ALMOST PERIODIC MILD SOLUTIONS OF A CLASS OF PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of almost periodic mild solutions of a class of partial functional differential equations via semilinear almost periodic abstract functional differential equations of the form

$$x' = f(t, x, x_t)\,. \tag{$*$}$$

To this end, we first associate with every almost periodic semilinear equation

$$x' = F(t, x) \tag{$**$}$$

a nonlinear semigroup in the space of almost periodic functions. We then give sufficient conditions (in terms of the accretiveness of the generator of this semigroup) for the existence of almost periodic mild solutions of $(**)$ as fixed points of the semigroup. Those results are then carried over to equation $(*)$. The main results are stated under accretiveness conditions of the function f in terms of x and Lipschitz conditions with respect to x_t.

1. Introduction

In this paper we are mainly concerned with the existence of almost periodic semilinear evolution equations of the form

$$\frac{dx}{dt} = Ax + f(t, x, x_t) \tag{1}$$

where A is the infinitesimal generator of a C_0-semigroup $T(t), t \geq 0$ and f is an everywhere defined continuous operator from $\mathbf{R} \times \mathbf{X} \times C$ to $\mathbf{X}$. Throughout this paper we will denote by $C = C_u((-\infty, 0], \mathbf{X})$ the space of all uniformly continuous and bounded functions from $(-\infty, 0]$ to $\mathbf{X}$, by $\mathbf{X}$ a

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given Banach space and by x_t the map $x(t+\theta) = x_t(\theta),\ \theta \in (-\infty, 0]$, where $x(\cdot)$ is defined on $(-\infty, a]$ for some $a > 0$.

As is well known (see e.g. [2], [5], [10], [12], [18-19], [24-27], [37] and the references therein) many problems on partial differential equations can be stated in the setting of abstract functional differential equations of the form (1). In passing we mention that the case $A = 0$ of equation (1) has been treated in [13-17], [28], [31]. In these papers the proofs of the main results are based on an existence theorem for bounded solutions due to Medvedev [20].

Recently, increasing interest in semilinear evolution equations can be observed (see e.g. [12], [18-19], [25-27], [32] and the references therein). This interest arises from a need to extend well-known results on ordinary differential equations to a class of partial differential equations. In this context we consider the existence of almost periodic mild solutions of equation (1). At this point we want to emphasize that our approach to the problem is somewhat different from that used in the papers [13-17], [28], [31]. In fact, we first associate with an equation without delay (equation (2) below) a semigroup of nonlinear operators which then plays a role similar to that of monodromy operator for equations with periodic coefficients. Then we use the results thus obtained in order to study equation (1). Our method, in the case $A = 0$, requires a condition on f which is somewhat stronger than that used in the previous papers in order to guarantee the existence of the associated semigroups. However, our method in turn allows to impose accretiveness conditions on the function f in a more general context which seems to be more suitable for equations whose right-hand side f depends explicitly on t.

2. Almost Periodic Solutions of Differential Equations without Delay

In this section we deal with almost periodic mild solutions to the semilinear evolution equation

$$\frac{dx}{dt} = Ax + f(t, x), \quad x \in \mathbf{X} \tag{2}$$

where $\mathbf{X}$ is a Banach space, A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$ of linear operators of type ω, i.e.

$$\|S(t)x - S(t)y\| \leq e^{\omega t}\|x - y\|,\ \forall\ t \geq 0,\ x, y \in \mathbf{X},$$

and B is an everywhere defined continuous operator from $\mathbf{R} \times \mathbf{X}$ to $\mathbf{X}$. Hereafter, by a mild solution $x(t), t \in [s, \tau]$ of equation (2) we mean a continuous solution of the integral equation

$$x(t) = S(t-s)x + \int_s^t S(t-\xi)B(\xi, x(\xi))d\xi,\ \forall s \leq t \leq \tau. \tag{3}$$

A strong solution (for the definition see [34], e.g.) of equation (2) is necessarily a mild solution of this equation. The inverse assertion, however, is not true for a general Banach space $\mathbf{X}$ (see [34] for a counter-example). On the other hand, under additional assumptions on $\mathbf{X}$ the mild solutions may be strong solutions. For more information on semilinear equations of the form (2) we refer to [12], [19], [24-25], [34].

