Stuffle Product Formulas of Multiple Zeta Values

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Abstract. Using the combinatorial descriptions of stuffle product, we obtain recursive formulas for the stuffle product of multiple zeta values and of multiple zeta-star values. Then we apply the formulas to prove several stuffle product formulas with one or two strings of z_p's. We also describe how to use our formulas in general cases.

1. Introduction and statement of main results

For integers $k_1 \geq 2$ and $k_2, \ldots, k_n \geq 1$, the multiple zeta value and the multiple zeta-star value are defined by

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

$$\zeta^\star(k_1, k_2, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

respectively. In case of $n = 1$, we get Riemann zeta values in both cases, which are special values of the Riemann zeta function at positive integer arguments.

Using the definitions we can express a product of two multiple zeta values (multiple zeta-star values) as a sum of multiple zeta values (multiple zeta-star values). For example, for multiple zeta values we have

$$\zeta(2)\zeta(3) = \sum_{m>0} \frac{1}{m^2} \times \sum_{n>0} \frac{1}{n^3} = \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0} \right) \frac{1}{m^2 n^3}$$
$$= \zeta(2,3) + \zeta(3,2) + \zeta(5),$$

and for multiple zeta-star values, we have

$$\zeta^\star(2)\zeta^\star(3) = \sum_{m\geq 1} \frac{1}{m^2} \times \sum_{n\geq 1} \frac{1}{n^3} = \left(\sum_{m\geq n\geq 1} + \sum_{n\geq m\geq 1} - \sum_{m=n>0} \right) \frac{1}{m^2 n^3}$$
$$= \zeta^\star(2,3) + \zeta^\star(3,2) - \zeta^\star(5).$$

Such products are called stuffle products (or harmonic shuffle products).

The algebraic settings are given in [3] for multiple zeta values, and in [6] for multiple zeta-star values. Since we only treat the stuffle product here, we give the following algebraic descriptions. Let $Y = \{z_k \mid k = 1, 2, \ldots\}$ be an alphabet with noncommutative letters, and let Y^* be the set of all words generated by letters in Y, which contains the empty word 1. Let $\mathfrak{h}^1 = \mathbb{Q}\langle Y \rangle$ be the noncommutative polynomial algebra over the rational number field $\mathbb{Q}$ generated by the alphabet Y, which is spanned by Y^* as a rational vector space. And let $\mathfrak{h}^0$ be the subalgebra of $\mathfrak{h}^1$ defined by

$$\mathfrak{h}^0 = \mathbb{Q} + \sum_{k=2}^{\infty} z_k \mathfrak{h}^1.$$

Corresponding to stuffle products of multiple zeta values and of multiple zeta-star values, we define two bilinear commutative products $*$ and $\bar{*}$ on $\mathfrak{h}^1$ by the rules

(1) $1 * w = w * 1 = w,$

(2) $z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2);$

(3) $1 \bar{*} w = w \bar{*} 1 = w,$

(4) $z_k w_1 \bar{*} z_l w_2 = z_k(w_1 \bar{*} z_l w_2) + z_l(z_k w_1 \bar{*} w_2) - z_{k+l}(w_1 \bar{*} w_2),$

where $w, w_1, w_2 \in Y^*$ and k, l are positive integers. Under each of these two products, $\mathfrak{h}^1$ becomes a commutative algebra, and $\mathfrak{h}^0$ is still a subalgebra of $\mathfrak{h}^1$.

To relate the algebra $\mathfrak{h}^0$ to multiple zeta values and to multiple zeta-star values, we define rational linear maps $Z\colon \mathfrak{h}^0 \to \mathbb{R}$ and $Z^\star\colon \mathfrak{h}^0 \to \mathbb{R}$ by $Z(1) = Z^\star(1) = 1$ and

$$Z(z_{k_1} z_{k_2} \cdots z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n), \quad Z^\star(z_{k_1} z_{k_2} \cdots z_{k_n}) = \zeta^\star(k_1, k_2, \ldots, k_n),$$

where $n, k_1, k_2, \ldots, k_n$ are positive integers with $k_1 > 1$, and $\mathbb{R}$ is the field of real numbers. Then it was shown in [3] and in [6] that $Z\colon (\mathfrak{h}^0, *) \to \mathbb{R}$ and $Z^\star\colon (\mathfrak{h}^0, \bar{*}) \to \mathbb{R}$ are algebra homomorphisms, respectively. More precisely, for any $w_1, w_2 \in \mathfrak{h}^0$, we have

$$Z(w_1 * w_2) = Z(w_1)Z(w_2),$$
$$Z^\star(w_1 \bar{*} w_2) = Z^\star(w_1)Z^\star(w_2).$$

Hence to get stuffle product formulas, we can work on $\mathfrak{h}^0$ first. For example, since

$$z_2 * z_3 = z_2 z_3 + z_3 z_2 + z_5 \quad \text{and} \quad z_2 \bar{*} z_3 = z_2 z_3 + z_3 z_2 - z_5,$$

applying the map Z and $Z^\star$ respectively, we get

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5), \quad \zeta^\star(2)\zeta^\star(3) = \zeta^\star(2,3) + \zeta^\star(3,2) - \zeta(5)$$

as proved above.

