\mathfrak{S}(n)\ll d(n)$.

(ii) *There exists an absolute positive constant $c^*>0$, such that, for $n\equiv 0 \pmod 2$,*

$$\mathfrak{S}(n)\geq c^*>0.$$

Proof. From Lemma 5.5, we know that $B(n,q)$ is multiplicative in q. Therefore, there holds

$$B(n,q)=\prod_{p^t\|q}B(n,p^t)=\prod_{p^t\|q}\sum_{\substack{a=1\\(a,p)=1}}^{p^t}\left(\prod_{k=2}^{4}C_k^2(p^t,a)\right)e\left(-\frac{an}{p^t}\right). \tag{5.4}$$

From (5.4) and Lemma 5.1, we deduce that $B(n,q)=\prod_{p\|q}B(n,p)$ and q is square-free. Thus, one has

$$\sum_{q=1}^{\infty}A(n,q)=\sum_{\substack{q=1\\q\text{ square-free}}}^{\infty}A(n,q). \tag{5.5}$$

Write

$$\mathcal{R}(p,a):=\prod_{k=2}^{4}C_k^2(p,a)-\prod_{k=2}^{4}S_k^2(p,a).$$

Then

$$A(n,p)=\frac{1}{(p-1)^6}\sum_{a=1}^{p-1}\left(\prod_{k=2}^{4}S_k^2(p,a)\right)e\left(-\frac{an}{p}\right)+\frac{1}{(p-1)^6}\sum_{a=1}^{p-1}\mathcal{R}(p,a)e\left(-\frac{an}{p}\right). \tag{5.6}$$

Applying Lemma 3.1 and noticing that $S_k(p,a)=C_k(p,a)+1$, we get $S_k(p,a)\ll p^{1/2}$, and thus $\mathcal{R}(p,a)\ll p^{5/2}$. Therefore, the second term in (5.6) is $\leq c_5p^{-5/2}$. On the other hand, from Lemma 5.3, we can see that the first term in (5.6) is $\le 2^6 \cdot 36 p^{-5/2} = 2304 p^{-5/2}$. Let $c_6 = \max(c_5, 2304)$. Then we have proved that, for $p \nmid n$, there holds

$$|A(n,p)| \le c_6 p^{-5/2}. \tag{5.7}$$

Moreover, if we use Lemma 3.1 directly, it follows that

$$\begin{aligned} |B(n,p)| &= \left| \sum_{a=1}^{p-1} \left(\prod_{k=2}^{4} C_k^2(p,a) \right) e\left(-\frac{an}{p} \right) \right| \le \sum_{a=1}^{p-1} \prod_{k=2}^{4} |C_k(p,a)|^2 \\ &\le (p-1) \cdot 2^6 \cdot p^3 \cdot 576 = 36864 p^3 (p-1), \end{aligned}$$

and therefore

$$|A(n,p)| = \frac{|B(n,p)|}{\varphi^6(p)} \le \frac{36864 p^3}{(p-1)^5} \le \frac{2^5 \cdot 36864 p^3}{p^5} = \frac{1179648}{p^2}. \tag{5.8}$$

Let $c_7 = \max(c_6, 1179648)$. Then, for square-free q, we have

$$\begin{aligned} |A(n,q)| &= \left(\prod_{\substack{p|q \\ p\nmid n}} |A(n,p)| \right) \left(\prod_{\substack{p|q \\ p|n}} |A(n,p)| \right) \le \left(\prod_{\substack{p|q \\ p\nmid n}} (c_7 p^{-5/2}) \right) \left(\prod_{\substack{p|q \\ p|n}} (c_7 p^{-2}) \right) \\ &= c_7^{\omega(q)} \left(\prod_{p|q} p^{-5/2} \right) \left(\prod_{p|(n,q)} p^{1/2} \right) \ll q^{-5/2+\varepsilon} (n,q)^{1/2}. \end{aligned}$$

