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Existence and Multiplicity of Solutions for a Quasilinear Elliptic Inclusion with a Nonsmooth Potential

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Abstract. This paper is concerned with a nonlinear elliptic inclusion driven by a multivalued subdifferential of nonsmooth potential and a nonlinear inhomogeneous differential operator. We obtain two multiplicity theorems in the Orlicz-Sobolev space. In the first multiplicity theorem, we produce three nontrivial smooth solutions. Two of these solutions have constant sign (one is positive, the other is negative). In the second multiplicity theorem, we derive an unbounded sequence of critical points for the problem. Our approach is variational, based on the nonsmooth critical point theory. We also show that C^1-local minimizers are also local minimizers in the Orlicz-Sobolev space for a large class of locally Lipschitz functions.

1. Introduction

In this paper, we deal with the following quasilinear elliptic problem with a nonsmooth potential:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) \in \partial F(x,u) & \text{for a.a. } x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$, $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable potential function, which is only locally Lipschitz and in general nonsmooth in the second variable. By $\partial F(x,u)$ we denote the generalized Clarke subdifferential of $u \mapsto F(x,u)$. In order to go further we introduce the functional space setting where problem (1.1) will be discussed. In this paper, we note that the operator in the divergence form is not homogenous and thus, we introduce the Orlicz-Sobolev space setting for problems of this type. As in [8, 14], the function a is such that $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} a(|t|)t & t \neq 0, \\ 0 & t = 0, \end{cases}$$

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which is an increasing homeomorphism from $\mathbb{R}$ into itself (such functions are called Young or N-function). If we set

$$G(t) = \int_0^t g(s)\,\mathrm{d}s, \quad \widetilde{G}(t) = \int_0^t g^{-1}(s)\,\mathrm{d}s,$$

then G and $\widetilde{G}$ are complementary N-functions (see [2,33]), which define the Orlicz spaces $L^G := L^G(\Omega)$ and $L^{\widetilde{G}} := L^{\widetilde{G}}(\Omega)$, respectively.

In order to construct an Orlicz-Sobolev space setting for problem (1.1), we assume the following conditions on $g(t)$:

(g_0) $a(t) \in C^1(0,+\infty)$, $a(t) > 0$ and a is a monotonic function for $t > 0$;

(g_1) $1 < g^- = \inf_{t>0} \frac{tg(t)}{G(t)} \leq g^+ = \sup_{t>0} \frac{tg(t)}{G(t)} < +\infty$;

(g_2) $0 < a^- = \inf_{t>0} \frac{tg'(t)}{g(t)} \leq a^+ = \sup_{t>0} \frac{tg'(t)}{g(t)} < +\infty$.

Under the condition (g_1) the function $G(t)$ satisfies Δ_2-condition, i.e.,

$$G(2t) \leq kG(t), \quad t > 0$$

for some constant $k > 0$. On the other hand, it follows from [16] that if G satisfies a global Δ_2 condition then there exists a best positive constant λ_1 such that

$$\lambda_1 \int_\Omega G(|u|)\,\mathrm{d}x \leq \int_\Omega G(|\nabla u|)\,\mathrm{d}x$$

for all $u \in W_0^{1,G}(\Omega)$. The Orlicz space L^G is the vectorial space of the measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\int_\Omega G(|u|)\,\mathrm{d}x < +\infty.$$

The space $L^G(\Omega)$ is a Banach space with the Luxemburg norm

$$|u|_G = \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{|u(x)|}{\lambda}\right)\mathrm{d}x \leq 1\right\},$$

and we shall denote by $W^{1,G}(\Omega)$ the corresponding Orlicz-Sobolev space, which consists of that whose distributional derivative Du also belongs to $L^G(\Omega)$, with the norm

$$\|u\|_{W^{1,G}(\Omega)} = |u|_G + |\nabla u|_G,$$

and $W_0^{1,G}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,G}(\Omega)$. The equivalent norm on $W_0^{1,G}(\Omega)$ can be defined by

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega G\left(\frac{|\nabla u(x)|}{\lambda}\right)\mathrm{d}x \leq 1\right\}.$$

The Orlicz-Sobolev conjugate function G_* of G is given by

$$G_*^{-1}(t) = \int_0^t \frac{G^{-1}(s)}{s^{(N+1)/N}}\,\mathrm{d}s$$

(see [2]), where we suppose that

$$\lim_{t\to 0}\int_t^1 \frac{G^{-1}(s)}{s^{(N+1)/N}}\,\mathrm{d}s < +\infty \quad\text{and}\quad \lim_{t\to\infty}\int_1^t \frac{G^{-1}(s)}{s^{(N+1)/N}}\,\mathrm{d}s = +\infty. \tag{1.2}$$

Let

$$g_*^- = \liminf_{t\to+\infty} \frac{tG_*'(t)}{G_*(t)}.$$

As in [10], by L'Hôpital's rule we have $g_*^- = Ng^-/(N-g^-)$. Throughout this paper, we suppose that g^+ and g_*^- satisfy the following condition

$$g^+ < g_*^-. \tag{1.3}$$

Remark 1.1. Below, we give two characteristic examples of functions which satisfy the conditions (g_0)–(g_2), (1.2) and (1.3).

(i) Set $G(t) = t^p/p$. Then, it is easy to check that $G(t)$ satisfies hypotheses (g_0)–(g_2), (1.2) and (1.3).

(ii) Let

$$g(t) = |t|^{p-2}t\log(1+|t|)$$

where $1 < p < N$. Then $g(t)$ satisfies hypotheses (g_0)–(g_2), (1.2) and (1.3).

When $F(x,u)$ is differentiable, problem (1.1) becomes

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x,u) & \text{for } x\in\Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$, $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Donaldson [13] and Gossez [22] firstly obtained the general existence results of problem (1.4) by the theory of monotone operators in the Orlicz-Sobolev spaces. In [10,16,23], the authors showed the existence results for problem (1.4) by means of monotone operator methods, variational techniques, or fixed point and degree theory arguments. Tan and Fan in [34], using the sub-super solution method and morse theory, proved the existence of multiple solutions for problem (1.1). In [3], Bonanno et al. discussed problem (1.4) with Neumann boundary condition by a critical points theorem. The reader may consult [2,14,32] and the references therein for more information about the Orlicz space.

While, in this study, the nonlinearity $f(x,u)$ can be discontinuous. The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased since many free boundary problems arising in mathematical physics can be stated in this form. Among these problems, we have the seepage problem, the obstacle problem and Elenbaas equation, see [5–7].

In order to enunciate the main results, we need the following hypotheses:

(H_0) (i) There exists an odd increasing homeomorphism h from $\mathbb{R}$ to $\mathbb{R}$, and nonnegative constants c_1, c_2 such that

$$|\omega(x,u)| \le c_1 + c_2 h(|u|)$$

for all $\omega(x,u) \in \partial F(x,u)$.

(ii)

$$\lim_{t\to+\infty} \frac{H(t)}{G_*(kt)} = 0, \quad \forall\, k > 0,$$

where $H(t) = \int_0^t h(s)\,\mathrm{d}s$.

Similar to condition (g_1), we also make the following condition on H.

(H_1) $1 < h^- = \inf_{t>0} th(t)/H(t) \le h^+ = \sup_{t>0} th(t)/H(t) < +\infty$.