There is another commutative product ⧢ among multiple zeta values, which is deduced from the iterated integral representation of multiple zeta values and is called the shuffle product (see [3, 7] for details). During the study of multiple zeta values, several shuffle product formulas were found. For example, as generalizations of Euler's decomposition formula, Eie and Wei essentially studied the shuffle product $z_k z_1^m \shuffle z_l z_1^n$ in [2], Lei, Guo and Ma obtained the formula for shuffle product $z_k z_1^m \shuffle z_{l_1} z_1^{n_1} z_{l_2} z_1^{n_2}$ in [4], and we studied the shuffle products of words with more strings of z_1's in [5]. For the stuffle product, Chen obtained a formula for the product $z_k z_p^m * z_l z_p^n$ in [1]. Chen expressed this product by simple products of the form $z_p^i * z_p^j$. We find that one can treat general products with more strings of z_p's such as

$$(1.1) \qquad z_{k_1} z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} \cdots z_{l_s} z_p^{n_s}$$

and

$$(1.2) \qquad z_{k_1} z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} \, \overline{\ast} \, z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} \cdots z_{l_s} z_p^{n_s}.$$

In fact, the general products can be expressed by simple products of the forms $z_p^i * z_p^j$ and $z_p^i \overline{\ast} z_p^j$, respectively. For the simple products, we have the following result.

Theorem 1.1. *For a positive integer p and two nonnegative integers a, b, we denote by G_b^a the set of all words in Y^* which contains z_p a times and z_{2p} b times. Then for any nonnegative integers m, n, we have*

$$(1.3) \qquad z_p^m * z_p^n = \sum_{i=0}^{\min\{m,n\}} \binom{m+n-2i}{m-i} \sum_{w \in G_i^{m+n-2i}} w$$

and

$$(1.4) \qquad z_p^m \, \overline{\ast} \, z_p^n = \sum_{i=0}^{\min\{m,n\}} (-1)^i \binom{m+n-2i}{m-i} \sum_{w \in G_i^{m+n-2i}} w.$$

Note that formula (1.3) is just [1, Lemma 1]. To study the general stuffle products mentioned above, we need the following recursive formulas, which are similar to [4, Lemma 2.1] for shuffle products.

Theorem 1.2. *Let $m, n, k_1, \ldots, k_m, l_1, \ldots, l_n$ be positive integers. Then for any integer j with $1 \le j \le m$, we have*

$$
\begin{aligned}
& z_{k_1} z_{k_2} \cdots z_{k_m} * z_{l_1} z_{l_2} \cdots z_{l_n} \\
(1.5) \quad &= \sum_{i=0}^{n} [(z_{k_1} z_{k_2} \cdots z_{k_{j-1}} * z_{l_1} z_{l_2} \cdots z_{l_i}) z_{k_j} + (z_{k_1} z_{k_2} \cdots z_{k_{j-1}} * z_{l_1} z_{l_2} \cdots z_{l_{i-1}}) z_{k_j + l_i}] \\
& \qquad \times (z_{k_{j+1}} \cdots z_{k_m} * z_{l_{i+1}} \cdots z_{l_n})
\end{aligned}
$$

and

$$(1.6) \quad \begin{aligned} &z_{k_1} z_{k_2} \cdots z_{k_m} \overline{\ast} z_{l_1} z_{l_2} \cdots z_{l_n} \\ &= \sum_{i=0}^{n} [(z_{k_1} z_{k_2} \cdots z_{k_{j-1}} \overline{\ast} z_{l_1} z_{l_2} \cdots z_{l_i}) z_{k_j} - (z_{k_1} z_{k_2} \cdots z_{k_{j-1}} \overline{\ast} z_{l_1} z_{l_2} \cdots z_{l_{i-1}}) z_{k_j + l_i}] \\ &\quad \times (z_{k_{j+1}} \cdots z_{k_m} \overline{\ast} z_{l_{i+1}} \cdots z_{l_n}). \end{aligned}$$

Here by convention, we let $z_{k_i} \cdots z_{k_{i-1}} = 1$ and $z_{k_i} \cdots z_{k_{i-2}} = 0$ for any integer i.

Applying the recursive formulas to $z_k z_p^m \ast z_l z_p^n$ and $z_k z_p^m \overline{\ast} z_l z_p^n$, we get the following stuffle product formulas.

Corollary 1.3. *Let k, l, p be positive integers and let m, n be nonnegative integers. Then we have*

$$(1.7) \quad \begin{aligned} z_k z_p^m \ast z_l z_p^n &= \sum_{i=1}^{m} z_k (z_p^i z_l + z_p^{i-1} z_{p+l})(z_p^{m-i} \ast z_p^n) \\ &\quad + \sum_{i=1}^{n} z_l (z_p^i z_k + z_p^{i-1} z_{p+k})(z_p^{n-i} \ast z_p^m) + (z_k z_l + z_l z_k + z_{k+l})(z_p^m \ast z_p^n) \end{aligned}$$

and

$$(1.8) \quad \begin{aligned} z_k z_p^m \overline{\ast} z_l z_p^n &= \sum_{i=1}^{m} z_k (z_p^i z_l - z_p^{i-1} z_{p+l})(z_p^{m-i} \overline{\ast} z_p^n) \\ &\quad + \sum_{i=1}^{n} z_l (z_p^i z_k - z_p^{i-1} z_{p+k})(z_p^{n-i} \overline{\ast} z_p^m) + (z_k z_l + z_l z_k - z_{k+l})(z_p^m \overline{\ast} z_p^n). \end{aligned}$$

Note that formula (1.7) is just [1, Theorem 3]. Applying the recursive formulas to $z_k z_p^m \ast z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$ and $z_k z_p^m \overline{\ast} z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$, we get the following stuffle product formulas.

