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Research Article

Least-Norm of the General Solution to Some System of Quaternion Matrix Equations and Its Determinantal Representations

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We constitute some necessary and sufficient conditions for the system $A_1X_1 = C_1$, $X_1B_1 = C_2$, $A_2X_2 = C_3$, $X_2B_2 = C_4$, $A_3X_1B_3 + A_4X_2B_4 = C_c$, to have a solution over the quaternion skew field in this paper. A novel expression of general solution to this system is also established when it has a solution. The least norm of the solution to this system is also researched in this article. Some former consequences can be regarded as particular cases of this article. Finally, we give determinantal representations (analogs of Cramer's rule) of the least norm solution to the system using row-column noncommutative determinants. An algorithm and numerical examples are given to elaborate our results.

1. Introduction

In the whole article, the notation $\mathbb{R}$ is reserved for the real number field and $\mathbb{H}^{m\times n}$ stands for the set of all $m\times n$ matrices over the quaternion skew field

$$\mathbb{H} = \{b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, b_0, b_1, b_2, b_3 \in \mathbb{R}\}. \tag{1}$$

$\mathbb{H}_r^{m\times n}$ specifies its subset of matrices with a rank r. For $A \in \mathbb{H}^{m\times n}$, let A^*, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ designate the conjugate transpose, the column right space and the left row space of A. $\dim \mathscr{R}(A)$ illustrates the size of $\mathscr{R}(A)$ and $\dim \mathscr{R}(A)=\dim \mathscr{N}(A)$ by [1], which is known as the rank of A denoted by $r(A)$.

Definition 1. The Moore-Penrose inverse of $A \in \mathbb{H}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the following four matrix equations

$$\begin{aligned} &(1)\ AXA = A,\\ &(2)\ XAX = X,\\ &(3)\ (AX)^* = AX,\\ &(4)\ (XA)^* = XA. \end{aligned} \tag{2}$$

Matrices satisfying (1) and (2) are known as reflexive inverses.

Note that the reflexive inverse is denoted most often by A_r^- but sometimes by A^+ (see, e.g., [2]) that is different from the denotation of the Moore-Penrose by $A^\dagger$. We will use the denotation A^+ for the reflexive inverse.

Suppose I refers an identity matrix with feasible size. In addition, $R_A = I - AA^\dagger$, $L_A = I - A^\dagger A$ represent a pair of orthogonal projectors induced by A, respectively, and $R_A^2 = R_A$, $R_A^* = R_A$, $L_A^2 = L_A$, $L_A^* = L_A$, and $R_{A^*} = L_A$.

Quaternions were invented by Hamilton in 1843. Zhang presented a detail survey on quaternion matrices in [3]. Quaternions provide a concise mathematical method for representing the automorphisms of three- and four-dimensional spaces. The representations by quaternions are more compact and quicker to compute than the representations by matrices [4]. For this reason, an increasing number of applications based on quaternions are found in various fields, such as color imaging, geometry, mechanics, linear adaptive filter, altitude control and computer science, signal processing, in particular as quaternion-valued neural networks, etc. [5–10].

The research of matrix equations have both applied and theoretical importance. In particular, the Sylvester-type matrix equations have far reaching applications in singular system control [11], system design [12], robust control [13], feedback [14], perturbation theory [15], linear descriptor systems [16], neural networks [17], and theory of orbits [18].