(H_2)

$$\lim_{u\to 0} \frac{F(x,u)}{|u|^{g^-}} = +\infty \quad \text{and} \quad \lim_{|u|\to+\infty} \frac{F(x,u)}{G(u)} < \lambda_1 \quad \text{for a.a. } x \in \Omega.$$

(H_3) For every $\delta > 0$, there exists $a_\delta > 0$, such that, if

$$\omega_*(x,u) = \min\{\omega(x,u) : \omega(x,u) \in \partial F(x,u)\},$$

then

$$\omega_* + a_\delta g(u) \ge 0 \quad \text{for a.a. } x \in \Omega, \text{ and all } u \in [-\delta,\delta].$$

(H_4) There exist $\theta > g^+$ and $M > 0$ such that

$$\langle \omega, u\rangle \ge \theta F(x,u) \ge 0$$

for a.a. $x \in \Omega$, all $|u| \ge M$, where $\omega(x,u) \in \partial F(x,u)$,

(H_5) $F(x,u) = F(x,-u)$ for a.a. $x \in \Omega$, all $u \in \mathbb{R}$.

Let $I\colon W_0^{1,G}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1), defined by

$$I(u) = \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega F(x,u(x))\,\mathrm{d}x \quad \forall\, u \in W_0^{1,G}(\Omega).$$

The main results of this paper are:

Theorem 1.2. *If hypotheses* (g_0)–(g_2) *and* (H_0)–(H_3) *hold, then problem* (1.1) *has at least three nontrivial solutions:* $u_0 \in \operatorname{int}(C_+)$, $v_0 \in -\operatorname{int}(C_+)$ *and* $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$ *and* u_0, v_0 *are local minimizers of* I.

Theorem 1.3. *If* $g^+ < h^-$, *hypotheses* (g_0)–(g_2), (H_0), (H_4) *and* (H_5) *hold, then problem* (1.1) *has a sequence of solutions* $\{u_k\}$ *such that* $I(u_k) \to \infty$ *as* $k \to \infty$.

Remark 1.4. There exist many functions F satisfying Theorems 1.2 and 1.3. For example, the following potential function F satisfies Theorem 1.2 (for the sake of simplicity, we drop the x-dependence). Set $g(t) = |t|^{p-2}t\log(1+|t|)$, $\eta \in (1, g^-)$, $0 < c_2 < \lambda_1$.

$$F(u) = \begin{cases} \frac{c_1}{\eta}|u|^\eta & \text{if } |u| \le 1, \\ c_2\left[\frac{1}{p}|t|^p \log(1+|t|) - \frac{1}{p}\int_0^{|t|} \frac{s^p}{1+s}\,\mathrm{d}s\right] & \text{if } |u| \ge 1, \end{cases}$$

where $c_1 = \frac{c_2\eta}{p}\left(\log 2 - \frac{1}{p}\int_0^1 \frac{s^p}{1+s}\,\mathrm{d}s\right)$.

In this work we extend the studies in the following sense:

(1) Unlike [3, 23], the lack of differentiability of the nonlinearity cause several technical difficulties. This means that the variational methods for C^1 functions are not suitable in our case. Therefore we will use a variational approach based on the nonsmooth critical point theory due to Clark [9] and Chang [7].

(2) Compared with [21], we have to prove that $C_0^1(\Omega)$ local minimizers are also local $W_0^{1,G}(\Omega)$-minimizers under certain conditions. While it is not easy to perform since the Orlicz-Sobolev space is more complicated than the Lebesgue-Sobolev space.

(3) Unlike [20, 25, 27], problem (1.1) possesses more complicated nonlinearities, e.g.,

   (i) plasticity: $G(t) = t^\alpha(\log(1+t))^\beta$, $\alpha \ge 1$, $\beta > 0$;

   (ii) nonlinear elasticity: $G(t) = (1+t^2)^\gamma - 1$, $\gamma > 1/2$;

   (iii) generalized Newtonian fluids: $G(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta\,\mathrm{d}s$, $0 \le \alpha \le 1$, $\beta > 0$.

Our paper is organized as follows. In Section 2, some necessary preliminary knowledge is presented. In Section 3, we assert that local minimizers in the space $C_0^1(\Omega)$ are also local minimizers in the Orlicz-Sobolev space for a large class of locally Lipschitz functions. In Section 4, employing suitable truncation techniques, we prove the existence of at least three nontrivial solutions for problem (1.1). In Section 5, using the nonsmooth fountain theorem, we obtain an unbounded sequence of critical points for problem (1.1).

2. Preliminaries

We first give some basic notations.

- $\rightharpoonup$ means weak convergence and $\to$ strong convergence.
- c and c_i $(i = 1, 2, \ldots)$ denote the estimated constants (the exact value may be different from line to line).
- $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual.

Next, we give some necessary definitions.

Definition 2.1. A function $I\colon X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $\nu, \eta \in U$

$$|I(\nu) - I(\eta)| \le L\|\nu - \eta\|.$$

Definition 2.2. Let $I\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of I in u along the direction ν is defined by

$$I^0(u;\nu) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau},$$

where $u, \nu \in X$.

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, \nu\rangle_X \le I^0(u;\nu) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \{I'(u)\}.$$

Clearly, these definitions extend the Gâteaux directional derivative and gradient.

Definition 2.3. We say that I satisfies the nonsmooth $(\mathrm{PS})_c$ if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad m^I(u_n) := \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*} \to 0 \quad \text{as } n \to +\infty$$

has a strongly convergent subsequence.

Definition 2.4. We say that $u \in W_0^{1,G}(\Omega)$ is a weak solution of problem (1.1), if for all $v \in W_0^{1,G}(\Omega)$, there exists a mapping $\Omega \ni x \mapsto \omega(x,u)$ with $\omega(x,u) \in \partial F(x,u)$ such that

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega \omega(x,u) v \,\mathrm{d}x.$$

Obviously, the critical points of I are weak solutions of problem (1.1).

Lemma 2.5. [2] *Under the condition* (g_1), *the spaces* $L^G(\Omega)$, $W_0^{1,G}(\Omega)$ *and* $W^{1,G}(\Omega)$ *are separable and reflexive Banach spaces.*

Lemma 2.6. [2] *Under the condition* $(H_0)(ii)$, *the embedding*

$$W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$$

is compact.

Lemma 2.7. [14] *Let* $\rho(u) = \int_\Omega G(u)\,\mathrm{d}x$, *we have*

(i) *if* $|u|_G < 1$, *then* $|u|_G^{g^+} \le \rho(u) \le |u|_G^{g^-}$;

(ii) *if* $|u|_G > 1$, *then* $|u|_G^{g^-} \le \rho(u) \le |u|_G^{g^+}$;

(iii) *if* $0 < t < 1$, *then* $t^{g^+} G(u) \le G(tu) \le t^{g^-} G(u)$;

(iv) *if* $t > 1$, *then* $t^{g^-} G(u) \le G(tu) \le t^{g^+} G(u)$.

Let $A\colon W_0^{1,G}(\Omega) \to (W_0^{1,G}(\Omega))^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle_{W_0^{1,G}(\Omega)} = \int_\Omega a(|\nabla u|)\nabla u \nabla v\,\mathrm{d}x \quad \forall\, u, v \in W_0^{1,G}(\Omega).$$

The following lemma can be found in [15].

Lemma 2.8. *The mapping* $A\colon W_0^{1,G}(\Omega) \to (W_0^{1,G}(\Omega))^*$ *is a bounded homeomorphism, and is of type* (S^+), *namely,* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ *imply that* $u_n \to u$ *in* $W_0^{1,G}(\Omega)$.