Corollary 1.4. *Let k, l_1, l_2, p be positive integers and let m, n_1, n_2 be nonnegative integers. Then we have*

$$(1.9) \quad \begin{aligned} &z_k z_p^m \ast z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} \\ &= \sum_{0 \leq i_1 \leq i_2 \leq m} z_k (z_p^{i_1} z_{l_1} + z_p^{i_1-1} z_{p+l_1})[(z_p^{i_2-i_1} \ast z_p^{n_1}) z_{l_2} + (z_p^{i_2-i_1-1} \ast z_p^{n_1}) z_{p+l_2}](z_p^{m-i_2} \ast z_p^{n_2}) \\ &\quad + \sum_{\substack{0 \leq i_1 \leq n_1 \\ 0 \leq i_2 \leq m}} z_{l_1} (z_p^{i_1} z_k + z_p^{i_1-1} z_{p+k})[(z_p^{i_2} \ast z_p^{n_1-i_1}) z_{l_2} + (z_p^{i_2-1} \ast z_p^{n_1-i_1}) z_{p+l_2}](z_p^{m-i_2} \ast z_p^{n_2}) \\ &\quad + z_{l_1} z_p^{n_1} z_{k+l_2}(z_p^m \ast z_p^{n_2}) + \sum_{0 \leq i \leq n_2} z_{l_1} z_p^{n_1} z_{l_2}(z_p^i z_k + z_p^{i-1} z_{p+k})(z_p^m \ast z_p^{n_2-i}) \\ &\quad + \sum_{0 \leq i \leq m} z_{k+l_1}[(z_p^i \ast z_p^{n_1}) z_{l_2} + (z_p^{i-1} \ast z_p^{n_1}) z_{p+l_2}](z_p^{m-i} \ast z_p^{n_2}) \end{aligned}$$

and

$$z_k z_p^m \bar{*} z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$$

$$(1.10)$$

$$= \sum_{0 \le i_1 \le i_2 \le m} z_k(z_p^{i_1} z_{l_1} - z_p^{i_1-1} z_{p+l_1})[(z_p^{i_2-i_1} \bar{*} z_p^{n_1})z_{l_2} - (z_p^{i_2-i_1-1} \bar{*} z_p^{n_1})z_{p+l_2}](z_p^{m-i_2} \bar{*} z_p^{n_2})$$

$$+ \sum_{\substack{0 \le i_1 \le n_1 \\ 0 \le i_2 \le m}} z_{l_1}(z_p^{i_1} z_k - z_p^{i_1-1} z_{p+k})[(z_p^{i_2} \bar{*} z_p^{n_1-i_1})z_{l_2} - (z_p^{i_2-1} \bar{*} z_p^{n_1-i_1})z_{p+l_2}](z_p^{m-i_2} \bar{*} z_p^{n_2})$$

$$- z_{l_1} z_p^{n_1} z_{k+l_2}(z_p^m \bar{*} z_p^{n_2}) + \sum_{0 \le i \le n_2} z_{l_1} z_p^{n_1} z_{l_2}(z_p^i z_k - z_p^{i-1} z_{p+k})(z_p^m \bar{*} z_p^{n_2-i})$$

$$- \sum_{0 \le i \le m} z_{k+l_1}[(z_p^i \bar{*} z_p^{n_1})z_{l_2} - (z_p^{i-1} \bar{*} z_p^{n_1})z_{p+l_2}](z_p^{m-i} \bar{*} z_p^{n_2}).$$

Here we set $z_p^{-1} = 0$.

Similarly, one can use the recursive formula (1.5) to the general product (1.1) and formula (1.6) to the general product (1.2). We find that all these type products can be expressed by simple products of the forms $z_p^i * z_p^j$ for the $*$ product, and by simple products of the forms $z_p^i \bar{*} z_p^j$ for the $\bar{*}$ product. However the complexity of the formula increases quickly when $m + n$ is larger. Hence we do not write the explicit formulas here.

After getting the formula of the product (1.1), one can apply the map Z to get a formula of the stuffle product

$$\zeta(k_1, \{p\}^{m_1}, \dots, k_r, \{p\}^{m_r})\zeta(l_1, \{p\}^{n_1}, \dots, l_s, \{p\}^{n_s}),$$

where $k_1, l_1 \ge 2$ and $\{p\}^m$ stands for a string of m p's. Similarly, from the formula of the product (1.2), one can obtain a corresponding formula of the stuffle product of multiple zeta-star values. For example, from Theorem 1.1, Corollaries 1.3 and 1.4, we immediately get the following corollary. Note that (1.11) is just [1, Theorem 1].

Corollary 1.5. *Let p be a positive integer. For nonnegative integers a, b, let A_b^a be the set of all strings containing a times p and b times $2p$. Then for any positive integers k, l, l_1, l_2 with $k, l, l_1 \ge 2$ and any nonnegative integers m, n, n_1, n_2, we have*

$$\zeta(k, \{p\}^m)\zeta(l, \{p\}^n)$$

$$(1.11)$$

$$= \sum_{\substack{1 \le i \le m \\ 0 \le j \le \min\{m-i,n\} \\ \mathbf{k} \in A_j^{m+n-i-2j}}} \binom{m+n-i-2j}{n-j} \left[\zeta(k, \{p\}^i, l, \mathbf{k}) + \zeta(k, \{p\}^{i-1}, p+l, \mathbf{k})\right]$$

$$+ \sum_{\substack{1 \le i \le n \\ 0 \le j \le \min\{m,n-i\} \\ \mathbf{k} \in A_j^{m+n-i-2j}}} \binom{m+n-i-2j}{m-j} \left[\zeta(l, \{p\}^i, k, \mathbf{k}) + \zeta(l, \{p\}^{i-1}, p+k, \mathbf{k})\right]$$

$$
+ \sum_{\substack{0 \le j \le \min\{m,n\} \\ \mathbf{k} \in A_j^{m+n-2j}}} \binom{m+n-2j}{m-j} \left[\zeta(k,l,\mathbf{k}) + \zeta(l,k,\mathbf{k}) + \zeta(k+l,\mathbf{k}) \right],
$$