Hence, by (5.5), we obtain

$$\begin{aligned} \sum_{q>Z} |A(n,q)| &\ll \sum_{q>Z} q^{-5/2+\varepsilon} (n,q)^{1/2} = \sum_{d|n} \sum_{q>Z/d} (dq)^{-5/2+\varepsilon} d^{1/2} = \sum_{d|n} d^{-2+\varepsilon} \sum_{q>Z/d} q^{-5/2+\varepsilon} \\ &\ll \sum_{d|n} d^{-2+\varepsilon} \left(\frac{Z}{d} \right)^{-3/2+\varepsilon} = Z^{-3/2+\varepsilon} \sum_{d|n} d^{-\frac{1}{2}+\varepsilon} \ll Z^{-3/2+\varepsilon} d(n). \end{aligned}$$

This proves (i) of Lemma 5.6.

To prove (ii) of Lemma 5.6, by Lemma 5.5, we first note that

$$\begin{aligned} \mathfrak{S}(n) &= \prod_{p} \left(1 + \sum_{t=1}^{\infty} A(n,p^t) \right) = \prod_{p} (1 + A(n,p)) \\ &= \left(\prod_{p \le c_7} (1 + A(n,p)) \right) \prod_{\substack{p>c_7 \\ p\nmid n}} (1 + A(n,p)) \Bigg) \left(\prod_{\substack{p>c_7 \\ p|n}} (1 + A(n,p)) \right). \end{aligned} \tag{5.9}$$

From (5.7), we have

$$\prod_{\substack{p>c_7 \\ p\nmid n}} (1 + A(n,p)) \ge \prod_{p>c_7} \left(1 - \frac{c_7}{p^{5/2}} \right) \ge c_8 > 0. \tag{5.10}$$

By (5.8), we obtain

(5.11)
$$\prod_{\substack{p>c_7\\p|n}}(1+A(n,p)) \geq \prod_{p>c_7}\left(1-\frac{c_7}{p^2}\right) \geq c_9 > 0.$$

On the other hand, it is easy to see that

$$1+A(n,p)=\frac{p\cdot\mathcal{L}(p,n)}{\varphi^6(p)}.$$

By Lemma 5.4, we know that $\mathcal{L}(p,n)>0$ for all p with $n\equiv 0 \pmod 2$, and thus $1+A(n,p)>0$. Therefore, there holds

(5.12)
$$\prod_{p\leq c_7}(1+A(n,p)) \geq c_{10} > 0.$$

Combining the estimates (5.9)–(5.11) and (5.12), and taking $c^* = c_8c_9c_{10} > 0$, we derive that

$$\mathfrak{S}(n) \geq c^* > 0.$$

This completes the proof Lemma 5.6. □

6. Proof of Proposition 2.2

We shall present some lemmas that will be used to prove Proposition 2.2. Define the multiplicative function $w_3(q)$ by

$$w_3(p^{3u+v})=\begin{cases}3p^{-u-1/2} & u\geq 0,\ v=1,\\ p^{-u-1} & u\geq 0,\ 2\leq v\leq 3.\end{cases}$$

Lemma 6.1. *Let c be a constant. For $x\geq 2$, one has*

$$\sum_{q\leq x} d^c(q)w_3^2(q) \ll \log^C x,$$

where C is an absolute constant.

Proof. See Lemma 2.1 of Zhao [17]. □

Lemma 6.2. *For $\gamma\in\mathbb{R}$, we define*

$$\mathcal{L}(\gamma)=\sum_{q\leq X_4}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{|\alpha-a/q|\leq X_3}\frac{w_3^2(q)d^c(q)\left|\sum_{X_4<p\leq 2X_4} e(p^4(\alpha+\gamma))\right|^2}{1+X_3^3|\alpha-a/q|}\,\mathrm{d}\alpha.$$