Next, we introduce the following Banach space:

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int}(C_+) = \left\{u \in C_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. The next theorem is a nonsmooth version of the classical mountain pass theorem.

Theorem 2.9. [18] *If X is a Banach space, $I\colon X \to \mathbb{R}$ is a locally Lipschitz functional which satisfies the nonsmooth PS-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r > 0$,*

$$\max\{I(u_0), I(u_1)\} \le \inf_{\|u-u_0\|=r} I(u) = \eta_0$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \ge \eta_0$ and c is a critical value of I.

For a given locally Lipschitz functional $I\colon X \to \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} K_I &= \{u \in X : 0 \in \partial I(u)\}, \\ K_I^c &= \{u \in X : I(u) = c, 0 \in \partial I(u)\}, \\ I^c &= \{u \in X : I(u) < c\}. \end{aligned}$$

The following theorem is a nonsmooth version of the so called second deformation theorem.

Theorem 2.10. [11] *Let X be a Banach space, $I\colon X \to \mathbb{R}$ be a locally Lipschitz functional satisfying the nonsmooth PS-condition, $a, b \in \mathbb{R}$ be numbers with $a < b$. Assume also that $K_I \cap \varphi^{-1}([a,b]) = \emptyset$ and K_I^a is a finite set containing only local minimizers of I. Then, there exists a continuous deformation $h\colon [0,1] \times I^b \to I^b$ such that*

(i) *$h(t,\cdot)|_{K_I^a} = \mathrm{id}|_{K_I^a}$ for all $t \in [0,1]$;*

(ii) *$h(1, I^b) \subseteq I^a \cup K_I^a$;*

(iii) *$I(h(t,u)) \le I(u)$ for all $t \in [0,1]$, $u \in I^b$.*

3. $W_0^{1,G}(\Omega)$ versus $C_0^1(\Omega)$ local minimizers

The next result relates to local $W_0^{1,G}(\Omega)$ and $C_0^1(\overline{\Omega})$-minimizers for a large class of smooth or nonsmooth functionals. The result was first proved for

$$G(y) = \frac{1}{2}\|y\|^2$$

and smooth (i.e., C^1) functionals by Brézis and Nirenberg [4]. It was extended to the case

$$G(y) = \frac{1}{p}\|y\|^p$$

with $1 < p < +\infty$, and smooth functionals by Guo-Zhang [24], and Azorero et al. [17]. The nonsmooth versions to these cases can be found in [21, 27]. Here we further extend all these results.

Let $F_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function, such that for a.a. $x \in \Omega$, the function $\zeta \mapsto F_0(x,\zeta)$ is locally Lipschitz and satisfies the condition (H_0). We define $\varphi\colon W_0^{1,G}(\Omega) \to \mathbb{R}$ by

$$\varphi(u) = \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega F_0(x,u)\,\mathrm{d}x, \quad \forall\, u \in W_0^{1,G}(\Omega).$$

From Clarke [9], we obtain that φ is Lipschitz on bounded sets, hence in particular locally Lipschitz.

Lemma 3.1. [34] *There exist constants d_1, d_2, depending on a^-, a^+, such that*

$$|a(|\eta|)\eta - a(|\xi|)\xi| \le d_1|\eta - \xi| a(|\eta| + |\xi|).$$

If $a(t)$ is decreasing for $t > 0$, we have

$$|a(|\eta|)\eta - a(|\xi|)\xi| \le d_2 g(|\eta - \xi|)$$

for all $\eta, \xi \in \mathbb{R}^N$.

Next, we give our main result, which shows the relationship between $C_0^1(\overline{\Omega})$ and $W_0^{1,G}(\Omega)$.

Theorem 3.2. *If hypotheses (g_0)–(g_3) hold, and $u_0 \in W_0^{1,G}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ, i.e., there exists $\rho_0 > 0$, such that*

$$\varphi(u_0) \le \varphi(u_0 + h) \quad \forall\, h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and it is also a local $W_0^{1,G}(\Omega)$-minimizer of φ, i.e., there exists $\rho_1 > 0$, such that

$$\varphi(u_0) \le \varphi(u_0 + h) \quad \forall\, h \in W_0^{1,G}(\Omega),\ \|h\| \le \rho_1.$$

Proof. Choosing $h \in C_0^1(\overline{\Omega})$, for $\tau > 0$ small enough we have

$$\varphi(u_0) \le \varphi(u_0 + \tau h),$$

hence

$$0 \le \varphi^0(u_0; h). \tag{3.1}$$

Since $h \in C_0^1(\overline{\Omega})$ is arbitrary, $\varphi^0(u_0;\cdot\,)$ is continuous and $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,G}(\Omega)$ from (3.1), we have

$$0 \le \varphi^0(u_0; h) \quad \forall\, h \in W_0^{1,G}(\Omega).$$

So

$$0 \in \partial\varphi(u_0)$$

and thus

$$A(u_0) = \omega_0, \tag{3.2}$$

where $\omega_0 \in \partial F(x, u_0)$ for a.a. $x \in \Omega$. It follows from (3.2) that

$$\begin{cases} -\operatorname{div}(a(|\nabla u_0|)\nabla u_0) = \omega_0 \in \partial F(x, u_0) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

From (g_0)–(g_2) we can easily obtain that

$$a(|t|)t^2 \geq g^- G(|t|) - c \quad \text{and} \quad a(|t|)t \leq g^+ g(|t|) + c.$$

From the above inequalities, invoking Theorem 3.1 and Corollary 3.1 in [34], we infer that $u_0 \in L^\infty(\Omega)$ and $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Now we prove that u_0 is also a local $W_0^{1,G}(\Omega)$-minimizer of φ. Define

$$J(u) = \int_\Omega G(|\nabla u - \nabla u_0|)\,\mathrm{d}x, \quad \forall\, u \in W_0^{1,G}(\Omega).$$

For $\epsilon \in (0, 1)$, set $D_\epsilon = \{u \in W_0^{1,G}(\Omega) : J(u) \leq \epsilon\}$. Then D_ϵ is a bounded, closed and convex subset of $W_0^{1,G}(\Omega)$ and it is a neighborhood of u_0 in $W_0^{1,G}(\Omega)$. On the hypothesis of F_0, the function $\varphi\colon W_0^{1,G}(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous and consequently $\inf_{D_\epsilon} \varphi$ is achieved at some $u_\epsilon \in D_\epsilon$. Invoking the nonsmooth Lagrange multiplier rule of Clarke [9], we can find $\lambda_\epsilon \leq 0$, such that

$$0 \in \partial\varphi(u_\epsilon) - \lambda_\epsilon J'(u_\epsilon).$$

This means that

$$-\operatorname{div}(a(|\nabla u_\epsilon|)\nabla u_\epsilon) + \lambda_\epsilon \operatorname{div}(a(|\nabla u_\epsilon - \nabla u_0|)(\nabla u_\epsilon - \nabla u_0)) = \omega_\epsilon \in \partial F(x, u_\epsilon). \tag{3.3}$$