$$
\zeta^\star(k,\{p\}^m)\zeta^\star(l,\{p\}^n)
$$
$$
= \sum_{\substack{1 \le i \le m \\ 0 \le j \le \min\{m-i,n\} \\ \mathbf{k} \in A_j^{m+n-i-2j}}} (-1)^j \binom{m+n-i-2j}{n-j} \left[\zeta^\star(k,\{p\}^i,l,\mathbf{k}) - \zeta^\star(k,\{p\}^{i-1},p+l,\mathbf{k}) \right]
$$
$$
+ \sum_{\substack{1 \le i \le n \\ 0 \le j \le \min\{m,n-i\} \\ \mathbf{k} \in A_j^{m+n-i-2j}}} (-1)^j \binom{m+n-i-2j}{m-j} \left[\zeta^\star(l,\{p\}^i,k,\mathbf{k}) - \zeta^\star(l,\{p\}^{i-1},p+k,\mathbf{k}) \right]
$$
$$
+ \sum_{\substack{0 \le j \le \min\{m,n\} \\ \mathbf{k} \in A_j^{m+n-2j}}} (-1)^j \binom{m+n-2j}{m-j} \left[\zeta^\star(k,l,\mathbf{k}) + \zeta^\star(l,k,\mathbf{k}) - \zeta^\star(k+l,\mathbf{k}) \right]
$$

and

$$
\zeta(k,\{p\}^m)\zeta(l_1,\{p\}^{n_1},l_2,\{p\}^{n_2})
$$
$$
= \sum_{\substack{0 \le i_1 \le i_2 \le m \\ 0 \le j_1 \le \min\{i_2-i_1,n_1\} \\ 0 \le j_2 \le \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} \binom{n_1+i_2-i_1-2j_1}{n_1-j_1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}
$$
$$
\times \left[\zeta(k,\{p\}^{i_1},l_1,\mathbf{k},l_2,\mathbf{l}) + \zeta(k,\{p\}^{i_1-1},p+l_1,\mathbf{k},l_2,\mathbf{l}) \right]
$$
$$
+ \sum_{\substack{0 \le i_1 \le i_2 \le m \\ 0 \le j_1 \le \min\{i_2-i_1-1,n_1\} \\ 0 \le j_2 \le \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1-1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} \binom{n_1+i_2-i_1-2j_1-1}{n_1-j_1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}
$$
$$
\times \left[\zeta(k,\{p\}^{i_1},l_1,\mathbf{k},p+l_2,\mathbf{l}) + \zeta(k,\{p\}^{i_1-1},p+l_1,\mathbf{k},p+l_2,\mathbf{l}) \right]
$$
$$
+ \sum_{\substack{0 \le i_1 \le n_1, \, 0 \le i_2 \le m \\ 0 \le j_1 \le \min\{i_2,n_1-i_1\} \\ 0 \le j_2 \le \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} \binom{n_1+i_2-i_1-2j_1}{i_2-j_1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}
$$
$$
\times \left[\zeta(l_1,\{p\}^{i_1},k,\mathbf{k},l_2,\mathbf{l}) + \zeta(l_1,\{p\}^{i_1-1},p+k,\mathbf{k},l_2,\mathbf{l}) \right]
$$

$$+ \sum_{\substack{0 \leq i_1 \leq n_1,\, 0 \leq i_2 \leq m \\ 0 \leq j_1 \leq \min\{i_2-1,n_1-i_1\} \\ 0 \leq j_2 \leq \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1-1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} \binom{n_1+i_2-i_1-2j_1-1}{i_2-j_1-1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}$$

$$\times \left[\zeta(l_1, \{p\}^{i_1}, k, \mathbf{k}, p+l_2, \mathbf{l}) + \zeta(l_1, \{p\}^{i_1-1}, p+k, \mathbf{k}, p+l_2, \mathbf{l}) \right]$$

$$+ \sum_{\substack{0 \leq j \leq \min\{m,n_2\} \\ \mathbf{k} \in A_j^{m+n_2-2j}}} \binom{m+n_2-2j}{m-j} \zeta(l_1, \{p\}^{n_1}, k+l_2, \mathbf{k})$$

$$+ \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq \min\{m,n_2-i\} \\ \mathbf{k} \in A_j^{m+n_2-i-2j}}} \binom{m+n_2-i-2j}{m-j}$$

$$\times \left[\zeta(l_1, \{p\}^{n_1}, l_2, \{p\}^i, k, \mathbf{k}) + \zeta(l_1, \{p\}^{n_1}, l_2, \{p\}^{i-1}, p+k, \mathbf{k}) \right]$$

$$+ \sum_{\substack{0 \leq i \leq m \\ 0 \leq j_1 \leq \min\{i,n_1\} \\ 0 \leq j_2 \leq \min\{m-i,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i-2j_1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i-2j_2}}} \binom{n_1+i-2j_1}{n_1-j_1} \binom{m+n_2-i-2j_2}{n_2-j_2} \zeta(k+l_1, \mathbf{k}, l_2, \mathbf{l})$$

$$+ \sum_{\substack{0 \leq i \leq m \\ 0 \leq j_1 \leq \min\{i-1,n_1\} \\ 0 \leq j_2 \leq \min\{m-i,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i-2j_1-1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i-2j_2}}} \binom{n_1+i-2j_1-1}{n_1-j_1} \binom{m+n_2-i-2j_2}{n_2-j_2} \zeta(k+l_1, \mathbf{k}, p+l_2, \mathbf{l}),$$

$$\zeta^\star(k, \{p\}^m) \zeta^\star(l_1, \{p\}^{n_1}, l_2, \{p\}^{n_2})$$

$$= \sum_{\substack{0 \leq i_1 \leq i_2 \leq m \\ 0 \leq j_1 \leq \min\{i_2-i_1,n_1\} \\ 0 \leq j_2 \leq \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} (-1)^{j_1+j_2} \binom{n_1+i_2-i_1-2j_1}{n_1-j_1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}$$