Then one has uniformly for $\gamma\in\mathbb{R}$ that

$$\mathcal{L}(\gamma) \ll X_4^2N^{-1+\varepsilon}.$$

Proof. We have

$$\mathcal{L}(\gamma) \le \sum_{q\le X_4} \int_{|\alpha-a/q|\le X_3} \frac{w_3^2(q)d^c(q)\sum_{a=1}^q \left|\sum_{X_4<p\le 2X_4} e(p^4(\alpha+\gamma))\right|^2}{1+X_3^3|\alpha-a/q|}\,\mathrm{d}\alpha.$$

For the sum in the above integral, there holds

$$\begin{aligned}
&\sum_{a=1}^q \left|\sum_{X_4<p\le 2X_4} e\left(p^4\frac{a}{q}+p^4(\beta+\gamma)\right)\right|^2\\
&=\sum_{a=1}^q \sum_{X_4<p_1,p_2\le 2X_4} e\left((p_1^4-p_2^4)\frac{a}{q}+(p_1^4-p_2^4)(\beta+\gamma)\right)\\
&=\sum_{X_4<p_1,p_2\le 2X_4} e\left((p_1^4-p_2^4)(\beta+\gamma)\right)\sum_{a=1}^q e\left(\frac{(p_1^4-p_2^4)a}{q}\right)\\
&=q\sum_{\substack{X_4<p_1,p_2\le 2X_4\\ p_1^4\equiv p_2^4 \pmod q}} e\left((p_1^4-p_2^4)(\beta+\gamma)\right).
\end{aligned}$$

Since $q\le X_4$ and $X_4<p\le 2X_4$, we have $(p,q)=1$. Then we obtain

$$\begin{aligned}
&\sum_{a=1}^q \left|\sum_{X_4<p\le 2X_4} e\left(p^4\frac{a}{q}+p^4(\beta+\gamma)\right)\right|^2\\
&\ll \frac{X_4^2}{q}\sum_{\substack{1\le n_1,n_2<q\\ n_1^4\equiv n_2^4 \pmod q\\ (n_1n_2,q)=1}} 1 \ll \frac{X_4^2}{q}\sum_{1\le n_1<q}\sum_{\substack{1\le n_2<q\\ n_1^4\equiv n_2^4 \pmod q}} 1 \ll X_4^2 d^c(q).
\end{aligned}$$

Therefore, from Lemma 6.1, we deduce that

$$\mathcal{L}(\gamma) \ll X_4^2\left(\sum_{q\le X_4} w_3^2(q)d^c(q)\right)\int_{|\beta|\le X_3}\frac{1}{1+|\beta|X_3^3}\,\mathrm{d}\beta \ll X_4^2N^{-1+\varepsilon}.$$

This completes the proof of Lemma 6.2. □

Lemma 6.3. *Suppose that α is a real number, and that $|\alpha-a/q|\le q^{-2}$ with $(a,q)=1$. Let $\beta=\alpha-a/q$. Then we have*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log x)^c\left(X_k^{1/2}\sqrt{q(1+N|\beta|)}+X_k^{4/5}+\frac{X_k}{\sqrt{q(1+N|\beta|)}}\right),$$

where $\delta_k=1/2+(\log k)/\log 2$ and c is a constant.

Proof. See Theorem 1.1 of Ren [12]. □

Lemma 6.4. *Suppose that α is a real number, and that there exist integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$(a, q) = 1, \quad 1 \le q \le N^{1/2}, \quad \text{and} \quad |q\alpha - a| \le N^{-1/2}.$$

Then for $k \in \{2, 3, 4\}$, we have

$$f_k(\alpha) \ll X_k^{1-\eta_k+\varepsilon} + \frac{X_k^{1+\varepsilon}}{\sqrt{q(1 + N|\alpha - a/q|)}},$$

where $\eta_2 = 1/8$, $\eta_3 = 1/14$ and $\eta_4 = 1/24$.

Proof. See Theorem 3 of Kumchev [6]. □

For $\mathscr{A} \subseteq (X_3, 2X_3] \cap \mathbb{N}$, we define

$$g(\alpha) = g_{\mathscr{A}}(\alpha) = \sum_{n \in \mathscr{A}} (\log n) e(n^3\alpha).$$

Then we have the following lemma.