Proceeding by contradiction, suppose that u_0 is not a local minimizer of φ in the $W_0^{1,G}(\Omega)$ topology, then for each $\epsilon \in (0, 1)$, there exists $u_\epsilon \neq u_0$ such that $\varphi(u_\epsilon) < \varphi(u_0)$. Note that $u_\epsilon \to u_0$ in $W_0^{1,G}(\Omega)$ as $\epsilon \to 0$. Below we need to prove that $u_\epsilon \to 0$ in $C_0^1(\overline{\Omega})$ as $\epsilon \to 0$, which contradicts the fact that u_0 is a local minimizer of φ in the C_0^1 topology. Dividing both sides of (3.3) by $1 - \lambda_\epsilon$, it follows that

$$-\operatorname{div}\left\{\frac{1}{1-\lambda_\epsilon}\left[a(|\nabla u_\epsilon|)\nabla u_\epsilon - \lambda_\epsilon a(|\nabla u_\epsilon - \nabla u_0|)(\nabla u_\epsilon - \nabla u_0)\right]\right\} = \frac{1}{1-\lambda_\epsilon}\omega_\epsilon \in \partial F(x, u_\epsilon).$$

Define $A_\epsilon\colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B_\epsilon\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} A_\epsilon(x, \zeta) &= \frac{1}{1-\lambda_\epsilon}\left[a(|\zeta|)\zeta - \lambda_\epsilon(a(|\zeta - \nabla u_0|)(\zeta - \nabla u_0))\right], \\ B_\epsilon(x, t) &= \frac{1}{1-\lambda_\epsilon}\omega(x, t). \end{aligned}$$

Then u_ϵ is a solution of the following problem:

$$\begin{cases} -\operatorname{div}(A_\epsilon(x,\nabla u)) = B_\epsilon(x,u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We can check that A_ϵ and B_ϵ satisfy the following conditions

$$A_\epsilon(x,\zeta)\zeta \geq c_0 G(|\zeta|) - c, \tag{3.4}$$

$$A_\epsilon(x,\zeta) \leq c_1 g(|\zeta|) + c, \tag{3.5}$$

$$B_\epsilon(x,u) \leq bh(|u|) + c, \tag{3.6}$$

where c_0, c_1, b and c are positive constants independent of $\epsilon \in (0,1)$. The verifications of (3.5) and (3.6) are simple, here we mainly give the proof of (3.4). From the definition of $A_\epsilon(x,\zeta)$ and (g_1), we derive

$$\begin{aligned} A_\epsilon(x,\zeta)\zeta &= \frac{1}{1-\lambda_\epsilon}[(a(|\zeta|)\zeta - \lambda_\epsilon a(|\zeta|)\zeta) - \lambda_\epsilon(a(|\zeta-\nabla u_0|)(\zeta-\nabla u_0) - a(|\zeta|)\zeta)]\zeta \\ &\geq \frac{1}{1-\lambda_\epsilon}[(1-\lambda_\epsilon)G(|\zeta|) - \lambda_\epsilon K], \end{aligned}$$

where $K = [a(|\zeta-\nabla u_0|)(\zeta-\nabla u_0|) - a(|\zeta|)\zeta]\zeta$. By virtue of Lemma 3.1, we have

$$\begin{aligned} |K| &= |a(|\zeta-\nabla u_0|)(\zeta-\nabla u_0) - a(|\zeta|)\zeta||\zeta| \\ &\leq c|\nabla u_0|a(|\zeta-\nabla u_0| + |\zeta|)|\zeta| \\ &\leq cg(|\zeta|) + c \\ &\leq \frac{1}{2}G(|\zeta|) + c, \end{aligned}$$

and when $a(t)$ is decreasing,

$$\begin{aligned} |K| &= |a(|\zeta-\nabla u_0|)(\zeta-\nabla u_0) - a(|\zeta|)\zeta||\zeta| \\ &\leq c|\nabla u_0|a(|\zeta-\nabla u_0| + |\zeta|)|\zeta| \\ &\leq ca(|\nabla u_0|)|\nabla u_0||\zeta| \\ &\leq \frac{1}{2}G(|\zeta|) + c, \end{aligned}$$

where c is a generic positive constant independent of ϵ. Hence, we derive

$$\begin{aligned} A_\epsilon(x,\zeta)\zeta &\geq \frac{1}{1-\lambda_\epsilon}[(1-\lambda_\epsilon)G(|\zeta|) - \lambda_\epsilon K] \\ &\geq \frac{1}{1-\lambda_\epsilon}\left[(1-\lambda_\epsilon)G(|\zeta|) - \lambda_\epsilon\left(\frac{1}{2}G(|\zeta|) + c\right)\right] \\ &= \frac{1}{1-\lambda_\epsilon}\left[\left(1-\frac{3}{2}\lambda_\epsilon\right)G(|\zeta|) - c\lambda_\epsilon\right] \\ &\geq \frac{1}{2}G(|\zeta|) - c, \end{aligned} \tag{3.7}$$

where c is a positive constant independent of ϵ. It follows from (3.7) that (3.4) is proved. From Theorem 3.1 in [34] we obtain that $u_\epsilon \in L^\infty$ and $|u_\epsilon|_{L^\infty}$ is bounded uniformly for $\epsilon \in (0,1)$.

Next, by employing the results in [28, 29], we will prove that $\|u\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c$ for some $\alpha \in (0,1)$ from two cases, respectively.

Case (i): Suppose that $\lambda_\epsilon \in [-1,0]$.

Recall that u_0 satisfies the equation

$$-\operatorname{div}[a(|\nabla u_0|)\nabla u_0] = \omega_0 \in \partial F(x,u_0). \tag{3.8}$$

Subtracting (3.8) from (3.3), we have

$$-\operatorname{div}(a(|\nabla u_\epsilon|)\nabla u_\epsilon - \lambda_\epsilon a(|\nabla u_\epsilon - \nabla u_0|)(\nabla u_\epsilon - \nabla u_0) - \lambda_\epsilon a(|\nabla u_0|)\nabla u_0) = \omega_\epsilon - \lambda_\epsilon \omega_0,$$

where $\omega_\epsilon \in \partial F(x,u_\epsilon)$ and $\omega_0 \in \partial F(x,u_0)$ for a.a. $x \in \Omega$. Define $\widetilde{A}_\epsilon \colon \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\widetilde{B}_\epsilon \colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}
\widetilde{A}_\epsilon &= a(|\zeta|)\zeta - \lambda_\epsilon a(|\zeta - \nabla u_0|)(\zeta - \nabla u_0) - \lambda_\epsilon a(|\nabla u_0|)\nabla u_0,\\
\widetilde{B}_\epsilon &= \omega_\epsilon - \lambda_\epsilon \omega_0.
\end{aligned}$$

It is easy to see that u_ϵ is a solution of the following problem:

$$\begin{cases}
-\operatorname{div}(\widetilde{A}_\epsilon(x,\nabla u)) = \widetilde{B}_\epsilon(x,u) & \text{in } \Omega,\\
u|_{\partial\Omega} = 0.
\end{cases}$$