Before proceeding we recall some notions and results which will be frequently used later on. We define the bracket $[\cdot,\cdot]$ in a Banach space $\mathbf{Y}$ as follows (see [5], [19])

$$[x,y] = \lim_{h\to+0} \frac{\|x+hy\|-\|y\|}{h} = \inf_{h>0} \frac{\|x+hy\|-\|y\|}{h}$$

Definition 1. *Suppose that F is a given operator in a Banach space $\mathbf{Y}$. Then $(F+\gamma I)$ is said to be* accretive *if and only if for every $\lambda > 0$ one of the following equivalent conditions is satisfied*

i) $(1-\lambda\gamma)\|x-y\| \leq \|x-y+\lambda(Fx-Fy)\|,\ \forall x,y \in D(F)$,
ii) $[x-y, Fx-Fy] \geq -\gamma\|x-y\|,\ \forall x,y \in D(F)$.

In particular, if $\gamma = 0$, then F is said to be accretive.

Remark. From this definition we may conclude that $(F+\gamma I)$ is accretive if and only if

$$\|x-y\| \leq \|x-y+\lambda(Fx-Fy)\| + \lambda\gamma\|x-y\| \tag{4}$$

for all $x,y \in D(F), \lambda > 0, 1 \geq \lambda\gamma$.

Definition 2. (Condition H1). Equation (2) is said to satisfy *condition H1* if

i) A is the infinitesimal generator of a linear semigroup S(t), $t \geq 0$ of type ω in $\mathbf{X}$,
ii) B is an everywhere defined continuous operator from $\mathbf{R}\times\mathbf{X}$ to $\mathbf{X}$,
iii) For every fixed $t \in \mathbf{R}$, the operator $(-B(t,\cdot)+\gamma I)$ is accretive in $\mathbf{X}$.

The following well-known results are quoted for the reader's convenience (see [12], [19], [34]).

Theorem 1. *Let equation (2) satisfy condition $H1$. Then for every fixed $s \in$ $\mathbf{R}$ and $x \in \mathbf{X}$ there exists a unique mild solution $x(\cdot)$ of equation (2) defined on $[s,+\infty)$. Moreover, if B is independent of t, then the mild solutions of equation (2) give rise to a semigroup of nonlinear operators $T(t), t \geq 0$ having the following properties:*

$$i)\quad T(t)x = S(t)x + \int_0^t S(t-\xi)BT^{\xi}x\,d\xi,\ \forall t \geq 0, x \in \mathbf{X}, \tag{5}$$

$$ii)\quad \|T(t)x - T(t)y\| \leq e^{(\omega+\gamma)t}\|x-y\|,\ \forall t \geq 0, x,y \in \mathbf{X}. \tag{6}$$

Throughout this paper we shall denote by $AP(\mathbf{X})$ the space of almost periodic $\mathbf{X}$-valued functions in Bohr's sense with supremum norm, i.e. the space of continuous functions f from $\mathbf{R}$ to $\mathbf{X}$ such that the set $\{f(\cdot+s) : \mathbf{R}\to\mathbf{X} \mid s \in \mathbf{R}\}$ is precompact in the space of functions $\{f : \mathbf{R}\to\mathbf{X} \mid f \text{ bounded and continuous}\}$ equipped with the supremum norm.

The following condition will be used frequently:

Definition 3. (Condition H2). Equation (2) is said to satisfy *condition H2* if for every $u \in AP(\mathbf{X})$ the function $B(\cdot, u(\cdot))$ belongs to $AP(\mathbf{X})$ and if the operator B_* taking u into $B(\cdot, u(\cdot))$ is continuous.