Some recent work on generalized Sylvester matrix equations and their systems can be observed in [19–31]. In 2014, Bao [32] examined the least-norm and extremal ranks of the least square solution to the quaternion matrix equations

$$\begin{aligned} A_1X &= C_1, \\ XB_1 &= C_2, \\ A_3XB_3 &= C_c. \end{aligned} \tag{3}$$

Wang et al. [33] examined the expression of the general solution to the system

$$\begin{aligned} A_1X_1 &= C_1, \\ A_2X_2 &= C_3, \\ A_3X_1B_3 + A_4X_2B_4 &= C_c, \end{aligned} \tag{4}$$

And, as an application, the P-symmetric and P-skew-symmetric solution to

$$\begin{aligned} A_aX &= C_a, \\ A_bXB_b &= C_b \end{aligned} \tag{5}$$

has been established. Li et al. [34] established a novel expression to the general solution of system (4) and they computed the least-norm of general solution to (4). In 2009, Wang et al. [35] constituted the expression of the general solution to

$$\begin{aligned} A_1X_1 &= C_1, \\ X_1B_1 &= C_2, \\ A_2X_2 &= C_3, \\ X_2B_2 &= C_4, \\ A_3X_1B_3 + A_4X_2B_4 &= C_c, \end{aligned} \tag{6}$$

and as an application they explored the (P, Q)-symmetric solution to the system

$$\begin{aligned} A_aX &= C_a, \\ XB_b &= C_b, \\ A_cXB_c &= C_c. \end{aligned} \tag{7}$$

Some latest findings on the least-norm of matrix equations and (P, Q)-symmetric matrices can be consulted in [36–40]. Furthermore, our main system (6) is a special case of the following system:

$$\begin{aligned} A_1X_1 &= C_1, \\ X_2B_1 &= D_1, \\ A_2X_3 &= C_2, \\ X_3B_2 &= D_2, \\ A_3X_4 &= C_3, \\ X_4B_3 &= D_3, \\ A_4X_1 + X_2B_4 + C_4X_3D_4 + C_5X_4D_5 &= C_c, \end{aligned} \tag{8}$$

which has been investigated by Zhang in 2014. But the expressions provided for the X_1, X_2, X_3, and X_4 in [41], we are in position to calculate the least-norm of the solutions with its determinantal representations. When some given matrices are zero in (8), then it becomes our system and we will give such kind of expressions in which the least-norm of the solutions can also be computed with its determinantal representations. It is worthy to note that Zhang examined (8) with complex settings and we will consider our system (6) with quaternion settings.

According to our best of knowledge, the least-norm of the general solution to system (6) is not investigated by any one. Motivated by the vast application of quaternion matrices and the latest interest of least-norm of matrix equations, we construct a novel expression of the general solution to system (6) and apply this to investigate the least-norm of the general solution to system (6) over $\mathbb{H}$ in this paper. Observing that systems (3) and (4) are particular cases of our system (6), solving system (6) will encourage the least-norm to a wide class of problems in the collected work.

Since the general solutions of considered systems are expressed in term of generalized inverses, another goal of the paper is to give determinantal representations of the least-norm of the general solution to system (6) based on determinantal representations of generalized inverses.

Determinantal representation of a solution gives a direct method of its finding analogous to the classical Cramer's rule that has important theoretical and practical significance. Through looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses even with the complex or real entries, in particular for the Moore-Penrose inverse (see, e.g., [42–44]). By virtue of noncommutativity of quaternions, the problem for determinantal representation of generalized quaternion inverses is more complicated, and only now it can be solved due to the theory of column-row determinants introduced in [45, 46]. Within the framework of the theory of noncommutative row-column determinants, determinantal representations of various generalized quaternion inverses and generalized inverse solutions to quaternion matrix equations have been derived by one of our authors (see, e.g.[47–54]) and by other researchers (see, e.g. [55–57]). Moreover, Song et al. [58] have just recently considered determinantal representations of general solution to the two-sided coupled generalized Sylvester matrix equation over $\mathbb{H}$ obtained using the theory of row-column determinants as well. But their proposed approach differs from our proposed. In [58], for determinantal representations of the general solution to the equation supplementary matrices have been used that not always easy to get. While, by proposed method only coefficient matrices of the equations are used. More detailed Cramer's rule to solutions and (skew-)Hermitian solutions of some systems of matrix equations and generalized Sylvester matrix equation over $\mathbb{H}$ are recently explored in [59, 60] and [61, 62], respectively.