$$\times \left[\zeta^\star(k, \{p\}^{i_1}, l_1, \mathbf{k}, l_2, \mathbf{l}) - \zeta^\star(k, \{p\}^{i_1-1}, p+l_1, \mathbf{k}, l_2, \mathbf{l}) \right]$$

$$- \sum_{\substack{0 \leq i_1 \leq i_2 \leq m \\ 0 \leq j_1 \leq \min\{i_2-i_1-1,n_1\} \\ 0 \leq j_2 \leq \min\{m-i_2,n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1+i_2-i_1-2j_1-1} \\ \mathbf{l} \in A_{j_2}^{m+n_2-i_2-2j_2}}} (-1)^{j_1+j_2} \binom{n_1+i_2-i_1-2j_1-1}{n_1-j_1} \binom{m+n_2-i_2-2j_2}{n_2-j_2}$$

$$\times \left[\zeta^\star(k, \{p\}^{i_1}, l_1, \mathbf{k}, p+l_2, \mathbf{l}) - \zeta^\star(k, \{p\}^{i_1-1}, p+l_1, \mathbf{k}, p+l_2, \mathbf{l}) \right]$$

$$
+ \sum_{\substack{0 \le i_1 \le n_1,\, 0 \le i_2 \le m \\ 0 \le j_1 \le \min\{i_2, n_1 - i_1\} \\ 0 \le j_2 \le \min\{m - i_2, n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1 + i_2 - i_1 - 2j_1} \\ \mathbf{l} \in A_{j_2}^{m + n_2 - i_2 - 2j_2}}} (-1)^{j_1 + j_2} \binom{n_1 + i_2 - i_1 - 2j_1}{i_2 - j_1} \binom{m + n_2 - i_2 - 2j_2}{n_2 - j_2}
$$

$$
\times \left[\zeta^\star(l_1, \{p\}^{i_1}, k, \mathbf{k}, l_2, \mathbf{l}) - \zeta^\star(l_1, \{p\}^{i_1 - 1}, p + k, \mathbf{k}, l_2, \mathbf{l}) \right]
$$

$$
- \sum_{\substack{0 \le i_1 \le n_1,\, 0 \le i_2 \le m \\ 0 \le j_1 \le \min\{i_2 - 1, n_1 - i_1\} \\ 0 \le j_2 \le \min\{m - i_2, n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1 + i_2 - i_1 - 2j_1 - 1} \\ \mathbf{l} \in A_{j_2}^{m + n_2 - i_2 - 2j_2}}} (-1)^{j_1 + j_2} \binom{n_1 + i_2 - i_1 - 2j_1 - 1}{i_2 - j_1 - 1} \binom{m + n_2 - i_2 - 2j_2}{n_2 - j_2}
$$

$$
\times \left[\zeta^\star(l_1, \{p\}^{i_1}, k, \mathbf{k}, p + l_2, \mathbf{l}) - \zeta^\star(l_1, \{p\}^{i_1 - 1}, p + k, \mathbf{k}, p + l_2, \mathbf{l}) \right]
$$

$$
- \sum_{\substack{0 \le j \le \min\{m, n_2\} \\ \mathbf{k} \in A_j^{m + n_2 - 2j}}} (-1)^j \binom{m + n_2 - 2j}{m - j} \zeta^\star(l_1, \{p\}^{n_1}, k + l_2, \mathbf{k})
$$

$$
+ \sum_{\substack{0 \le i \le n_2 \\ 0 \le j \le \min\{m, n_2 - i\} \\ \mathbf{k} \in A_j^{m + n_2 - i - 2j}}} (-1)^j \binom{m + n_2 - i - 2j}{m - j}
$$

$$
\times \left[\zeta^\star(l_1, \{p\}^{n_1}, l_2, \{p\}^i, k, \mathbf{k}) - \zeta^\star(l_1, \{p\}^{n_1}, l_2, \{p\}^{i-1}, p + k, \mathbf{k}) \right]
$$

$$
- \sum_{\substack{0 \le i \le m \\ 0 \le j_1 \le \min\{i, n_1\} \\ 0 \le j_2 \le \min\{m - i, n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1 + i - 2j_1} \\ \mathbf{l} \in A_{j_2}^{m + n_2 - i - 2j_2}}} (-1)^{j_1 + j_2} \binom{n_1 + i - 2j_1}{n_1 - j_1} \binom{m + n_2 - i - 2j_2}{n_2 - j_2} \zeta^\star(k + l_1, \mathbf{k}, l_2, \mathbf{l})
$$

$$
+ \sum_{\substack{0 \le i \le m \\ 0 \le j_1 \le \min\{i - 1, n_1\} \\ 0 \le j_2 \le \min\{m - i, n_2\} \\ \mathbf{k} \in A_{j_1}^{n_1 + i - 2j_1 - 1} \\ \mathbf{l} \in A_{j_2}^{m + n_2 - i - 2j_2}}} (-1)^{j_1 + j_2} \binom{n_1 + i - 2j_1 - 1}{n_1 - j_1} \binom{m + n_2 - i - 2j_2}{n_2 - j_2} \zeta^\star(k + l_1, \mathbf{k}, p + l_2, \mathbf{l}).
$$

Also, if we combine the stuffle product formulas obtained here together with shuffle product formulas obtained in [2, 5], we can get some double shuffle relations of multiple zeta values for some special types. Here we leave the explicit formulas to the reader.

Section 2 contains proofs of Theorems 1.1, 1.2, Corollaries 1.3 and 1.4. In Section 3, we describe how to apply our recursive formulas to express the products (1.1) and (1.2) by simple products, and take the cases $r = s = 2$ as an example.