The main point of our study is to associate with equation (2) an evolution semigroup which plays a role similar to that of the monodromy operator for equations with periodic cofficients. Hereafter we will denote by $U(t,s)$, $t \geq s$, the evolution operator corresponding to equation (2) which satisfies the assumptions of Theorem 1, i.e. $U(t,s)x$ is the unique solution of equation (3).

Proposition 1. *Let the conditions $H1$ and $H2$ be satisfied. Then with equation (2) one can associate an evolution semigroup $T^h, h \geq 0$ acting on $AP(\mathbf{X})$, defined as*

$$[T^h v](t) = U(t, t-h)v(t-h), \forall h \geq 0,\ t \in \mathbf{R}, v \in AP(\mathbf{X}).$$

Moreover, this semigroup has the following properties:

i) *$T^h, h \geq 0$ is strongly continuous, and*

$$T^h u = S^h u + \int_0^h S^{h-\xi} B_*(T^\xi u) d\xi,\ \forall h \geq 0, u \in AP(\mathbf{X}),$$

where $(S^h u)(t) = S(h)u(t-h), \forall h \geq 0, t \in \mathbf{R}, u \in AP(\mathbf{X})$.

ii)

$$\|T^h u - T^h v\| \leq e^{(\omega+\gamma)h} \|u - v\|,\ \forall h \geq 0, u, v \in AP(\mathbf{X}).$$

Proof. We first look at the solutions to the equation

$$w(t) = S^{t-a} z + \int_a^t S^{t-\xi} B_*(w(\xi)) d\xi\ \forall z \in AP(\mathbf{X}), t \geq a \in \mathbf{R}. \tag{7}$$

It may be noted that $S^h, h \geq 0$ is a strongly continuous semigroup of linear oparators in $AP(\mathbf{X})$ of type ω. Furthermore, for $\lambda > 0, \lambda\gamma < 1$ and $u, v \in AP(\mathbf{X})$, from the accretiveness of the operators $-B(t,\cdot) + \gamma I$ we get

$$\begin{aligned}(1-\lambda\gamma)\|x - y\| &= (1-\lambda\gamma) \sup_t \|u(t) - v(t)\| \\ &= \sup_t (1-\lambda\gamma)\|u(t) - v(t)\| \\ &\leq \sup_t \|u(t) - v(t) - \lambda[B(t, u(t)) - B(t, v(t))]\| \\ &= \|u - v - \lambda(B_* u - B_* v)\|.\end{aligned} \tag{8}$$

This shows that $(-B_* + \gamma I)$ is accretive . In virtue of Theorem 1 there exists a semigroup $T^h, h \geq 0$ such that

$$T^h u = S^h u + \int_0^h S^{h-\xi} B_* T^\xi u d\xi,$$

$$\|T^h u - T^h v\| \leq e^{(\omega+\gamma)h} \|u - v\|,\ \forall h \geq 0, u, v \in AP(\mathbf{X}).$$

From this we get

$$[T^h u](t) = [S^h u](t) + \int_0^h [S^{h-\xi} B_*(T^\xi u)](t) d\xi, \forall t \in \mathbf{R}.$$

Thus

$$\begin{aligned}[T^h u](t) &= S(h)u(t-h) + \int_0^h S(h-\xi)[B_*(T^\xi u](t-h+\xi)d\xi \\ &= S(h)u(t-h) + \int_0^h S(h-\xi)B(t+\xi-h,[T^u](t+\xi-h))d\xi \\ &= S(h)u(t-h) + \int_{t-h}^t S(t-\eta)B(\eta,[T^{\eta-(t-h)}u](\eta)d\eta.\end{aligned}$$

If we denote $[T^{t-s}u](t)$ by $x(t)$, we get

$$x(t) = S(t-s)z + \int_s^t S(t-\xi)B(\xi,x(\xi))d\xi, \forall t \geq s, \tag{9}$$

where $z = u(s)$. Consequently, from the uniqueness of mild solutions of equation (2) we get $[T^{t-s}u](t) = x(t) = U(t,s)u(s)$ and $[T^h u](t) = U(t,t-h)u(t-h)$ for all $t \geq s, u \in AP(Q)$. This completes the proof of the proposition. ∎

The main idea underlying our approach is the following assertion.