The remainder of our article is directed as follows. In Section 2, we commence with some needed known results about systems of matrix equations and determinantal representations of the Moore-Penrose inverse and of solutions to the quaternion matrix equations. In Section 3, we provide a new expression of the general solution to our system (6) and present an algorithm with an example. We discuss the least-norm of the general solution to (6) over $\mathbb{H}$ in Section 4. In Section 5, determinantal representations of the general solution to (6) are derived and other example to elaborate obtained Cramer's Rule to system (6) with data from the example in Section 3 is given. As expected, we get the same solution. Finally, in Section 6, the conclusions are drawn.

2. Preliminaries

We commence with the following lemmas which have crucial function in the construction of the chief outcomes of the following sections.

2.1. The General Solution to System (6)

Lemma 2 (see [63]). *Let $A \in \mathbb{H}^{s\times t}$, $B \in \mathbb{H}^{s\times k}$, and $C \in \mathbb{H}^{l\times t}$ be given. Then*

(1) $r(A) + r(R_A B) = r(B) + r(R_B A) = r\begin{bmatrix} A & B \end{bmatrix}$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix} A \\ C \end{bmatrix}$.

(3) $r(B) + r(C) + r(R_B A L_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$.

Lemma 3 (see [64]). *Let A, B, and C be known matrices over $\mathbb{H}$ with right sizes. Then*

(1) $A^{\dagger} = (A^*A)^{\dagger}A^* = A^*(AA^*)^{\dagger}$.

(2) $L_A = L_A^2 = L_A^*$, $R_A = R_A^2 = R_A^*$.

(3) $L_A(BL_A)^{\dagger} = (BL_A)^{\dagger}$, $(R_AC)^{\dagger}R_A = (R_AC)^{\dagger}$.

Lemma 4 (see [65]). *Let Φ, Ω be matrices over $\mathbb{H}$ and*

$$\begin{aligned} \Phi &= \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \\ \Omega &= \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix}, \\ F &= \Phi_2 L_{\Phi_1}, \\ T &= R_{\Omega_1}\Omega_2. \end{aligned} \tag{9}$$

Then

$$\begin{aligned} L_{\Phi} &= L_{\Phi_1} L_F, \\ L_{\Omega} &= \begin{bmatrix} L_{\Omega_1} & -\Omega_1^{\dagger}\Omega_2 L_T \\ 0 & L_T \end{bmatrix}, \\ R_{\Omega} &= R_T R_{\Omega_1}, \\ R_{\Phi} &= \begin{bmatrix} R_{\Phi_1} & 0 \\ -R_F \Phi_2 \Phi_1^{\dagger} & R_F \end{bmatrix}, \end{aligned} \tag{10}$$

where Φ_1^{+}, Ω_1^{+} are any fixed reflexive inverses, L_{Φ_1} and R_{Ω_1} stand for the projectors $L_{\Phi_1} = I - \Phi_1^{+}\Phi_1$, $R_{\Omega_1} = I - \Omega_1\Omega_1^{+}$ induced by Φ_1, Ω_1, respectively.

Remark 5. Since Moore-Penrose inverses are reflexive inverses, this lemma can be used for Moore-Penrose inverses without any changes. It has taken place in ([64], Lemma 2.4). But for more credibility, we prove this lemma below for the Moore-Penroses inverse as well.

