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b-generalized (α, β)-derivations and b-generalized (α, β)-biderivations of Prime Rings

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Abstract. Let R be a ring, α and β two automorphisms of R. An additive mapping $d\colon R \to R$ is called an (α, β)-derivation if $d(xy) = d(x)\alpha(y)+\beta(x)d(y)$ for any $x, y \in R$. An additive mapping $G\colon R \to R$ is called a generalized (α, β)-derivation if $G(xy) = G(x)\alpha(y) + \beta(x)d(y)$ for any $x, y \in R$, where d is an (α, β)-derivation of R. In this paper we introduce the definitions of b-generalized (α, β)-derivation and b-generalized (α, β)-biderivation. More precisely, let $d\colon R \to R$ and $G\colon R \to R$ be two additive mappings on R, α and β automorphisms of R and $b \in R$. G is called a b-generalized (α, β)-derivation of R, if $G(xy) = G(x)\alpha(y) + b\beta(x)d(y)$ for any $x, y \in R$.

Let now $D\colon R\times R \to R$ be a biadditive mapping. The biadditive mapping $\Delta\colon R\times R \to R$ is said to be a b-generalized (α, β)-biderivation of R if, for every $x, y, z \in R$, $\Delta(x, yz) = \Delta(x, y)\alpha(z) + b\beta(y)D(x, z)$ and $\Delta(xy, z) = \Delta(x, z)\alpha(y) + b\beta(x)D(y, z)$.

Here we describe the form of any b-generalized (α, β)-biderivation of a prime ring.

1. Introduction

Let R be a prime ring with center $Z(R)$, right Martindale quotient ring Q_r and extended centroid C. An additive mapping $d\colon R \to R$ is said to be a *derivation* of R if

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a *generalized derivation* of R if there exists a derivation d of R such that

$$F(xy) = F(x)y + xd(y)$$

for all $x, y \in R$. The derivation d is uniquely determined by F, which is called an *associated derivation* of F.

In a recent paper [9], Koşan and Lee propose the following new definition. Let $d\colon R \to Q_r$ be an additive mapping and $b \in Q_r$. An additive mapping $F\colon R \to Q_r$ is called a *left b-generalized derivation*, with an associated mapping d, if $F(xy) = F(x)y + bxd(y)$, for all

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$x, y \in R$. In the same paper it is proved that, if R is prime ring, then d is a derivation of R. In the present paper this mapping F will be called a *b-generalized derivation* with an associated pair (b, d). Clearly, any generalized derivation with an associated derivation d is a b-generalized derivation with an associated pair $(1, d)$.

Let α be an automorphism of R. An additive mapping $d\colon R \to R$ is said to be a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. The automorphisms α is called an *associated automorphism* of d. An additive mapping $F\colon R \to R$ is called a *generalized skew derivation* of R if there exists a skew derivation d of R with an associated automorphism α such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$.

Let now α and β be two automorphisms of R. An additive mapping $d\colon R \to R$ is said to be a (α, β)*-derivation* of R if

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y)$$

for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a *generalized* (α, β)*-derivation* of R if there exists an (α, β)-derivation d of R such that

$$F(xy) = F(x)\alpha(y) + \beta(x)d(y)$$

for all $x, y \in R$.

There arises the question of whether there exists a unified definition of b-generalized derivation and generalized (α, β)-derivation. In view of this idea, we now give a definition which is a common generalization of the previous two definitions:

Definition 1.1. Let R be an associative algebra, $b \in Q_r$, d an additive mapping of R and α, β be two automorphisms of R. A linear mapping $F\colon R \to R$ is called a *b-generalized* (α, β)*-derivation* of R, with an associated word (b, α, β, d) if

$$F(xy) = F(x)\alpha(y) + b\beta(x)d(y)$$

holds for all $x, y \in R$.

Let now $D\colon R \times R \to R$ be a biadditive map. D is called a *biderivation* if $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$. In this case we have that $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$.

The concept of a biderivation was introduced in [10] by Maksa. In [3] Brešar, Martindale III and Miers characterized biderivations of noncommutative rings and proved that any biderivation D of a prime ring R has the following form: $D(x, y) = \lambda[x, y]$ for any $x, y \in R$, where λ is a fixed element of C.