We need to prove that for all $x,y \in \overline{\Omega}$, $\zeta \in \mathbb{R}^N \setminus \{0\}$, $\xi \in \mathbb{R}^N$, $t \in \mathbb{R}$ the following conditions hold:

$$\widetilde{A}_\epsilon(x,0) = 0, \tag{3.9}$$

$$\sum_{i,j=1}^N \frac{\partial(\widetilde{A}_\epsilon)_j}{\partial \zeta_i}(x,\zeta)\xi_i\xi_j \geq \frac{g(|\zeta|)}{|\zeta|}|\xi|^2, \tag{3.10}$$

$$\sum_{i,j=1}^N \frac{\partial(\widetilde{A}_\epsilon)_j}{\partial \zeta_i}(x,\zeta)|\zeta| \leq c(1 + g(|\zeta|)), \tag{3.11}$$

$$|\widetilde{A}_\epsilon(x,\zeta) - \widetilde{A}_\epsilon(y,\zeta)| \leq c(1 + g(|\zeta|))(|x-y|^\theta) \quad \text{for some } \theta \in (0,1), \tag{3.12}$$

$$|\widetilde{B}_\epsilon(x,t)| \leq c + ch(|t|). \tag{3.13}$$

(3.9) and (3.13) are obvious. Inequalities (3.10) and (3.11) follow from Lemma 3.1 and the following derivative

$$\begin{aligned}
D_\zeta(a(|\zeta|)\zeta) &= a'(|\zeta|)\frac{\zeta \otimes \zeta}{|\zeta|} + a(|\zeta|)\,\mathrm{id}\\
&= a(|\zeta|)\left(\mathrm{id} + \frac{a'(|\zeta|)|\zeta|}{a(|\zeta|)}\frac{\zeta \otimes \zeta}{|\zeta|^2}\right).
\end{aligned}$$

Inequality (3.12) follows from Lemma 3.1 and the fact that $\nabla u_0(x)$ is Hölder continuous. According to the regularity results in [28, 29], from (3.9)–(3.13), one has

$$u_\epsilon \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_\epsilon\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c, \tag{3.14}$$

where the positive constant c is independent of $\lambda_\epsilon \in [-1, 0]$.

Case (ii): Suppose that $\lambda_\epsilon < -1$.

Let $v_\epsilon = u_\epsilon - u_0$. By virtue of (3.3) and (3.8) we know that v_ϵ satisfies the equation

$$\begin{aligned} &- \operatorname{div} \left[a(|\nabla v_\epsilon|)\nabla v_\epsilon + \frac{1}{|\lambda_\epsilon|} a(|\nabla v_\epsilon + \nabla u_0|)(\nabla v_\epsilon + \nabla u_0) - \frac{1}{|\lambda_\epsilon|} a(|\nabla u_0|)\nabla u_0 \right] \\ &= \frac{1}{|\lambda_\epsilon|} [\omega(x, v_\epsilon + u_0) - \omega(x, u_0)], \end{aligned}$$

where $\omega(x, v_\epsilon + u_0) \in \partial F(x, v_\epsilon + u_0)$ and $\omega(x, u_0) \in \partial F(x, u_0)$ for a.a. $x \in \Omega$. Set

$$\begin{aligned} \widehat{A}_\epsilon(x, \zeta) &= a(|\zeta|)\zeta + \frac{1}{|\lambda_\epsilon|} a(|\zeta + \nabla u_0|)(\zeta + \nabla u_0) - \frac{1}{|\lambda_\epsilon|} a(|\nabla u_0|)\nabla u_0, \\ \widehat{B}_\epsilon(x, t) &= \frac{1}{|\lambda_\epsilon|} [\omega(x, t + u_0) - \omega(x, u_0)], \end{aligned}$$

where $\omega(x, t + u_0) \in \partial F(x, t + u_0)$ and $\omega(x, u_0) \in \partial F(x, u_0)$ for a.a. $x \in \Omega$.

In a similar way, we can prove that $\widehat{A}_\epsilon$ and $\widehat{B}_\epsilon$ satisfy the corresponding conditions (3.9)–(3.13). So from the regularity results in [28, 29], we obtain

$$v_\epsilon \in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|v_\epsilon\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c. \tag{3.15}$$

Let $\epsilon \downarrow 0$. Due to the fact that for every $\alpha \in (0, 1)$ the embedding $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ is compact, from (3.14) and (3.15), we can find a subsequence u_{ϵ_n} of u_ϵ such that $u_{\epsilon_n} \to \widetilde{u}$ in $C_0^1(\overline{\Omega})$. From the construction we have $u_{\epsilon_n} \to u_0$ in $W_0^{1,G}(\Omega)$, which means $\widetilde{u} = u_0$. So, for n sufficiently large, say $n \geq n_0$, we obtain

$$\|u_{\epsilon_n} - u_0\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq r_1,$$

which implies

$$\varphi(u_0) \leq \varphi(u_{\epsilon_n}). \tag{3.16}$$

However, the choice of the sequence $\{u_{\epsilon_n}\}$ means

$$\varphi(u_{\epsilon_n}) < \varphi(u_0), \quad \forall\, n \geq n_0,$$

which is a contradiction to (3.16). Hence the proof is completed. $\square$

4. First multiplicity theorem

In this section we prove Theorem 1.2, which produces three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative).

In order to obtain solutions with constant sign we will use truncations. Let us introduce the notations $r_\pm = \max\{\pm r, 0\}$ and the truncation functions $\tau_\pm \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_\pm(u) = \begin{cases} u & \text{if } \pm u \geq 0, \\ 0 & \text{if } \pm u < 0. \end{cases}$$

Proof of Theorem 1.2. Let $F_\pm(x,u) = F(x, \tau_\pm(u))$ for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$. Evidently both $F_\pm \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable. Moreover, for a.a. $x \in \Omega$, we have that $F_\pm$ are locally Lipschitz and from the nonsmooth chain rule (see Clarke [9])

$$\partial F_\pm(x,u) = \begin{cases} \partial F(x,u) & \text{if } \pm u > 0, \\ \{t\omega : t \in [0,1], \omega \in \partial F(x,0)\} & \text{if } u = 0, \\ 0 & \text{if } \pm u < 0. \end{cases} \tag{4.1}$$

Let $I_\pm \colon W_0^{1,G}(\Omega) \to \mathbb{R}$ be the locally Lipschitz functions defined by

$$I_\pm(u) = \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega F_\pm(x,u)\,\mathrm{d}x \quad \forall\, u \in W_0^{1,G}(\Omega).$$

By virtue of hypothesis (H_2), there exist $\epsilon \in (0,1)$ and $M_0 > 0$ large enough such that

$$F(x,u) \leq \lambda_1(1-\epsilon)G(|u|) \quad \text{for all } |u| \geq M.$$

From hypothesis (H_0), when $|u| < M$, we have

$$|G(x,u)| \leq c_6.$$

Then, when $\|u\| > 1$, we have

$$\begin{aligned} I(u) &= \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x \\ &\geq \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \lambda_1(1-\epsilon)\int_\Omega G(|u|)\,\mathrm{d}x - c_6 \\ &\geq \epsilon \|u\|^{g^-} - c_6. \end{aligned}$$

It follows from the above inequality that I is coercive. Also, using the compactness of the embedding of $W_0^{1,G}(\Omega)$ into $L^H(\Omega)$, we can easily obtain that I is sequentially weakly lower semicontinuous. Hence, from the Weierstrass theorem, we have $u_0 \in W_0^{1,G}(\Omega)$, such that

$$I_+(u_0) = \inf_{u \in W_0^{1,G}(\Omega)} I_+(u) = m_+. \tag{4.2}$$

For $u \in \operatorname{int}(C_+)$, we have

$$\lim_{|t|\to 0} \frac{I(tu)}{|t|^{g^-}} = -\infty.$$

In fact, for any $\{t_n\} \subset \mathbb{R}$ with $t_n \to 0$, set $v_n = t_n u$. Then

$$v_n \to 0 \text{ in } W_0^{1,G}(\Omega), \quad v_n \to 0 \text{ a.a. } x \in \Omega.$$