2. Proofs

In this section, we give proofs of the results mentioned in Section 1.

2.1. Proof of Theorem 1.1

For the simple product $z_p^m * z_p^n$, Chen used induction on $m+n$ to prove formula (1.3) in [1]. Here we give another proof, which seems simple and intuitive. We use the combinatorial description of stuffle product $*$. By the definition of stuffle product $*$, we have

$$z_{k_1} \cdots z_{k_n} * z_{k_{n+1}} \cdots z_{k_{n+m}} = \sum_{i=0}^{\min\{n,m\}} \sum_{\sigma \in \mathfrak{S}_{n,m,i}} z_{\sigma^{-1}(1)} z_{\sigma^{-1}(2)} \cdots z_{\sigma^{-1}(n+m-i)},$$

where $\mathfrak{S}_{n,m,i}$ is the set of all surjections $\sigma \colon \{1,2,\ldots,n+m\} \twoheadrightarrow \{1,2,\ldots,n+m-i\}$ with conditions

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \quad \sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m),$$

and for any $\sigma \in \mathfrak{S}_{n,m,i}$ and any $j \in \{1,2,\ldots,n+m-i\}$,

$$z_{\sigma^{-1}(j)} = \begin{cases} z_k & \text{if } \sigma^{-1}(j) = \{k\}, \\ z_{k+l} & \text{if } \sigma^{-1}(j) = \{k,l\}. \end{cases}$$

Now if we write

$$z_p^m * z_p^n = \sum_{w \in Y^*} c_w w,$$

then we only need to determine the coefficients c_w. By the combinatorial description of the product $*$ described above, it is easy to see that if $c_w \neq 0$, then all the letters in the word w must be z_p or z_{2p}, and the number of z_{2p}'s is less than or equal to $\min\{m,n\}$. Hence we can write

$$z_p^m * z_p^n = \sum_{i=0}^{\min\{m,n\}} \sum_{w \in G_i^{m+n-2i}} c_i^w w.$$

For any $w \in G_i^{m+n-2i}$, we look for the possibility to get w. There are $m+n-2i$ times z_p in w, in which $m-i$ times z_p are from z_p^m and others are from z_p^n. Hence the possibility is $\binom{m+n-2i}{m-i}$. In other words, for any $w \in G_i^{m+n-2i}$, we have

$$c_i^w = \binom{m+n-2i}{m-i}.$$

Then we get (1.3).

Similarly, to prove formula (1.4) for the simple product $z_p^m \overline{\ast} z_p^n$, we use the combinatorial description of stuffle product $\overline{\ast}$. We have

$$z_{k_1} \cdots z_{k_n} \overline{\ast} z_{k_n+1} \cdots z_{k_{n+m}} = \sum_{i=0}^{\min\{n,m\}} \sum_{\sigma \in \mathfrak{S}_{n,m,i}} z_{\sigma^{-1}(1)}^\star z_{\sigma^{-1}(2)}^\star \cdots z_{\sigma^{-1}(n+m-i)}^\star,$$

where for any $\sigma \in \mathfrak{S}_{n,m,i}$ and any $j \in \{1, 2, \ldots, n+m-i\}$,

$$z_{\sigma^{-1}(j)}^\star = \begin{cases} z_k & \text{if } \sigma^{-1}(j) = \{k\}, \\ -z_{k+l} & \text{if } \sigma^{-1}(j) = \{k, l\}. \end{cases}$$

By the combinatorial description, we can write

$$z_p^m \overline{\ast} z_p^n = \sum_{i=0}^{\min\{m,n\}} \sum_{w \in G_i^{m+n-2i}} d_i^w w.$$

In this case, for any $w \in G_i^{m+n-2i}$, the possibility to get w is also $\binom{m+n-2i}{m-i}$. While every letter z_{2p} in w will bring a negative sign. Hence for any $w \in G_i^{m+n-2i}$, we have

$$d_i^w = (-1)^i \binom{m+n-2i}{m-i}.$$

Then we get (1.4).

For more details, see [5], which uses the same idea to deal with shuffle products.

2.2. Proof of Theorem 1.2

We only prove the recursive formula (1.5). The recursive formula (1.6) can be proven in a similar way. To get the result, one can use induction on $m + n$, while here we use the combinatorial description of the stuffle product.

The left-hand side of (1.5) is a sum over surjections

$$\sigma \colon \{1, 2, \ldots, m+n\} \twoheadrightarrow \{1, 2, \ldots, m+n-p\},$$

which are strictly increasing on $\{1, \ldots, m\}$ and on $\{m+1, \ldots, m+n\}$ separately, and p is an integer with the condition $0 \le p \le \min\{m, n\}$. Now for a fixed $j \in \{1, 2, \ldots, m\}$, there are two different types of such σ's, one satisfies the condition $\sigma(i) < \sigma(j) < \sigma(i+1)$ for some $i \in \{m, m+1, \ldots, m+n\}$, and the other one satisfies the condition $\sigma(j) = \sigma(i)$ for some $i \in \{m+1, m+2, \ldots, m+n\}$. Summing over the first type of σ's we get the term

$$\sum_{i=0}^{n} (z_{k_1} z_{k_2} \cdots z_{k_{j-1}} \ast z_{l_1} z_{l_2} \cdots z_{l_i}) z_{k_j} (z_{k_{j+1}} \cdots z_{k_m} \ast z_{l_{i+1}} \cdots z_{l_n}).$$

And summing over the second type of σ's we obtain the term

$$\sum_{i=1}^{n}(z_{k_1}z_{k_2}\cdots z_{k_{j-1}}*z_{l_1}z_{l_2}\cdots z_{l_{i-1}})z_{k_j+l_i}(z_{k_{j+1}}\cdots z_{k_m}*z_{l_{i+1}}\cdots z_{l_n}).$$

Then we get the recursive formula (1.5).