Corollary 1. *Let all assumptions of Proposition 1 be satisfied. Then a mild solution $x(t)$ of equation (1), defined on the whole real line $\mathbf{R}$, is almost periodic if and only if it is a common fixed point of the evolution semigroup $T^h, h \geq 0$ defined in Proposition 1 above.*

Proof. Suppose that $x(t)$, defined on the real line $\mathbf{R}$, is an almost periodic mild solution of equation (2). Then from the uniqueness of mild solutions we get

$$x(t) = U(t,t-h)x(t-h) = [T^h x](t), \ \forall t \in \mathbf{R}.$$

This shows that x is a fixed point of T^h for every $h > 0$. Conversely, suppose that $y(\cdot)$ is any common fixed point of $T^h, h \geq 0$. Then

$$y(t) = [T^{t-s}y](t) = U(t,s)y(s), \ \forall t \geq s.$$

This shows that $y(\cdot)$ is a mild solution of equation (2). ∎

We now apply Corollary 1 in order to get sufficient conditions for the existence of almost periodic mild solutions of equation (2).

Corollary 2. *Let all conditions of Proposition* 1 *be satisfied. Furthermore, let $\omega + \mu$ be negative and $-B_* - \mu I$ be accretive. Then there exists a unique almost periodic mild solution of equation (2).*

Proof. It is obvious that there exists a unique common fixed point of the semigroup $T^h, h \geq 0$. The assertion now follows from Corollary 1. ∎

Remark. A particular case in which we can check the accretiveness of $-B_* - \mu I$ is $\omega + \gamma < 0$. In fact, this follows easily from the above estimates for $\|u - v\|$ (see the estimate (8)).

Another "hyperbolic" case in which there exists a unique common fixed point for the semigroup $T^h, h \geq 0$ can be described as follows.

Definition 4. A semigroup $S(t), t \geq 0$ of linear operators in a Banach space $\mathbf{X}$ is said to be *hyperbolic* if there exist positive constants K, α and a bounded projection P of $\mathbf{X}$ with the following properties:

i) $PS(t) = S(t)P$, $\forall t \geq 0$,

ii) $(I-P)S(t)(I-P)$ is a homeomorphism of $\mathrm{Ker}P$, $\forall t \geq 0$,

iii) $\|[(I-P)S(t)(I-P)]^{-1}\| \leq Ke^{-\alpha t}$ and $\|PS(t)P\| \leq Ke^{\alpha t}$, $\forall t \geq 0$.

Corollary 3. *Let all conditions of Proposition 1 be satisfied. Moreover, let A be the infinitesimal generator of a hyperbolic semigroup and let $B(t,x)$ satisfy the estimate*

$$\|B(t,x) - B(t,y)\| \leq \delta\|x-y\|, \ \forall t \in \mathbf{R}, x, y \in \mathbf{X}.$$

Then, for δ sufficiently small, equation (2) has a unique almost periodic mild solution.

Proof. It may be seen that

$$\|U(t,s)x - U(t,s)y\| \leq e^{\omega(t-s)}\|x-y\| + \int_s^t e^{\omega(t-\xi)}\delta\|U(\xi,s)x - U(\xi,s)y\|d\xi.$$

Using Gronwall's inequality we get

$$\|U(t,s)x - U(t,s)y\| \leq e^{(\omega+\delta)(t-s)}\|x-y\|, \ \forall t \geq s, \ x, y \in \mathbf{X}.$$