Later in [1] Argaç introduced the notion of generalized biderivation. More precisely, let $D\colon R \times R \to R$ be a biderivation. A biadditive mapping $\Delta\colon R \times R \to R$ is said to be a *generalized biderivation* if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of R associated with D as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of R associated with D, i.e., $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$. Argaç also proved that any generalized biderivation D of a prime ring R has the following form: $D(x, y) = \lambda[x, y]$ for any $x, y \in R$, where λ is a fixed element of C.

Let now $D\colon R \times R \to R$ be a biadditive mapping, α an automorphism of R. D is said to be a *skew biderivation* associated with α if for every $x \in R$, the map $y \mapsto D(x, y)$ is a skew derivation of R associated with α as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a skew derivation of R associated with α, i.e., $D(x, yz) = D(x, y)z + \alpha(y)D(x, z)$ and $D(xy, z) = D(x, z)y + \alpha(x)D(y, z)$ for all $x, y, z \in R$. In [2], Brešar determined the form of any skew biderivation of a prime ring R. More precisely, if D is a skew biderivation with an associated automorphism α, then there exists an invertible element q of Q such that $\alpha(x) = qxq^{-1}$ and $D(x, y) = q[x, y]$ for any $x, y \in R$.

More recently, in [6] Fošner described the form of generalized skew biderivations in a prime ring. More precisely, if $D\colon R \times R \to R$ is a skew biderivation of R, associated with the automorphism α of R, then the biadditive mapping $\Delta\colon R \times R \to R$ is said to be a *generalized skew biderivation* associated with α and D, if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized skew derivation of R associated with α and D, as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized skew derivation of R associated with α and D, i.e., $\Delta(x, yz) = \Delta(x, y)z + \alpha(y)D(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + \alpha(x)D(y, z)$ for all $x, y, z \in R$. In [6, Theorem 1] it is proved that if Δ is a generalized skew biderivation with an associated automorphism α, then there exists an invertible element q of Q such that $\alpha(x) = qxq^{-1}$ and $\Delta(x, y) = q[x, y]$ for any $x, y \in R$.

In light of Definition 1.1, here we would like to introduce the following concepts, which generalize the previous cited ones:

Definition 1.2. Let R be an associative algebra, $b \in Q_r$, $D\colon R \times R \to R$ a biadditive mapping of R and α, β be two automorphisms of R. D is said to be an (α, β)*-biderivation* of R if for every $x \in R$, the map $y \mapsto D(x, y)$ is an (α, β)-derivation of R, as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is an (α, β)-derivation of R, i.e.,

(a) $D(x, yz) = D(x, y)\alpha(z) + \beta(y)D(x, z)$ for any $x, y, z \in R$;

(b) $D(xy, z) = D(x, z)\alpha(y) + \beta(x)D(y, z)$ for any $x, y, z \in R$.

Definition 1.3. Let R be an associative algebra, $b \in Q_r$, $D\colon R \times R \to R$ a biadditive mapping of R and α, β be two automorphisms of R. The biadditive mapping $\Delta\colon R \times R \to R$ is said to be a *b-generalized* (α, β)*-biderivation* associated with the word (b, α, β, D) if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a b-generalized (α, β)-derivation of R associated with the word (b, α, β, D), as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a b-generalized (α, β)-derivation of R associated with the word (b, α, β, D), i.e.,

(a) $\Delta(x, yz) = \Delta(x, y)\alpha(z) + b\beta(y)D(x, z)$ for any $x, y, z \in R$;

(b) $\Delta(xy, z) = \Delta(x, z)\alpha(y) + b\beta(x)D(y, z)$ for any $x, y, z \in R$.

Here we will describe the structure of an arbitrary b-generalized (α, β)-biderivation in a prime ring and prove the following:

Theorem 1.4. *Let R be a non-commutative prime ring, $b \in Q_r$, $D\colon R \times R \to R$ a biadditive mapping of R and α, β be two automorphisms of R. If Δ is a non-zero b-generalized (α, β)-biderivation of R, associated with the word (b, α, β, D), then D is an (α, β)-biderivation of R and there exists $q \in Q$ such that $\alpha^{-1}\beta(x) = qxq^{-1}$ for any $x \in R$, and $D(x, y) = \alpha(q)[\alpha(x), \alpha(y)]$, $\Delta(x, y) = b\alpha(q)[\alpha(x), \alpha(y)]$ for all $x, y \in R$.*

2. Characterization of b-generalized (α, β)-derivations

In this section we would like to describe the general form of b-generalized (α, β)-derivations in prime rings.