Lemma 6.5. *Let $\mathscr{M}$ be the union of the intervals $\mathscr{M}(q, a)$ for $1 \le a \le q \le X_3^{3/4}$ and $(a, q) = 1$, where*

$$\mathscr{M}(q, a) = \left\{ \alpha : |q\alpha - a| \le X_3^{-9/4} \right\}.$$

Suppose that $G(\alpha)$ and $h(\alpha)$ are integrable functions of period one. Then for any measurable set $\mathfrak{m} \subseteq [0, 1]$, we have

$$\int_{\mathfrak{m}} g(\alpha)G(\alpha)h(\alpha)\, \mathrm{d}\alpha \ll N^{1/3} \mathcal{J}_0^{1/4} \left(\int_{\mathfrak{m}} |G(\alpha)|^2 \, \mathrm{d}\alpha \right)^{1/4} \mathscr{J}^{1/2}(\mathfrak{m}) + N^{7/24+\varepsilon} \mathscr{J}(\mathfrak{m}),$$

where

$$\mathscr{J}(\mathfrak{m}) = \int_{\mathfrak{m}} |G(\alpha)h(\alpha)| \, \mathrm{d}\alpha, \quad \mathcal{J}_0 = \sup_{\beta \in [0,1]} \int_{\mathscr{M}} \frac{w_3^2(q)|h(\alpha + \beta)|^2}{\left(1 + X_3^3|\alpha - a/q|\right)^2} \, \mathrm{d}\alpha.$$

Proof. See Lemma 3.1 of Zhao [17]. □

Define

$$\mathfrak{N}(q, a) = \left[\frac{a}{q} - \frac{1}{qN^{5/6}}, \frac{a}{q} + \frac{1}{qN^{5/6}} \right], \quad \mathfrak{N} = \bigcup_{q \le N^{1/6}} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \mathfrak{N}(q, a).$$

We write $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{N}$ and $\mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{N}$.

Lemma 6.6. *Suppose that $\alpha \in \mathfrak{m}_1$. Then we have*

$$f_2(\alpha) \ll N^{71/160+\varepsilon} \quad \text{and} \quad f_3(\alpha) \ll N^{133/480+\varepsilon}.$$

Proof. For $\alpha \in \mathfrak{m}_1$, there exist integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$1 \le a \le q \le N^{1/6}, \quad |q\alpha - a| \le N^{-5/6}, \quad (a,q) = 1.$$

Since $\alpha \notin \mathfrak{M}$, we have either $q > P$ or $|q\alpha - a| > Q^{-1}$. Therefore, from Lemma 6.3, we deduce that

$$f_2(\alpha) \ll N^{71/160+\varepsilon} \quad \text{and} \quad f_3(\alpha) \ll N^{133/480+\varepsilon}.$$

This completes the proof of Lemma 6.6. □

Lemma 6.7. *Suppose that $\alpha \in \mathfrak{m}_2$. Then we have*

$$f_2(\alpha) \ll X_2^{1-1/8+\varepsilon}, \quad f_3(\alpha) \ll X_3^{1-1/14+\varepsilon}, \quad f_4(\alpha) \ll X_4^{1-1/24+\varepsilon}.$$

Proof. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$1 \le q \le N^{1/2}, \quad |q\alpha - a| \le N^{-1/2}, \quad (a,q) = 1.$$

Since $\alpha \in \mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{N}$, we get either $q > N^{1/6}$ or $N|q\alpha - a| > N^{1/6}$. Therefore, the conclusions follow from Lemma 6.4. □

Lemma 6.8. *Let $f_k(\alpha)$ be defined as above. Then we have*

(i) $\int_0^1 \left|f_2^2(\alpha) f_3^4(\alpha)\right| \mathrm{d}\alpha \ll N^{4/3+\varepsilon}$;

(ii) $\int_0^1 \left|f_2^2(\alpha) f_3^6(\alpha)\right| \mathrm{d}\alpha \ll N^{2+\varepsilon}$;

(iii) $\int_0^1 \left|f_2^2(\alpha) f_4^8(\alpha)\right| \mathrm{d}\alpha \ll N^{2+\varepsilon}$.