Consequently,

$$\begin{aligned}
&\|(U(t,s)x - S(t-s)x) - (U(t,s)y - S(t-s)y)\| \leq \\
&\leq \delta \int_s^t e^{\omega(t-\xi)}\|U(\xi,s)x - U(\xi,s)y\|d\xi \leq \\
&\leq \delta \int_s^t e^{\omega(t-\xi)}e^{(\omega+\delta)(\xi-s)}d\xi \, \|x-y\| = \\
&= \delta(t-s)e^{t-s}\|x-y\|, \ \forall t \geq s, \ x, y \in \mathbf{X}.
\end{aligned}$$

Applying this in order to estimate the Lipschitz constant of $T^1 - S^1$ we get

$$\begin{aligned}
&\|(T^1 - S^1)u - (T^1 - S^1)v\| = \\
&= \sup_t \|[U(t,t-1)u(t-1) - S(1)u(t-1)] - \\
&\quad - [U(t,t-1)v(t-1) - S(1)v(t-1)\| \leq \\
&\leq \sup_t \delta e^{\omega}\|u(t) - v(t)\| \leq \delta\|u-v\|, \ \forall u, v \in AP(\mathbf{X}).
\end{aligned}$$

Since S^1 is hyperbolic, 1 does not belong to the spectrum $\sigma(S^1)$. Consequently, applying the Inverse Function Theorem for Lipschitz continuous mappings (see e.g. [19, Proposition 2.3, p. 67]) we observe that if $\delta e^{\omega} < \|(S^1 - I)^{-1}\|$, the maping $T^1 - I$ is invertible and then it has a unique fixed point. Finally, this is a unique common fixed point for the semigroup $T^h, h \geq 0$. Now the assertion of the corollary follows from Corollary 1. ∎

3. Almost Periodic Solutions of Differential Equations with Delays

In this section we apply the reults of the previous section in order to study the existence of almost periodic mild solutions of the equation

$$\frac{dx}{dt} = Ax + f(t, x, x_t) \tag{1}$$

where A is defined as in Section 2, and f is an everywhere defined continuous mapping from $\mathbf{R} \times \mathbf{X} \times C$ to $\mathbf{X}$. Hereafter we call a continuous function $x(t)$ defined on the real line $\mathbf{R}$ a mild solution of equation (1) if

$$x(t) = S(t-s)x(s) + \int_s^t S(t-\xi)f(\xi, x(\xi), x_\xi)d\xi, \ \forall t \geq s.$$

We refer the reader to [32] for more information on the existence and uniqueness of mild solutions of equations of the form (1). We should emphasize that our study is concerned only with the existence of almost periodic mild solutions of equation (1), and not with mild solutions in general. Consequently, the conditions guaranteeing the existence and uniqueness of mild solutions of equation (1) as general as in [32] are not supposed to be *a priori* conditions.

Definition 5. (Condition H3). Equation (1) is said to satisfy *Condition H3* if the following is true:

i) For every $g \in AP(\mathbf{X})$ the mapping $F(t, x) = f(t, x, g_t)$ satisfies conditions $H1$ and $H2$ with the same constant γ.
ii) There exists a constant μ with $\omega - \mu < 0$ such that $-(\mu I + F_*)$ is accretive for every $g \in AP(\mathbf{X})$.
iii) $[x - y, f(t, x, \phi) - f(t, y, \phi')] \leq \gamma\|x - y\| + \delta\|\phi - \phi'\|$, $\forall t \in \mathbf{R}, x, y \in \mathbf{X}, \phi, \phi' \in C$.

Theorem 2. *Let condition $H3$ hold. Then for δ sufficiently small (see the estimate (13) below), equation (1) has an almost periodic mild solution.*

Proof. First we fix a function $g \in AP(\mathbf{X})$. In view of Proposition 1 we observe that the equation

$$\frac{dx}{dt} = Ax + F(t, x)$$

has a unique almost periodic mild solution, where $F(t, x) = f(t, x, g_t)$. We denote this solution by Tg. Thus, we have defined an operator T acting on $AP(\mathbf{X})$. We now prove that T is a strict contraction mapping. In fact, let us denote by $U(t, s)$ and $V(t, s)$ the Cauchy operators

$$U(t,s)x = S(t-s)x + \int_s^t S(t-\xi)f(\xi, U(\xi,s)x, g_\xi)d\xi, \tag{10}$$

$$V(t,s)x = S(t-s)x + \int_s^t S(t-\xi)f(\xi, V(\xi,s)x, h_\xi)d\xi, \tag{11}$$

for given $g, h \in AP(\mathbf{X}), x \in \mathbf{X}, t \geq s$.