2.3. Proof of Corollary 1.3

Here we prove formula (1.7), and one can prove formula (1.8) in a similar way. We remark that the strategy of our proof is slightly different from that of [1, Theorem 3]. We apply the recursive formula (1.5) with $j = 1$ to get

$$z_k z_p^m * z_l z_p^n = z_k z_l(z_p^m * z_p^n) + z_{k+l}(z_p^m * z_p^n) + \sum_{i=0}^{n}(z_l z_p^i z_k + z_l z_p^{i-1} z_{p+k})(z_p^m * z_p^{n-i}),$$

which is just (1.7).

2.4. Proof of Corollary 1.4

Here we prove formula (1.9), and one can prove formula (1.10) in a similar way. Applying the recursive formula (1.5) with $j = 1$, we get

$$
\begin{aligned}
&z_k z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} \\
&= z_k(z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}) + z_{k+l_1}(z_p^m * z_p^{n_1} z_{l_2} z_p^{n_2}) \\
&\quad + \sum_{i=0}^{n_1}(z_{l_1} z_p^i z_k + z_{l_1} z_p^{i-1} z_{p+k})(z_p^m * z_p^{n_1-i} z_{l_2} z_p^{n_2}) + z_{l_1} z_p^{n_1} z_{k+l_2}(z_p^m * z_p^{n_2}) \\
&\quad + \sum_{i=0}^{n_2}(z_{l_1} z_p^{n_1} z_{l_2} z_p^i z_k + z_{l_1} z_p^{n_1} z_{l_2} z_p^{i-1} z_{p+k})(z_p^m * z_p^{n_2-i}).
\end{aligned}
$$

(2.1)

Then we have to compute the products $z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$ and $z_p^m * z_p^{n_1-i} z_{l_2} z_p^{n_2}$. For the second one, we have the following general result.

Lemma 2.1. *For positive integers l, p and nonnegative integers m, n_1, n_2, we have*

$$z_p^m * z_p^{n_1} z_l z_p^{n_2} = \sum_{i=0}^{m}[(z_p^i * z_p^{n_1})z_l + (z_p^{i-1} * z_p^{n_1})z_{p+l}](z_p^{m-i} * z_p^{n_2}). \tag{2.2}$$

Proof. We get (2.2) by applying the recursive formula (1.5) with $j = n_1 + 1$. □

And for the first one, we have the following result.

Lemma 2.2. *For positive integers p, l_1, l_2 and nonnegative integers m, n_1, n_2, we have*

(2.3)
$$z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$$
$$= \sum_{0 \le i_1 \le i_2 \le m} (z_p^{i_1} z_{l_1} + z_p^{i_1-1} z_{p+l_1})[(z_p^{i_2-i_1} * z_p^{n_1}) z_{l_2} + (z_p^{i_2-i_1-1} * z_p^{n_1}) z_{p+l_2}](z_p^{m-i_2} * z_p^{n_2}).$$

Proof. Applying the recursive formula (1.5) with $j = 1$, we get

$$z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} = \sum_{i=0}^m (z_p^i z_{l_1} + z_p^{i-1} z_{p+l_1})(z_p^{m-i} * z_p^{n_1} z_{l_2} z_p^{n_2}).$$

Then using (2.2), we have

$$z_p^m * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$$
$$= \sum_{i=0}^m \sum_{t=0}^{m-i} (z_p^i z_{l_1} + z_p^{i-1} z_{p+l_1})[(z_p^t * z_p^{n_1}) z_{l_2} + (z_p^{t-1} * z_p^{n_1}) z_{p+l_2}](z_p^{m-i-t} * z_p^{n_2}).$$

Let $i_1 = i$ and $i_2 = i + t$ in the summation of the right-hand side of the above equation, we get the result. $\square$

Now formula (1.9) follows from (2.1), (2.2) and (2.3). We then finish the proof of Corollary 1.4.

2.5. Remark

In fact, one can get Corollaries 1.3 and 1.4 in the same way as the proofs of Theorems 1.1 and 1.2 given above. For example, $z_k z_p^m * z_l z_p^n$ is a sum over surjections

$$\sigma \colon \{1, 2, \ldots, m+n+2\} \twoheadrightarrow \{1, 2, \ldots, m+n+2-j\},$$

which are strictly increasing on $\{1, \ldots, m+1\}$ and on $\{m+2, \ldots, m+n+2\}$ respectively. We can separate such σ's into three different types according to

$$\sigma(1) < \sigma(m+2), \quad \sigma(m+2) < \sigma(1) \quad \text{or} \quad \sigma(1) = \sigma(m+2),$$

which correspond to the three sums in the right-hand side of (1.7). Then we get formula (1.7).

But as $r + s$ increases in (1.1), the possibilities of different types of σ's similar to above will increase rapidly. Hence in the following section we discuss the general products by using the recursive formulas.