Lemma 2.1. *Let R be a prime ring, $\alpha, \beta \in \mathrm{Aut}(R)$, $0 \neq b \in Q_r$, $d\colon R \to R$ be an additive mapping of R and F be the b-generalized (α, β)-derivation of R with an associated word (b, α, β, d). Then d is an (α, β)-derivation of R.*

Proof. For any $x, y, z \in R$, we have both

$$F(xyz) = F(xy)\alpha(z) + b\beta(xy)d(z) = F(x)\alpha(y)\alpha(z) + b\beta(x)d(y)\alpha(z) + b\beta(x)\beta(y)d(z)$$

and

$$F(xyz) = F(x)\alpha(y)\alpha(z) + b\beta(x)d(yz).$$

Comparing the above two relations, it follows that

$$0 = b\beta(x)d(y)\alpha(z) + b\beta(x)\beta(y)d(z) - b\beta(x)d(yz).$$

That is,

$$b\beta(R)\big(d(yz) - d(y)\alpha(z) - \beta(y)d(z)\big) = 0.$$

Therefore, by the primeness of R and since $b \neq 0$, we get $d(yz) = d(y)\alpha(z) + \beta(y)d(z)$ for all $y, z \in R$, as required. □

Fact 2.2. Let R be a prime ring, then the following statements hold:

(a) Any automorphism of R can be uniquely extended to Q_r (see [5, Fact 2]).

(b) Every generalized skew derivation of R can be uniquely extended to Q_r (see [4, Lemma 2]).

Proposition 2.3. *Let R be a prime ring, $\alpha,\beta \in \mathrm{Aut}(R)$, $b \in Q_r$, $d\colon R \to R$ be an additive mapping of R and F be the b-(α,β)-derivation of R with an associated word (b,α,β,d). Then F can be uniquely extended to Q_r and assumes the form $F(x) = a\alpha(x) + bd(x)$, where $a \in Q_r$.*

Proof. First we recall that, for any $x \in Q_r$, there exists an ideal I_x of R such that $xI_x \subseteq R$.

In case $b = 0$, then $F(xy) = F(x)\alpha(y)$. Thus F can be extended to Q_r by $F(xy) = F(x)\alpha(y)$ for all $y \in I_x$.

Let us consider the case of $b \neq 0$. Define $T\colon R \to R$ such that $T(x) = F(x) - bd(x)$. Since d is an (α,β)-derivation of R, we have

$$\begin{aligned} T(xy) &= F(x)\alpha(y) + b\beta(x)d(y) - bd(x)\alpha(y) - b\beta(x)d(y) \\ &= \big(F(x) - bd(x)\big)\alpha(y) = T(x)\alpha(y) \end{aligned}$$

for all $x, y \in R$. As above, T can be extended to Q_r by $T(xy) = T(x)\alpha(y)$ for all $y \in I_x$. Since $F(x) = T(x) + bd(x)$ and both T and d can be uniquely extended to Q_r, we know that F can be uniquely extended to Q_r.

Moreover, for any $x \in Q_r$, $F(x) = F(1\cdot x) = F(1)\alpha(x) + b\beta(1)d(x) = a\alpha(x) + bd(x)$, where $a = F(1) \in Q_r$. $\square$