Putting $u(t) = U(t,s)x, v(t) = V(t,s)x$ for given s, x, from the assumptions we have

$$[u(t) - v(t), f(t, u(t), g_t) - f(t, v(t), h_t)] \leq m(t, \|u(t) - v(t)\|),$$

where $m(t, \|u(t) - v(t)\|) = \gamma\|u(t) - v(t)\| + \delta\|h - g\|$. Using this we get

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|u(t-\eta) - v(t-\eta)\| + \eta m(t, \|u(t) - v(t)\|) + \\ &+ \int_{t-\eta}^{t} \|S(t-\xi)f(\xi, u(\xi), h_\xi) - f(t, u(t), h_t)\| d\xi + \\ &+ \int_{t-\eta}^{t} \|S(t-\xi)f(\xi, v(\xi), g_\xi) - f(t, v(t), g_t)\| d\xi. \end{aligned}$$

Now let us fix arbitrary real numbers $a \leq b$. Since the functions $S(t-\xi)f(\xi, u(\xi), h_\xi)$ and $S(t-\xi)f(\xi, v(\xi), g_\xi)$ are uniformly continuous on the set $a \leq \xi \leq t \leq b$, for every $\epsilon > 0$ there exists an $\eta_0 = \eta_0(\epsilon)$ such that

$$\begin{aligned} \|S(t-\xi)f(\xi, u(\xi), h_\xi) - f(t, u(t), h_t)\| &< \epsilon, \\ \|S(t-\xi)f(\xi, v(\xi), g_\xi) - f(t, v(t), g_t)\| &< \epsilon, \end{aligned}$$

for all $\|t - \xi\| < \eta_0$ and $t \leq \xi \in [a, b]$. Hence, denoting $\|u(t) - v(t)\|$ by $\alpha(t)$, for $\eta < \eta_0$ we have

$$\alpha(t) - e^{\omega\eta}\alpha(t-\eta) \leq \eta m(t, \alpha(t)) + 2\eta\epsilon. \tag{12}$$

Applying this estimate repeatedly, we get

$$\alpha(t) - e^{\omega(t-s)} \leq \sum_{i=1}^{n} e^{\omega(t-t_i)} m(t_i, \alpha(t_i))\Delta_i + 2\epsilon \sum_{i=1}^{n} e^{\omega(t-t_i)}\Delta_i,$$

where $t_0 = s < t_1 < t_2 < ... < t_n = t$ and $|t_i - t_{i-1}| = \Delta_i$. Thus, since ϵ is arbitrary, and since the function m is continuous, we get

$$\begin{aligned} \alpha(t) - e^{\omega(t-s)}\alpha(s) &\leq \int_s^t e^{\omega(t-\xi)} m(\xi, \alpha(\xi)) d\xi \\ &= \int_s^t e^{\omega(t-\xi)} (\gamma\alpha(\xi) + \delta\|h - g\|) d\xi. \end{aligned}$$

Applying Gronwall's inequality we get

$$\alpha(t) \leq e^{(\gamma+\omega)(t-s)}\alpha(s) + e^{\gamma(t-s)+\omega t}\left(\frac{e^{-\omega s} - e^{-\omega t}}{\omega}\right)\delta\|h - g\|.$$

Because of the identity $\alpha(s) = \|u(s) - v(s)\| = \|U(s,s)x - V(s,s)x\| = 0$, from the above estimate we obtain

$$\sup_{t-1 \leq \xi \leq t} \|U(\xi, t-1)x - V(\xi, t-1)x\| \leq \frac{e^{\gamma+\omega} - e^{\gamma}}{\omega}\delta\|h - g\|.$$