Due to hypothesis (H_2) and Fatou Lemma, we derive

$$\liminf_{n\to\infty} \int_\Omega \frac{F(x, v_n)}{|v_n|^{g^-}} |u|^{g^-} \,\mathrm{d}x \geq \int_\Omega \lim_{n\to\infty} \frac{F(x, v_n)}{|v_n|^{g^-}} |u|^{g^-} \,\mathrm{d}x = +\infty. \tag{4.3}$$

So

$$\begin{aligned}
\frac{I(t_n u)}{|t_n|^{g^-}} &= \frac{\int_\Omega G(|\nabla t_n u|)\,\mathrm{d}x - \int_\Omega F(x, t_n u)\,\mathrm{d}x}{|t_n|^{g^-}} \\
&\leq \frac{|t_n|^{g^-} \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega F(x, t_n u)\,\mathrm{d}x}{|t_n|^{g^-}} \\
&\leq \int_\Omega G(|\nabla u|)\,\mathrm{d}x - \int_\Omega \frac{F(x, v_n)}{|v_n|^{g^-}} |u|^{g^-} \,\mathrm{d}x \\
&\to -\infty
\end{aligned} \tag{4.4}$$

as $n \to \infty$. From (4.3), we know that for any $u \in \operatorname{int}(C_+)$, there exists $t_0 \in (0,1)$ such that

$$I(tu) < 0 \quad \text{for } t \in (0, t_0).$$

So

$$I_+(u_0) = m_+ < 0 = I_+(0),$$

i.e.,

$$u_0 \neq 0.$$

It follows from (4.2) that

$$0 \in \partial I_+(u_0).$$

So

$$A(u_0) = \omega_0, \tag{4.5}$$

where $\omega_0 \in \partial F_+(x, u_0(x))$ for a.a. $x \in \Omega$. We act on (4.5) with the test function $-u_0^- \in W_0^{1,G}(\Omega)$. Then, it follows from (4.1) that

$$\int_{\Omega_-} a(|\nabla u_0|) |\nabla u_0|^2 \,\mathrm{d}x = \int_{\Omega_-} \omega_0 u_0 \,\mathrm{d}x = 0,$$

where $\Omega_- = \{x \in \Omega : u_0(x) < 0\}$ and $\omega_0 \in \partial F(x, u_0)$. This means that

$$u_0^- = 0, \quad \text{i.e., } u_0 \geq 0,\ u_0 \neq 0.$$

From (4.5), we obtain

$$\begin{cases} -\operatorname{div}(a(|\nabla u_0|)\nabla u_0) = \omega_0 & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

Noting that

$$\nabla u_0(x) = 0 \quad \text{on } \{u_0 = 0\}$$

(Stampacchia theorem (see [19, p. 195])), we deduce that

$$\omega_0(x, u_0) \in \partial F(x, u_0(x)) \quad \text{for a.a. } x \in \Omega.$$

So, u_0 is a nontrivial positive solution of problem (1.1). As before, from the nonlinear regularity theory, we have $u_0 \in C_+ \setminus \{0\}$. Let $\delta = \|u_0\|_\infty$ and $a_\delta > 0$ be as postulated by hypothesis (H_3). Then

$$-\operatorname{div}(a(|\nabla u_0|)\nabla u_0) + a_\delta g(u_0) = \omega_0 + a_\delta g(u_0) \geq 0$$

for a.a. $x \in \Omega$. So

$$\operatorname{div}(a(|\nabla u_0|)\nabla u_0) \leq a_\delta g(u_0)$$

for a.a. $x \in \Omega$ and it follows from hypotheses (g_0)–(g_2) that $u_0 \in \operatorname{int}(C_+)$ (see Montenegro [31, Theorem 6]). If

$$W_+ = \{u \in W_0^{1,G}(\Omega) : u(x) \geq 0 \text{ for a.a. } x \in \Omega\},$$

then clearly

$$I|_{W_+} = I_+|_{W_+}.$$

Hence, u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of I. According to Theorem 3.2, we deduce that u_0 is a local $W_0^{1,G}(\Omega)$-minimizer of I.

Similarly, working with the functional I_-, we can have another constant sign smooth solution $v_0 \in -\operatorname{int}(C_+)$ of problem (1.1), which is a local minimizer of the functional I.

Next, we will prove the existence of the third nontrivial solution for problem (1.1). Without any loss of generality, we may suppose that $I(v_0) \leq I(u_0)$.

Note that u_0 is a global minimizer of I_+, and we distinguish two cases. Suppose that there exists another nontrivial critical point $y_0 \in W_0^{1,G}(\Omega) \setminus \{0, u_0\}$ of I_+. Then, as in the above proof, we can obtain that $y_0 \in \operatorname{int}(C_+)$ and it solves problem (1.1). Hence we have derived a third nontrivial solution (which in fact is positive).

So, we may assume that u_0 is the only nontrivial critical point of I_+, and similarly that v_0 is the only nontrivial critical point of I_-. Reasoning as in the proof of Proposition 29 in [30], we can find $r \in (0,1)$ small, such that

$$I(v_0) \le I(u_0) < \inf\{I(u) : \|u - u_0\| = r\} = \eta_r, \quad \|u_0 - v_0\| > r. \tag{4.6}$$

In a similar way, from hypotheses (H_0) and (H_2), we can prove that I is coercive, and so it satisfies the nonsmooth $(\mathrm{PS})_c$. From (4.6) and the nonsmooth mountain pass theorem, we can find $y_0 \in W_0^{1,G}(\Omega)$, such that

$$I(v_0) \le I(u_0) < \eta_r \le I(y_0) \tag{4.7}$$

and

$$0 \in \partial I(y_0). \tag{4.8}$$

(4.7) means that $y_0 \notin \{v_0, u_0\}$ and it follows from (4.8) that

$$A(y_0) = \omega(x, y_0),$$

where $\omega(x, y_0) \in \partial F(x, y_0)$ for a.a. $x \in \Omega$. Thus y_0 is a solution of problem (1.1) and $y_0 \in C_0^1(\overline{\Omega})$ from the nonlinear regularity theory. It only remains to prove that $y_0 \neq 0$. By virtue of Theorem 2.9, we have

$$c = I(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)), \tag{4.9}$$

where

$$\Gamma = \{\gamma \in C([0,1]; W_0^{1,G}(\Omega)) : \gamma(0) = v_0, \gamma(1) = u_0\}.$$

According to (4.9), if we can find $\gamma_* \in \Gamma$, such that

$$I(\gamma_*(t)) < 0 \quad \forall\, t \in [0,1],$$

then

$$c = I(y_0) < 0 = I(0)$$

and so $y_0 \neq 0$. So our aim is to find such a path $\gamma_* \in \Gamma$.

With this aim in mind, set

$$\Gamma_c = \{\gamma \in C([0,1]; C_0^1(\overline{\Omega})) : \gamma(0) = v_0, \gamma(1) = u_0\}.$$

Due to the density of the embedding of $C_0^1(\overline{\Omega})$ into $W_0^{1,G}(\Omega)$, we have that Γ_c is dense in Γ. We can find $\widetilde{\gamma} \in \Gamma_c$ such that $0 \notin \widetilde{\gamma}([0,1])$. Noting that

$$\widetilde{\gamma}([0,1]) \subseteq C_0^1(\overline{\Omega}) \quad \text{and} \quad 0 \notin \widetilde{\gamma}([0,1]),$$

we can find $\lambda \in (0,1)$ small enough, for all $u \in \widetilde{\gamma}([0,1])$, as doing in (4.4), such that

$$\frac{I(\lambda u)}{\lambda^{p^-}} \to -\infty \quad \text{as } \lambda \to 0^+. \tag{4.10}$$

From (4.10), there exists $\lambda_0 \in (0,1)$ such that

$$I(\lambda u) < 0 \tag{4.11}$$

for all $\lambda \in (0,\lambda_0)$ and all $u \in \widetilde{\gamma}([0,1])$. Setting $\widehat{\gamma} = \lambda\widetilde{\gamma}$, from (4.11), we see that

$$I|_{\widehat{\gamma}} < 0 \tag{4.12}$$

and $\widehat{\gamma}$ is a continuous path in $W_0^{1,G}(\Omega)$ which connects λv_0 and λu_0.