Example 2.4. Let R be an associative algebra, α and β be two automorphisms of R, $a, b, c \in R$. The following mapping

$$G\colon R \to R, \quad x \mapsto a\alpha(x) + b\beta(x)c$$

is a b-generalized (α,β)-derivation of R with an associated word (b,α,β,d), where $d(x) = \beta(x)c - c\alpha(x)$ for all $x \in R$. Indeed, for all $x, y \in R$,

$$\begin{aligned} G(xy) &= a\alpha(x)\alpha(y) + b\beta(x)\beta(y)c \\ &= a\alpha(x)\alpha(y) - b\beta(x)c\alpha(y) + b\beta(x)c\alpha(y) + b\beta(x)\beta(y)c \\ &= \big(a\alpha(x) + b\beta(x)c\big)\alpha(y) + b\beta(x)\big(\beta(y)c - c\alpha(y)\big) \\ &= G(x)\alpha(y) + b\beta(x)d(y), \end{aligned}$$

where $d(y) := \beta(y)c - c\alpha(y)$ is an inner (α,β)-derivation of R induced by the element $c \in R$, with two associated automorphisms α and β. Such b-generalized (α,β)-derivations are called *inner b-generalized (α,β)-derivations.*

Example 2.5. Let R be an associative algebra, $b \in R$, α, β two automorphisms of R and d an (α, β)-derivation of R. Then the following mapping

$$G\colon R \to R, \quad x \mapsto b(\alpha - d)(x)$$

is a b-generalized (α, β)-derivation of R. Indeed, for all $x, y \in R$,

$$\begin{aligned} G(xy) &= b(\alpha - d)(xy) = b\alpha(x)\alpha(y) - bd(xy) \\ &= b\alpha(x)\alpha(y) - bd(x)\alpha(y) - b\beta(x)d(y) \\ &= \big(b\alpha(x) - bd(x)\big)\alpha(y) - b\beta(x)d(y) \\ &= G(x)\alpha(y) - b\beta(x)d(y). \end{aligned}$$

Thus G is a b-generalized (α, β)-derivation of R, with an associated word $(-b, \alpha, \beta, d)$.

Example 2.6. Let R be an associative algebra, $b \in R$, α, β two automorphisms of R and d an (α, β)-derivation of R. Then the following mapping

$$G\colon R \to R, \quad x \mapsto b(\beta - d)(x)$$

is a b-generalized (α, β)-derivation of R. Indeed, for all $x, y \in R$,

$$\begin{aligned} G(xy) &= b(\beta - d)(xy) = b\beta(x)\beta(y) - bd(xy) \\ &= b\beta(x)\beta(y) - bd(x)\alpha(y) - b\beta(x)d(y) \\ &= b\beta(x)\beta(y) - bd(x)\alpha(y) - b\beta(x)d(y) + b\beta(x)\alpha(y) - b\beta(x)\alpha(y) \\ &= \big(b\beta(x) - bd(x)\big)\alpha(y) + b\beta(x)\big(\beta(y) - \alpha(y) - d(y)\big) \\ &= G(x)\alpha(y) - b\beta(x)g(y) \end{aligned}$$

where it is easy to see that $g(y) = \beta(y) - \alpha(y) - d(y)$ is an (α, β)-derivation of R. Thus G is a b-generalized (α, β)-derivation of R, with an associated word (b, α, β, g).

3. b-generalized (α, β)-biderivations of prime rings

We permit the following:

Lemma 3.1. *Let R be a prime ring, $\Delta\colon R \times R \to R$ a non-zero biadditive mapping of R and α be an automorphism of R. Assume that:*

(a) $\Delta(x, yz) = \Delta(x, y)\alpha(z)$ *for any* $x, y, z \in R$*;*

(b) $\Delta(xy, z) = \Delta(x, z)\alpha(y)$ *for any* $x, y, z \in R$*.*

Then R is commutative.

Proof. For any $x, y, z, t \in R$ we have both

$$\Delta(xy, zt) = \Delta(x, zt)\alpha(y) = \Delta(x, z)\alpha(t)\alpha(y) \tag{3.1}$$

and

$$\Delta(xy, zt) = \Delta(xy, z)\alpha(t) = \Delta(x, z)\alpha(y)\alpha(t). \tag{3.2}$$

Comparing (3.1) with (3.2) one has $\Delta(x, z)[\alpha(y), \alpha(t)] = 0$ for all $x, y, z, t \in R$. Replacing y by ry, for any $r \in R$, we get $\Delta(x, z)r[\alpha(y), \alpha(t)] = 0$ for all $x, y, z, t, r \in R$. By the primeness of R and since $\Delta \neq 0$, it follows that $[\alpha(y), \alpha(t)] = 0$ for any $y, t \in R$, that is R is commutative. □

Lemma 3.2. *Let R be a non-commutative prime ring, $b \in Q_r$, $D\colon R\times R \to R$ a biadditive mapping of R and α, β be two automorphisms of R. If Δ is a non-zero b-generalized (α,β)-biderivation of R, associated with the word (b, α, β, D), then D is an (α,β)-biderivation of R.*

Proof. Since R is not commutative and in light of Lemma 3.1, we may assume $b \neq 0$.