Proof. We only give the details of the proof of (iii), since the proofs of (i) and (ii) are similar to that of (iii). The conclusion can be deduced by counting the number of solutions of the underlying Diophantine equation:

$$x_1^2 - x_2^2 = y_1^4 + y_2^4 + y_3^4 + y_4^4 - y_5^4 - y_6^4 - y_7^4 - y_8^4$$

with $X_2 < x_1, x_2 \le 2X_2$ and $X_4 < y_i \le 2X_4$ for $i = 1, 2, \ldots, 8$. If $x_1 \ne x_2$, the contribution is bounded by $X_4^{8+\varepsilon}$. If $x_1 = x_2$, the contribution is bounded by

$$\ll X_2 \cdot \int_0^1 |f_4(\alpha)|^8 \, \mathrm{d}\alpha.$$

By Lemma 2.5 of Vaughan [15], we have

$$\int_0^1 |f_4(\alpha)|^8 \, \mathrm{d}\alpha \ll X_4^{5+\varepsilon},$$

and thus the contribution with $x_1 = x_2$ is $\ll X_2 \cdot X_4^{5+\varepsilon} \ll N^{7/4+\varepsilon}$. Combining above two cases, we deduce that

$$\int_0^1 \left|f_2^2(\alpha) f_4^8(\alpha)\right| \mathrm{d}\alpha \ll X_4^{8+\varepsilon} + X_2 \cdot X_4^{5+\varepsilon} \ll N^{2+\varepsilon}.$$

This completes the proof of Lemma 6.8. □

Proof of Proposition 2.2. Define

$$\mathcal{I}(t) = \int_{\mathfrak{m}_2} \left|f_2^4(\alpha) f_3^t(\alpha) f_4^4(\alpha)\right| \mathrm{d}\alpha, \quad t \geq 1.$$

Taking

$$g(\alpha) = f_3(\alpha), \quad h(\alpha) = f_4(\alpha), \quad G(\alpha) = |f_2(\alpha)|^4 |f_3(\alpha)|^2 f_3(-\alpha) f_4(-\alpha) |f_4(\alpha)|^2$$

in Lemma 6.5, we obtain

$$\text{(6.1)} \quad \mathcal{I}(4) \ll N^{1/3} \mathscr{J}_0^{1/4} \left(\int_{\mathfrak{m}_2} |f_2(\alpha)|^8 |f_3(\alpha)|^6 |f_4(\alpha)|^6 \, \mathrm{d}\alpha \right)^{1/4} (\mathcal{I}(3))^{1/2} + N^{7/24+\varepsilon} \cdot \mathcal{I}(3),$$

where

$$\mathscr{J}_0 = \sup_{\beta \in [0,1]} \sum_{q \leq X_3^{3/4}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathscr{M}(q,a)} \frac{w_3^2(q) |h(\alpha+\beta)|^2}{\left(1 + X_3^3 |\alpha - a/q|\right)^2} \mathrm{d}\alpha$$

with

$$\mathscr{M}(q,a) = \left\{\alpha : |q\alpha - a| \leq X_3^{-9/4}\right\}.$$

By Lemma 6.2, we get

$$\text{(6.2)} \qquad \mathscr{J}_0 \ll \mathcal{L}(\gamma) \ll N^{-1/2+\varepsilon}.$$