In the following, we will find a continuous path in $W_0^{1,G}(\Omega)$ which connects λu_0 and u_0 and along which I is strictly negative. For this purpose, note that

$$m_+ = \inf_{u \in W_0^{1,G}(\Omega)} I_+(u) < 0 = I_+(0).$$

Also, we may suppose that $K_{I_+}^{m_+} = \{u_0\}$ or otherwise we already obtain a second positive solution. By virtue of Theorem 2.10, we can find a continuous deformation $h\colon [0,1] \times I_+^0 \to I_+^0$, such that

$$h(1, I_+^0) \subseteq I_+^{m_+} \cup K_{I_+}^{m_+} = I_+^{m_+} \cup \{u_0\} = \{u_0\} \tag{4.13}$$

(since $I_+^{m_+} = \emptyset$) and

$$I_+(h(t,u)) \le I_+(u) \quad \forall\, t \in [0,1],\ u \in I_+^0. \tag{4.14}$$

Consider the continuous path $\gamma_+\colon [0,1] \to W_0^{1,G}(\Omega)$, defined by

$$\gamma_+(t) = h(t, \lambda u_0)^+ \quad \forall\, t \in [0,1].$$

Then

$$\gamma_+(0) = h(0,\lambda u_0)^+ = (\lambda u_0)^+ = \lambda u_0,$$

and

$$\gamma_+(1) = h(1,\lambda u_0)^+ = u_0$$

(see (4.13)). So γ_+ is a continuous path in $W_0^{1,G}(\Omega)$, which connects λu_0 and u_0. Furthermore, by (4.14) and $I|_{W^+} = I_+|_{W_+}$, we obtain

$$\begin{aligned} I(\gamma_+(t)) &= I(h(t,\lambda u_0)^+) = I_+(h(t,\lambda u_0)^+) \\ &\le I_+(\lambda u_0) = I(\lambda u_0) < 0 \quad \forall\, t \in [0,1] \end{aligned}$$

(see (4.12)). Hence

$$I|_{\gamma_+} < 0.$$

In a similar way, we can produce a continuous path γ_- in $W_0^{1,G}(\Omega)$ which connects λv_0 and v_0 and

$$I|_{\gamma_-} < 0.$$

Concatenating γ_-, $\widehat{\gamma}$ and γ_+, we have $\gamma_* \in \Gamma$, such that

$$I|_{\gamma_*} < 0,$$

hence, $y_0 \neq 0$. So $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$ is the third nontrivial smooth solution of problem (1.1). $\square$

5. Second multiplicity theorem

In this section, we will prove Theorem 1.3. Let $X = W_0^{1,G}(\Omega)$. Since X is a reflexive and separable Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \ldots\}},$$

and

$$\langle e_j, e_i^* \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$.

Definition 5.1. Assume that the compact group G acts diagonally on V^k

$$g(v_1, \ldots, v_k) = (gv_1, \ldots, gv_k),$$

where V is a finite dimensional space. The action of G is admissible if every continuous equivariant map $\partial U \to V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in V^k, $k \geq 2$, has a zero.

The antipodal action $G = \mathbb{Z}_2$ on $V = \mathbb{R}$ is admissible.

(A$_1$) The compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{m \in \mathbb{N}} X_m}$, the space X_m are invariant and there exists a finite dimensional space V such that, for every $m \in \mathbb{N}$, $X_m \simeq V$ and the action of G on V is admissible.

The following lemma is very important when we use the nonsmooth fountain theorem to prove infinite solutions for problem (1.1).

Lemma 5.2. *If hypothesis* (H_0) *is satisfied, then we have*

$$\beta_k = \sup_{u \in Z_k, \|u\|=1} |u|_H \to 0, \quad k \to \infty.$$

Proof. It is obvious that $0 < \beta_{k+1} \leq \beta_k$. So there exists $\beta \geq 0$ such that $\beta_k \to \beta$ as $k \to \infty$. We need to prove $\beta = 0$. From the definition of β_k, for every $k \geq 0$ there exists $u_k \in Z_k$ such that $\|u_k\| = 1$, $0 \leq \beta - |u_k|_H < 1/k$. Then, there exists a subsequence of $\{u_k\}$, which still denote by u_k, such that

$$u_k \rightharpoonup u \text{ in } X, \quad \text{and} \quad \langle e_j^*, u \rangle = \lim_{k \to \infty} \langle e_j^*, u_k \rangle = 0, \quad j = 1, 2, \ldots,$$

which means that $u = 0$ and $u_k \rightharpoonup 0$ in X. Since the Sobolev embedding $X \hookrightarrow L^H(\Omega)$ is compact then $u_k \to 0$ in $L^H(\Omega)$. Thus we obtain $\beta = 0$. □

The next theorem is the nonsmooth fountain theorem, which was proved by Dai in [12].

Lemma 5.3. *Under assumption* (A_1)*, let* $I\colon X \to \mathbb{R}$ *be an invariant locally Lipschitz functional. If for every* $k \in \mathbb{N}$*, there exist* $\rho_k > r_k > 0$ *such that*

(A_2) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0$*;*

(A_3) $b_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \to \infty$*,* $k \to \infty$*;*

(A_4) *I satisfies the nonsmooth* $(PS)_c$ *condition for every* $c > 0$*,*

then I has an unbounded sequence of critical values.

Proof of Theorem 1.3. Noting that I is a locally Lipschitz function on X, and considering (H_5) we can employ the nonsmooth version of the fountain theorem with the antipodal action of $\mathbb{Z}_2$ to prove Theorem 1.3.

Claim. I satisfies the nonsmooth $(PS)_c$.

Suppose that $\{u_n\} \subset X$ is a sequence, such that

$$I(u_n) \to c \quad \text{and} \quad m^I(u_n) \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

We first prove that the sequence $\{u_n\}$ is bounded in X. Let $u_n^* \in \partial I(u_n)$ such that $m^I(u_n) = \|u_n^*\|_{X^*}$. Then from (5.1) we have

$$-\langle u_n^*, u_n \rangle = -\langle A(u_n), u_n \rangle + \int_\Omega \omega(x, u_n) u_n \,\mathrm{d}x \leq \varepsilon_n \|u_n\| \tag{5.2}$$

with $\varepsilon_n \downarrow 0$ and

$$\int_\Omega \theta G(|\nabla u_n|) \,\mathrm{d}x - \int_\Omega \theta F(x, u_n) \,\mathrm{d}x \leq \theta c. \tag{5.3}$$