3. The general products

In this section, we discuss the general products (1.1) and (1.2). Here we only treat the $*$ product. Let $r, s, k_1, \ldots, k_r, l_1, \ldots, l_s, p$ be positive integers and let $m_1, \ldots, m_r, n_1, \ldots, n_s$ be nonnegative integers. Applying the recursive formula (1.5) with $j = 1$ to the product (1.1), we find (1.1) can be expressed by products

$$(T_1) \quad z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2} \cdots z_{l_s} z_p^{n_s},$$

$$(T_2) \quad z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_p^{i_1} z_{l_2} z_p^{n_2} \cdots z_{l_s} z_p^{n_s}, \qquad 0 \le i_1 \le n_1,$$

$$(T_3) \quad z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_p^{i_2} z_{l_3} \cdots z_{l_s} z_p^{n_s}, \qquad 0 \le i_2 \le n_2,$$

$$\vdots$$

$$(T_s) \quad z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_p^{i_{s-1}} z_{l_s} z_p^{n_s}, \qquad 0 \le i_{s-1} \le n_{s-1},$$

$$(T_{s+1}) \quad z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_p^{i_s}, \qquad 0 \le i_s \le n_s.$$

We may assume that $r > 1$. For $i = 1, 2, \ldots, s$, using the recursive formula (1.5) with $j = m_1 + 1$ to the product (T_i), we find that (T_i) can be expressed by products of forms $(T_{i+1}), \ldots, (T_{s+1})$. For the product (T_{s+1}), we also apply the recursive formula (1.5) with $j = m_1 + 1$ to get

$$z_p^{m_1} z_{k_2} z_p^{m_2} \cdots z_{k_r} z_p^{m_r} * z_p^{i_s}$$

$$= \sum_{i=0}^{i_s} [(z_p^{m_1} * z_p^i) z_{k_2} + (z_p^{m_1} * z_p^{i-1}) z_{p+k_2}](z_p^{m_2} z_{k_3} \cdots z_{k_r} z_p^{m_r} * z_p^{i_s - i}).$$

Hence (T_{s+1}) can be expressed by products of forms (T_{s+1}) with smaller r. Then we see that we can express the product (1.1) by simple products of the form $z_p^i * z_p^j$.

Let's take the case $r = s = 2$, that is $z_{k_1} z_p^{m_1} z_{k_2} z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$, as an example. Applying the recursive formula (1.5) with $j = 1$, we get

(3.1)
$$z_{k_1} z_p^{m_1} z_{k_2} z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$$

$$= z_{k_1}(z_p^{m_1} z_{k_2} z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}) + z_{k_1+l_1}(z_p^{m_1} z_{k_2} z_p^{m_2} * z_p^{n_1} z_{l_2} z_p^{n_2})$$

$$+ \sum_{i=0}^{n_1}(z_{l_1} z_p^i z_{k_1} + z_{l_1} z_p^{i-1} z_{p+k_1})(z_p^{m_1} z_{k_2} z_p^{m_2} * z_p^{n_1 - i} z_{l_2} z_p^{n_2})$$

$$+ z_{l_1} z_p^{n_1} z_{k_1+l_2}(z_p^{m_1} z_{k_2} z_p^{m_2} * z_p^{n_2})$$

$$+ \sum_{i=0}^{n_2}(z_{l_1} z_p^{n_1} z_{l_2} z_p^i z_{k_1} + z_{l_1} z_p^{n_1} z_{l_2} z_p^{i-1} z_{p+k_1})(z_p^{m_1} z_{k_2} z_p^{m_2} * z_p^{n_2 - i}).$$

Hence we need to compute products of the forms

(i) $z_p^{m_1} z_k z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$,

(ii) $z_p^{m_1} z_k z_p^{m_2} * z_p^{n_1} z_l z_p^{n_2}$,

(iii) $z_p^{m_1} z_k z_p^{m_2} * z_p^n$.

Applying the recursive formula (1.5) with $j = 1$ to the product (i), we get

$$z_p^{m_1} z_k z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$$
$$= \sum_{i=0}^{m_1} (z_p^i z_{l_1} + z_p^{i-1} z_{p+l_1})(z_p^{m_1-i} z_k z_p^{m_2} * z_p^{n_1} z_{l_2} z_p^{n_2}) + z_p^{m_1} z_{k+l_1}(z_p^{m_2} * z_p^{n_1} z_{l_2} z_p^{n_2})$$
$$+ \sum_{i=0}^{m_2} (z_p^{m_1} z_k z_p^i z_{l_1} + z_p^{m_1} z_k z_p^{i-1} z_{p+l_1})(z_p^{m_2-i} * z_p^{n_1} z_{l_2} z_p^{n_2}).$$

Hence the product (i) can be expressed by products of the forms (ii) and (iii). For the product (ii), applying the recursive formula (1.5) with $j = m_1 + 1$, we get

$$z_p^{m_1} z_k z_p^{m_2} * z_p^{n_1} z_l z_p^{n_2}$$
$$= \sum_{i=0}^{n_1} [(z_p^{m_1} * z_p^i) z_k + (z_p^{m_1} * z_p^{i-1}) z_{p+k}](z_p^{m_2} * z_p^{n_1-i} z_l z_p^{n_2})$$
$$+ (z_p^{m_1} * z_p^{n_1}) z_{k+l}(z_p^{m_2} * z_p^{n_2})$$
$$+ \sum_{i=0}^{n_2} [(z_p^{m_1} * z_p^{n_1} z_l z_p^i) z_k + (z_p^{m_1} * z_p^{n_1} z_l z_p^{i-1}) z_{p+k}](z_p^{m_2} * z_p^{n_2-i}).$$

Then the product (ii) can be expressed by products of the form (iii). Finally, applying the recursive formula (1.5) with $j = m_1 + 1$ to the product (iii), we get

$$z_p^{m_1} z_k z_p^{m_2} * z_p^n = \sum_{i=0}^{n} [(z_p^{m_1} * z_p^i) z_k + (z_p^{m_1} * z_p^{i-1}) z_{p+k}](z_p^{m_2} * z_p^{n-i}),$$

which is expressed by simple products.

Putting all these results into (3.1), we can finally obtain a formula which writes the product $z_{k_1} z_p^{m_1} z_{k_2} z_p^{m_2} * z_{l_1} z_p^{n_1} z_{l_2} z_p^{n_2}$ by simple products of the form $z_p^i * z_p^j$. While there are too many terms to write, so we omit this formula.

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