From Lemma 6.7, we obtain

$$\text{(6.3)} \qquad \begin{aligned} &\int_{\mathfrak{m}_2} \left|f_2^8(\alpha) f_3^6(\alpha) f_4^6(\alpha)\right| \mathrm{d}\alpha \\ &\ll \left(\sup_{\alpha \in \mathfrak{m}_2} |f_2(\alpha)|^4\right) \left(\sup_{\alpha \in \mathfrak{m}_2} |f_3(\alpha)|^2\right) \left(\sup_{\alpha \in \mathfrak{m}_2} |f_4(\alpha)|^2\right) \cdot \mathcal{I}(4) \\ &\ll (X_2^{7/8+\varepsilon})^4 (X_3^{13/14+\varepsilon})^2 (X_4^{23/24+\varepsilon})^2 \cdot \mathcal{I}(4) \ll N^{319/112+\varepsilon} \cdot \mathcal{I}(4). \end{aligned}$$

Putting (6.2) and (6.3) into (6.1), we derive that

$$\text{(6.4)} \qquad \mathcal{I}(4) \ll N^{1237/1344+\varepsilon} (\mathcal{I}(4))^{1/4} (\mathcal{I}(3))^{1/2} + N^{7/24+\varepsilon} \mathcal{I}(3).$$

It follows from Hölder's inequality that

(6.5)
$$\begin{aligned}\mathcal{I}(3) &\ll \left(\int_{\mathfrak{m}_2} \left|f_2^{4/3}(\alpha)f_3^{4/3}(\alpha)f_4^{4/3}(\alpha)\right|^3 \mathrm{d}\alpha\right)^{1/3} \left(\int_{\mathfrak{m}_2} \left|f_2^{8/3}(\alpha)f_3^{5/3}(\alpha)f_4^{8/3}(\alpha)\right|^{3/2} \mathrm{d}\alpha\right)^{2/3} \\ &= (\mathcal{I}(4))^{1/3} \cdot \left(\mathcal{I}\left(\frac{5}{2}\right)\right)^{2/3}.\end{aligned}$$

According to Lemmas 6.7, 6.8 and Hölder's inequality, we obtain

(6.6)
$$\begin{aligned}\mathcal{I}\left(\frac{5}{2}\right) &\ll \sup_{\alpha\in\mathfrak{m}_2} |f_2(\alpha)|^2 \times \int_{\mathfrak{m}_2} \left|\left(f_2^{1/2}(\alpha)f_3^{3/2}(\alpha)\right)\cdot\left(f_2^{1/2}(\alpha)f_3(\alpha)\right)\cdot\left(f_2(\alpha)f_4^4(\alpha)\right)\right| \mathrm{d}\alpha \\ &\ll \left(\sup_{\alpha\in\mathfrak{m}_2} |f_2(\alpha)|^2\right)\left(\int_0^1 \left|f_2^2(\alpha)f_3^6(\alpha)\right| \mathrm{d}\alpha\right)^{1/4} \left(\int_0^1 \left|f_2^2(\alpha)f_3^4(\alpha)\right| \mathrm{d}\alpha\right)^{1/4} \\ &\quad\times \left(\int_0^1 \left|f_2^2(\alpha)f_4^8(\alpha)\right| \mathrm{d}\alpha\right)^{1/2} \\ &\ll N^{7/8+\varepsilon}\cdot(N^{2+\varepsilon})^{1/4}(N^{4/3+\varepsilon})^{1/4}(N^{2+\varepsilon})^{1/2} \ll N^{65/24+\varepsilon}.\end{aligned}$$

Putting (6.6) into (6.5), we derive that

$$\mathcal{I}(3) \ll N^{65/36+\varepsilon}\cdot(\mathcal{I}(4))^{1/3}. \tag{6.7}$$

Inserting (6.7) into (6.4), we have

$$\mathcal{I}(4) \ll N^{7351/4032+\varepsilon}(\mathcal{I}(4))^{5/12} + N^{151/72+\varepsilon}(\mathcal{I}(4))^{1/3},$$

which implies

$$\mathcal{I}(4) \ll N^{7351/2352+\varepsilon} + N^{151/48+\varepsilon} \ll N^{151/48+\varepsilon} = N^{7/3+1-3/16+\varepsilon}. \tag{6.8}$$