Let x, y, z, t be arbitrary elements of R. Then

$$\Delta(x(yt), z) = \Delta(x, z)\alpha(y)\alpha(t) + b\beta(x)D(yt, z). \tag{3.3}$$

On the other hand

$$\begin{aligned} \Delta((xy)t, z) &= \Delta(xy, z)\alpha(t) + b\beta(x)\beta(y)D(t, z) \\ &= \Delta(x, z)\alpha(y)\alpha(t) + b\beta(x)D(y, z)\alpha(t) + b\beta(x)\beta(y)D(t, z). \end{aligned} \tag{3.4}$$

Relations (3.3) and (3.4) imply that

$$b\beta(x)\big(D(yt, z) - D(y, z)\alpha(t) - \beta(y)D(t, z)\big) = 0, \quad \forall\, x, y, z, t \in R. \tag{3.5}$$

By the primeness of R and since $b \neq 0$, relation (3.5) implies that $D(yt, z) = D(y, z)\alpha(t) + \beta(y)D(t, z)$.

By using the same argument one may prove that $D(y, tz) = D(y, t)\alpha(z) + \beta(t)D(y, z)$ for any $y, z, t \in R$, that is D is an (α,β)-biderivation, as required. □

Proposition 3.3. *Let R be a non-commutative prime ring, α, β be two automorphisms of R, $D\colon R \times R \to R$ a non-zero (α,β)-biderivation of R. Then there exists $q \in Q_r$ such that $\alpha^{-1}\beta(x) = qxq^{-1}$ for any $x \in R$, and $D(x, y) = \alpha(q)[\alpha(x), \alpha(y)]$ for all $x, y \in R$.*

Proof. For any $x, y, z, t \in R$ we have both

$$\begin{aligned} D(xy, zt) &= D(x, zt)\alpha(y) + \beta(x)D(y, zt) \\ &= D(x, z)\alpha(t)\alpha(y) + \beta(z)D(x, t)\alpha(y) + \beta(x)D(y, z)\alpha(t) \\ &\quad + \beta(x)\beta(z)D(y, t) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} D(xy, zt) &= D(xy, z)\alpha(t) + \beta(z)D(xy, t) \\ &= D(x, z)\alpha(y)\alpha(t) + \beta(x)D(y, z)\alpha(t) + \beta(z)D(x, t)\alpha(y) \\ &\quad + \beta(z)\beta(x)D(y, t). \end{aligned} \tag{3.7}$$

Comparing (3.6) with (3.7) we have that

$$D(x, z)[\alpha(t), \alpha(y)] + [\beta(x), \beta(z)]D(y, t) = 0, \quad \forall\, x, y, z, t \in R. \tag{3.8}$$

Replacing y by uy in (3.8), it follows that

$$\begin{aligned} &D(x, z)[\alpha(t), \alpha(u)]\alpha(y) + D(x, z)\alpha(u)[\alpha(t), \alpha(y)] \\ &+ [\beta(x), \beta(z)]D(u, t)\alpha(y) + [\beta(x), \beta(z)]\beta(u)D(y, t) = 0, \quad \forall\, x, y, z, t, u \in R. \end{aligned} \tag{3.9}$$

By using (3.8) in (3.9), one has

$$D(x, z)\alpha(u)[\alpha(y), \alpha(t)] - [\beta(x), \beta(z)]\beta(u)D(y, t) = 0, \quad \forall\, x, y, z, t, u \in R. \tag{3.10}$$

We remark that, since R is not commutative and $D \neq 0$, then there exist $x_0, y_0, z_0, t_0 \in R$ such that

$$[\alpha(y_0), \alpha(t_0)] \neq 0 \quad \text{and} \quad D(x_0, z_0) \neq 0.$$

Therefore, by (3.10), $[\beta(x_0), \beta(z_0)]\beta(u)D(y_0, t_0) \neq 0$ for some element $u \in R$, that is both $[\beta(x_0), \beta(z_0)] \neq 0$ and $D(y_0, t_0) \neq 0$.