Now let us denote by $T_h^t, T_g^t, t \geq 0$ the respective evolution semigroups corresponding to equations (10) and (11). Since Th and Tg are defined as the unique fixed points u_0, v_0 of T_h^1, T_g^1, respectively, we have

$$\begin{aligned}\|Th - Tg\| = \|u_0 - v_0\| &= \|T_h^1 u_0 - T_g^1 v_0\| \le \|T_h^1 u_0 - T_g^1 u_0\| + \\ &\quad + \|T_g^1 u_0 - T_g^1 v_0\| \\ &\le \frac{e^{\gamma+\omega} - e^{\gamma}}{\omega}\delta\|h-g\| + e^{\omega-\mu}\|u_0 - v_0\| \\ &= N\delta\|h-g\| + e^{\omega-\mu}\|Th - Tg\|,\end{aligned}$$

where $N = (e^{\gamma+\omega} - e^{\gamma})/\omega$. Finally, we have

$$\|Th - Tg\| \le \frac{e^{\gamma}(e^{\omega}-1)}{\omega(1-e^{\omega-\mu})}.$$

Thus, if the estimate

$$\delta < \frac{\omega(1-e^{\omega-\mu})}{e^{\gamma}(e^{\omega}-1)} \tag{13}$$

holds true, then T is a strict contraction mapping in $AP(\mathbf{X})$. By virtue of the Contraction Mapping Principle T has a unique fixed point. It is easy to see that this fixed point is an almost periodic mild solution of equation (1). This completes the proof of the theorem. ∎

Remarks. 1) In case $\omega = 0, \gamma = -\mu$ we get the estimate

$$\delta < e^{\mu} - 1 = \mu + \mu^2/2 + \ldots$$

which guarantees the existence of the fixed point of T.

2) If $\omega + \gamma < 0$, then we can choose $\mu = -\gamma$, and therefore we get the accretiveness condition on $-(F_* + \mu I)$. However, in general, the condition $\omega + \gamma < 0$ is a very strong restriction on the coefficients of equation (1), if f depends explicitly on t.

4. Examples

In applications one frequently encounters functions f from $\mathbf{R}\times\mathbf{X}\times C \to \mathbf{X}$ of the form

$$f(t,x,g_t) = F(t,x) + G(t,g_t),\ \forall t \in \mathbf{R},\ x \in \mathbf{X},\ g_t \in C,$$

where F satisfies condition ii) of Definition 5 and $G(t,y)$ is Lipschitz continuous with respect to $y \in C$, i.e.

$$\|G(t,y) - G(t,z)\| \le \delta\|y - z\|,\ \forall t \in \mathbf{R},\ y,z \in C$$

for some positive constant δ. With f in this form numerous examples of partial functional differential equations fitting into our abstract framework can be found (see e.g. [2], [18], [19], [26], [27], in particular the recent book by Wu [35]).

In order to describe a concrete example we consider a bounded domain Ω in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and suppose that

$$A(x,D)u = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha u$$

is a strongly elliptic differential operator in Ω. Then, defining the operator

$$Au = A(x,D)u,\ \forall u \in D(A) = W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$$

we know from [27, Theorem 3.6] that the operator $-A$ is the infinitesimal generator of an analytic semigroup of contractions on $L^2(\Omega)$. Now let $f, g : \mathbf{R}\times\Omega\times\mathbf{R} \to \mathbf{R}$ be Lipschiz continuous and define the operators $F(t,w)(x) = f(t,x,w(x))$ and $G(t,w)(x) = g(t,x,w(x))$ where $t \in \mathbf{R}$, $x \in \Omega$ and $w \in L^2(\Omega)$. Then, for any positive constant r, the boundary value problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= A(x,D)u(t,x) + f(t,x,u(t,x)) + g(t,x,u(t-r,x)) \quad \text{in } \Omega\ , \\ u(t,x) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

fits into the abstract setting of equation (1).

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