Adding (5.2) and (5.3), we obtain

$$\int_\Omega (\theta G(|\nabla u_n|) - g(|\nabla u_n|)\nabla u_n)\,\mathrm{d}x - \int_\Omega (\theta F(x,u_n) - \omega(x,u_n))u_n\,\mathrm{d}x \le \theta c + \varepsilon_n \|u_n\|, \tag{5.4}$$

where $\theta > g^+$, $\omega(x,u_n) \in \partial F(x,u_n)$ for a.a. $x \in \Omega$. We proceed by contradiction and thus suppose that there exists a subsequence such that $\|u_n\| > n$. In this case (see [1]) we have that

$$\int_\Omega G(|\nabla u_n|)\,\mathrm{d}x \ge \|u_n\| \tag{5.5}$$

for all $n \in \mathbb{N}$. So, from (5.4) and (5.5), we obtain that

$$\begin{aligned} &\int_\Omega (\theta G(|\nabla u_n|) - g(|\nabla u_n|)|\nabla u_n|)\,\mathrm{d}x - \int_\Omega (\theta F(x,u_n) - \omega(x,u_n))u_n\,\mathrm{d}x \\ &\le \theta c + \varepsilon_n \int_\Omega G(|\nabla u_n|)\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

By virtue of hypotheses (g_1) and (H_4), choosing $\rho > 0$ small enough so that $g^+ + \rho < \theta$, when $|u| \ge M$, we have

$$0 < ug(u) \le (g^+ + \rho)G(u), \tag{5.7}$$

and

$$0 \le \theta F(x,u) \le \langle \omega, u\rangle, \tag{5.8}$$

for a.a. $x \in \Omega$. Set

$$\Omega_{1,n} = \{x \in \Omega : |\nabla u_n(x)| < M\}, \qquad \Omega_{2,n} = \{x \in \Omega : |\nabla u_n(x)| \ge M\},$$
$$\Omega_{3,n} = \{x \in \Omega : |u_n(x)| < M\}, \qquad \Omega_{4,n} = \{x \in \Omega : |u_n(x)| \ge M\}.$$

Then, it follows from hypothesis (H_0) and (5.6) that there exists a positive constant $\widetilde{c}$ independent of n, such that

$$\begin{aligned} &\int_{\Omega_{2,n}} (\theta G(|\nabla u_n|) - g(|\nabla u_n|)\nabla u_n)\,\mathrm{d}x - \int_{\Omega_{4,n}} (\theta F(x,u_n) - \omega(x,u_n))u_n\,\mathrm{d}x \\ &\le \widetilde{c} + \varepsilon_n \int_\Omega G(|\nabla u_n|)\,\mathrm{d}x, \end{aligned} \tag{5.9}$$

where $\omega(x,u_n) \in \partial F(x,u_n)$ for a.a. $x \in \Omega$. From (5.7) and (5.8), one has

$$\int_{\Omega_{2,n}} (\theta G(|\nabla u_n|) - g(|\nabla u_n|)\nabla u_n)\,\mathrm{d}x \ge (\theta - g^+ - \rho)\int_{\Omega_{2,n}} G(|\nabla u_n|)\,\mathrm{d}x \tag{5.10}$$

and

$$\int_{\Omega_{4,n}} (\theta F(x,u_n) - \omega(x,u_n))u_n\,\mathrm{d}x \le 0. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), we deduce that

$$\int_{\Omega_{2,n}} G(|\nabla u_n|)\,\mathrm{d}x \le c + \varepsilon_n \int_\Omega G(|\nabla u_n|)\,\mathrm{d}x.$$

From the fact $\Omega = \Omega_{1,n} \cup \Omega_{2,n}$, we have

$$\int_\Omega G(|\nabla u_n|)\,\mathrm{d}x \le c + \varepsilon_n \int_\Omega G(|\nabla u_n|)\,\mathrm{d}x.$$

Taking n large enough, say $n \ge n_0$, $\varepsilon_n < 1/2$, we can obtain

$$\int_\Omega G(|\nabla u_n|)\,\mathrm{d}x \le c \quad \text{for } n \ge n_0,$$

and hence, it follows from (5.5) that

$$\|u_n\| \le c$$

for all $n \in \mathbb{N}$, a contradiction to $\|u_n\| > n$.

Since $\{u_n\}$ is bounded, we may assume that $u_n \rightharpoonup u$ in X, and $u_n \to u$ in $L^H(G)$ (since the embedding of X into $L^H(G)$ is compact). Then from (5.1) and (5.2) we have

$$\left| \langle A(u_n), u_n - u\rangle + \int_\Omega \omega(x, u_n)(u_n - u)\,\mathrm{d}x \right| \le \|u_n^*\|_{X^*} \|u_n - u\|.$$

Note that $\omega(x, u_n)$ is bounded in $(L^H(\Omega))^*$ (see (H_0)) which means that

$$\left| \int_\Omega \omega(x, u_n)(u_n - u)\,\mathrm{d}x \right| \le \|\omega(x, u_n)\|_{(L^H(G))^*} \|u_n - u\|_{L^H(G)} \to 0.$$

Since $\|u_n^*\|_{X^*} \to 0$ and $\{u_n\}$ is bounded in X, we have

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Invoking Lemma 2.8, we infer that $u_n \to u$, which proves our Claim.

According to hypothesis (H_4), we can obtain that there exist $R > 0$ and $c > 0$ such that (see [26])

$$F(x, u) \ge c|u|^\theta \quad \text{for } |u| \ge R,\ x \in \Omega,\ u \in X. \tag{5.12}$$

By virtue of (5.12), when $\|u\| > 1$ we have

$$\begin{aligned} I(u) &\le \int_\Omega G(|\nabla u|)\,\mathrm{d}x - c\int_\Omega |u|^\theta\,\mathrm{d}x + c \\ &\le \|u\|^{g^+} - c\int_\Omega |u|^\theta\,\mathrm{d}x + c. \end{aligned}$$

Since on the finite dimensional space Y_k all norms are equivalent, and $\theta > g^+$, relation (A_2) is satisfied for every $\|u\| = \rho_k > 1$ large enough. By virtue of hypothesis (H_0), on $\mathbb{Z}_2$, we have

$$\begin{aligned} I(u) &\geq \int_\Omega G(|\nabla u|)\,\mathrm{d}x - c\int_\Omega H(|u|)\,\mathrm{d}x - c \\ &\geq \begin{cases} \|u\|^{g^+} - c\beta_k^{h^-}\|u\|^{h^-} - c & \text{if } \|u(x)\| < 1, \\ \|u\|^{g^-} - c\beta_k^{h^+}\|u\|^{h^+} - c & \text{if } \|u(x)\| \geq 1. \end{cases} \end{aligned}$$

If $\|u\| < 1$, choosing $r_k = [g^+/(ch^-\beta_k^{h^-})]^{1/(h^- - g^+)}$, form Lemma 5.2, we have

$$I(u) \geq \left(1 - \frac{g^+}{h^-}\right)\left(\frac{g^+}{ch^-\beta_k^{h^-}}\right)^{g^+/(h^- - g^+)} - c \to \infty \tag{5.13}$$

as $k \to \infty$. If $\|u\| \geq 1$, choosing $r_k = [g^-/(ch^+\beta_k^{h^+})]^{1/(h^+ - g^-)}$, form Lemma 5.2, we also have

$$I(u) \geq \left(1 - \frac{g^-}{h^+}\right)\left(\frac{g^-}{ch^+\beta_k^{h^+}}\right)^{g^-/(h^+ - g^-)} - c \to \infty \tag{5.14}$$

as $k \to \infty$. So, from (5.13) and (5.14), relation (A_3) is proved.

According to the nonsmooth fountain theorem (Lemma 5.3), we obtain that problem (1.1) has an unbounded sequence of critical points. The proof is completed. □

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