Now we fix x_0, z_0, y_0, t_0, with $[x_0, z_0] \neq 0$, $[y_0, t_0] \neq 0$, $D(x_0, z_0) \neq 0$ and $D(y_0, t_0) \neq 0$. For simplicity of notation we write $a_1 = D(x_0, z_0) \neq 0$, $a_2 = [\alpha(y_0), \alpha(t_0)] \neq 0$, $a_3 = [\beta(x_0), \beta(z_0)] \neq 0$ and $a_4 = D(y_0, t_0) \neq 0$, so that, by relation (3.10) we get

$$a_1\alpha(u)a_2 - a_3\beta(u)a_4 = 0, \quad \forall\, u \in R \tag{3.11}$$

that is R satisfies the following generalized polynomial identity with automorphisms α and β:

$$a_1\alpha(X)a_2 - a_3\beta(X)a_4. \tag{3.12}$$

Suppose that α and β are mutually outer, that is $\alpha^{-1}\beta$ is not an inner automorphism of Q_r. In this case, by [7, Theorem 4] and relation (3.12), it follows that $a_1Xa_2 - a_3Ya_4$ is a generalized polynomial identity for R, that is $a_1r_1a_2 - a_3r_2a_4 = 0$ for any $r_1, r_2 \in Q_r$. In particular, for $r_1 = 0$ (respectively for $r_2 = 0$) we have $a_3r_2a_4 = 0$ for any $r_2 \in Q_r$ (respectively $a_1r_1a_2 = 0$ for any $r_1 \in Q_r$). Hence, by the primeness of Q_r, either $a_1 = 0$ or $a_2 = 0$ (respectively either $a_3 = 0$ or $a_4 = 0$), which is a contradiction, since a_1, a_2, a_3, a_4 are not zeros.

Hence we may assume that $\alpha^{-1}\beta$ is an inner automorphism of Q_r, that is there exists an invertible element of Q_r such that $\alpha^{-1}\beta(x) = pxp^{-1}$ for any $x \in R$. Now we apply automorphism α^{-1} to relation (3.11):

$$\alpha^{-1}(a_1)u\alpha^{-1}(a_2) - \alpha^{-1}(a_3)pup^{-1}\alpha^{-1}(a_4) = 0, \quad \forall\, u \in R.$$

Since $\alpha^{-1}(a_1) \neq 0$, $\alpha^{-1}(a_2) \neq 0$, $\alpha^{-1}(a_3)p \neq 0$ and $p^{-1}\alpha^{-1}(a_4) \neq 0$ and by using the result in [8, Lemma 1.3.2], it follows that there exists an element $\lambda \in C$, depending on the choice of x_0, z_0, y_0 and t_0, such that $\alpha^{-1}(a_1) = \lambda\alpha^{-1}(a_3)p$ and $p^{-1}\alpha^{-1}(a_4) = \lambda\alpha^{-1}(a_2)$. Hence $\alpha^{-1}(D(x_0, z_0)) = \lambda p[x_0, z_0]$ and $\alpha^{-1}(D(y_0, t_0)) = \lambda p[y_0, t_0]$.

By repeating the same process for y_1, t_1 elements of R such that $[y_1, t_1] \neq 0$ and $D(y_1, t_1) \neq 0$, it follows that there exist $\lambda' \in C$, depending on the choice of x_0, z_0, y_1 and t_1, such that $\alpha^{-1}(D(x_0, z_0)) = \lambda' p[x_0, z_0]$ and $\alpha^{-1}(D(y, t)) = \lambda' p[y, t]$.

Thus $\lambda' p[x_0, z_0] = \lambda p[x_0, z_0]$ and, since $0 \neq p$ is invertible and $[x_0, z_0] \neq 0$, one has $\lambda = \lambda'$. In other words, there exists a unique $\lambda \in C$ such that

$$[x, z] \neq 0 \quad \Longrightarrow \quad \alpha^{-1}(D(x, z)) = \lambda p[x, z]. \tag{3.13}$$

Finally consider two elements $x_1, z_1 \in R$ such that $[x_1, z_1] = 0$. Then, by (3.10),

$$D(x_1, z_1)\alpha(u)[\alpha(y), \alpha(t)] = 0, \quad \forall\, y, t, u \in R. \tag{3.14}$$

By the primeness of R and since R is not commutative, relation (3.14) implies $D(x_1, z_1) = 0$.