For the contribution from $\mathfrak{m}_1$, by Lemmas 6.6 and 6.8, we deduce that

(6.9)
$$\begin{aligned}&\int_{\mathfrak{m}_1} \left|f_2^4(\alpha)f_3^4(\alpha)f_4^4(\alpha)\right| \mathrm{d}\alpha \\ &\ll \left(\sup_{\alpha\in\mathfrak{m}_1} |f_2(\alpha)|^2\right)\left(\sup_{\alpha\in\mathfrak{m}_1} |f_3(\alpha)|^{3/2}\right)\int_{\mathfrak{m}_1} \left|f_2^2(\alpha)f_3^{5/2}(\alpha)f_4^4(\alpha)\right| \mathrm{d}\alpha \\ &\ll N^{71/80+\varepsilon}\cdot N^{133/320+\varepsilon}\cdot\left(\int_0^1 \left|f_2^2f_3^6\right| \mathrm{d}\alpha\right)^{1/4}\left(\int_0^1 \left|f_2^2f_3^4\right| \mathrm{d}\alpha\right)^{1/4}\left(\int_0^1 \left|f_2^2f_4^8\right| \mathrm{d}\alpha\right)^{1/2} \\ &\ll N^{71/80+\varepsilon}\cdot N^{133/320+\varepsilon}\cdot(N^{2+\varepsilon})^{1/4}\cdot(N^{4/3+\varepsilon})^{1/4}\cdot(N^{2+\varepsilon})^{1/2} \\ &\ll N^{3011/960+\varepsilon} \ll N^{7/3+1-3/16+\varepsilon}.\end{aligned}$$

Combining (6.8) and (6.9), we obtain the conclusion of Proposition 2.2. This completes the proof of Proposition 2.2. □

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References

[1] S. K.-K. Choi and A. V. Kumchev, *Mean values of Dirichlet polynomials and applications to linear equations with prime variables*, Acta Arith. **123** (2006), no. 2, 125–142.

[2] H. Davenport, *Multiplicative Number Theory*, Second edition, Graduate Texts in Mathematics **74**, Springer-Verlag, New York, 1980.

[3] P. X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. **11** (1970), no. 4, 329–339.

[4] L. K. Hua, *Additive Theory of Prime Numbers*, Translations of Mathematical Monographs **13**, American Mathematical Society, Providence, R.I., 1965.

[5] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), no. 2, 164–170.

[6] A. V. Kumchev, *On Weyl sums over primes and almost primes*, Michigan Math. J. **54** (2006), no. 2, 243–268.

[7] J. Liu, *On Lagrange's theorem with prime variables*, Q. J. Math. **54** (2003), no. 4, 453–462.

[8] Z. Liu, *Goldbach-Linnik type problems with unequal powers of primes*, J. Number Theory **176** (2017), 439–448.

[9] X. D. Lü, *Waring-Goldbach problem: two squares, two cubes and two biquadrates*, Chinese Ann. Math. Ser. A **36** (2015), no. 2, 161–174.

[10] C. D. Pan and C. B. Pan, *Goldbach's Conjecture*, Science Press, Beijing, 1981.

[11] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin, 1957.

[12] X. Ren, *On exponential sums over primes and application in Waring-Goldbach problem*, Sci. China Ser. A **48** (2005), no. 6, 785–797.

[13] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second edition, (Revised by D. R. Heath-Brown), Oxford University Press, Oxford, 1986.

[14] R. C. Vaughan, *On the representation of numbers as sums of powers of natural numbers*, Proc. London Math. Soc. (3) **21** (1970), 160–180.

[15] ———, *The Hardy-Littlewood Method*, Second edition, Cambridge University Press, Cambridge, 1997.

[16] I. M. Vinogradov, *Elements of Number Theory*, Dover Publications, New York, 1954.

[17] L. Zhao, *On the Waring-Goldbach problem for fourth and sixth powers*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 6, 1593–1622.

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