Notice that in a similar way one may prove that $D(x_2, z_2) = 0$ implies $[x_2, z_2] = 0$. Hence it is proved that

$$[x, z] = 0 \quad \Longleftrightarrow \quad D(x, z) = 0. \tag{3.15}$$

From (3.13) and (3.15) it follows that there exists $\lambda \in C$ such that

$$\alpha^{-1}(D(x, z)) = \lambda p[x, z], \forall\, x, z \in R. \tag{3.16}$$

Notice that $\alpha^{-1}\beta(x) = pxp^{-1} = (\lambda p)x(\lambda p)^{-1}$. Hence, if we denote $q = \lambda p$ then (3.16) reduces to

$$D(x, z) = \alpha(q)[\alpha(x), \alpha(z)], \forall\, x, z \in R$$

and we are done. □

Proof of Theorem 1.4. For any $x, y, z, t \in R$ we have both

$$\begin{aligned} \Delta(xy, zt) &= \Delta(x, zt)\alpha(y) + b\beta(x)D(y, zt) \\ &= \Delta(x, z)\alpha(t)\alpha(y) + b\beta(z)D(x, t)\alpha(y) + b\beta(x)D(y, z)\alpha(t) \\ &\quad + b\beta(x)\beta(z)D(y, t) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \Delta(xy, zt) &= \Delta(xy, z)\alpha(t) + b\beta(z)D(xy, t) \\ &= \Delta(x, z)\alpha(y)\alpha(t) + b\beta(x)D(y, z)\alpha(t) + b\beta(z)D(x, t)\alpha(y) \\ &\quad + b\beta(z)\beta(x)D(y, t). \end{aligned} \tag{3.18}$$

Comparing (3.17) with (3.18) we have that

$$\Delta(x, z)[\alpha(t), \alpha(y)] + b[\beta(x), \beta(z)]D(y, t) = 0, \quad \forall\, x, y, z, t \in R. \tag{3.19}$$

In light of Proposition 3.3, there exists $q \in Q_r$ such that $\alpha^{-1}\beta(x) = qxq^{-1}$ and $D(y, t) = \alpha(q)[\alpha(y), \alpha(t)]$. Thus we may write relation (3.19) as follows:

$$\Delta(x, z)[\alpha(t), \alpha(y)] + b[\beta(x), \beta(z)]\alpha(q)[\alpha(y), \alpha(t)] = 0, \quad \forall\, x, y, z, t \in R,$$

that is

$$\big(\Delta(x, z) - b[\beta(x), \beta(z)]\alpha(q)\big)[\alpha(t), \alpha(y)] = 0, \quad \forall\, x, y, z, t \in R.$$

Replacing t by $t't$, for any $t' \in R$, we have

$$\begin{aligned} 0 &= \big(\Delta(x, z) - b[\beta(x), \beta(z)]\alpha(q)\big)[\alpha(t')\alpha(t), \alpha(y)] \\ &= \big(\Delta(x, z) - b[\beta(x), \beta(z)]\alpha(q)\big)\alpha(t')[\alpha(t), \alpha(y)], \quad \forall\, x, y, z, t, t' \in R, \end{aligned}$$

that is

$$\big(\Delta(x, z) - b[\beta(x), \beta(z)]\alpha(q)\big)R[\alpha(t), \alpha(y)] = 0, \quad \forall\, x, y, z, t \in R.$$

By the primeness of R and since R is not commutative, it follows that $\Delta(x, z) = b[\beta(x), \beta(z)]\alpha(q)$. Finally, since $\alpha^{-1}\beta(x) = qxq^{-1}$ implies $\beta(x)\alpha(q) = \alpha(q)\alpha(x)$ for all $x \in R$, then $\Delta(x, z) = b\alpha(q)[\alpha(x), \alpha(z)]$, as required. □

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