Lemma 6 (see [66]). *Suppose that*

$$B_1 X C_1 + B_2 Y C_2 = A \tag{11}$$

is consistent linear matrix equation, where $B_1 \in \mathbb{H}^{m\times p}$, $C_1 \in \mathbb{H}^{q\times n}$, $B_2 \in \mathbb{H}^{m\times s}$, $C_2 \in \mathbb{H}^{t\times n}$ and $A \in \mathbb{H}^{m\times n}$, respectively. Then

(1) *The general solution of the homogeneous equation,*

$$B_1 X C_1 + B_2 Y C_2 = 0, \tag{12}$$

can be expressed by

$$\begin{aligned} X &= X_1 X_2 + X_3, \\ Y &= Y_1 Y_2 + Y_3, \end{aligned} \tag{13}$$

where $X_1 - X_3$ and $Y_1 - Y_3$ are general solution of the following four homogeneous matrix expressions

$$\begin{aligned} B_1 X_1 &= -B_2 Y_1, \\ X_2 C_1 &= Y_2 C_2, \\ B_1 X_3 C_1 &= 0, \\ B_2 Y_3 C_2 &= 0. \end{aligned} \tag{14}$$

By computing the value of unknowns in the above equations and using them in X and Y, we have

$$\begin{aligned} X &= S_1 L_G U R_H T_1 + L_{B_1} V_1 + V_2 R_{C_1}, \\ Y &= S_2 L_G U R_H T_2 + L_{B_2} W_1 + W_2 R_{C_2}, \end{aligned} \tag{15}$$

where $S_1 = [I_p, \ 0]$, $S_2 = [0, \ I_s]$, $T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}$, $G = [B_1, \ B_2]$, and $H = \begin{bmatrix} C_1 \\ -C_2 \end{bmatrix}$; the matrices U, V_1, V_2, W_1 and W_2 are free to vary over $\mathbb{H}$.

(2) *Assume that the matrix expression (11) is solvable, then its general solution can be expressed as*

$$X = X_0 + X_1 X_2 + X_3,$$
$$Y = Y_0 + Y_1 Y_2 + Y_3, \tag{16}$$

where X_0 and Y_0 are any pair of particular solutions to (11).

It can also be written as

$$X = X_0 + S_1 L_G U R_H T_1 + L_{B_1} V_1 + V_2 R_{C_1},$$
$$Y = Y_0 + S_2 L_G U R_H T_2 + L_{B_2} W_1 + W_2 R_{C_2}. \tag{17}$$

Lemma 7 (see [67]). *Let $A_1 \in \mathbb{H}^{m_1\times n_1}$, $B_1 \in \mathbb{H}^{r_1\times s_1}$, $C_1 \in \mathbb{H}^{m_1\times r_1}$, and $C_2 \in \mathbb{H}^{n_1\times s_1}$ be given and $X_1 \in \mathbb{H}^{n_1\times r_1}$ to be determined. Then the system*

$$A_1 X_1 = C_1,$$
$$X_1 B_1 = C_2, \tag{18}$$

is consistent if and only if

$$R_{A_1} C_1 = 0,$$
$$C_2 L_{B_1} = 0, \tag{19}$$
$$A_1 C_2 = C_1 B_1.$$

Under these conditions, the general solution to (18) can be established as

$$X_1 = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + L_{A_1} U_1 R_{B_1}, \tag{20}$$

where U_1 is a free matrix over $\mathbb{H}$ with accordant dimension.

Lemma 8 (see [35]). *Let $A_1 \in \mathbb{H}^{m_1\times p_1}$, $B_1 \in \mathbb{H}^{q_1\times n_1}$, $C_1 \in \mathbb{H}^{m_1\times q_1}$, $C_2 \in \mathbb{H}^{p_1\times n_1}$, $A_2 \in \mathbb{H}^{m_2\times p_2}$, $B_2 \in \mathbb{H}^{q_2\times n_2}$, $C_3 \in \mathbb{H}^{m_2\times q_2}$, $C_4 \in \mathbb{H}^{p_2\times n_2}$,, $A_3 \in \mathbb{H}^{s\times p_1}$, $B_3 \in \mathbb{H}^{q_1\times t}$, $A_4 \in \mathbb{H}^{s\times p_2}$, $B_4 \in \mathbb{H}^{q_2\times t}$, $C_c \in \mathbb{H}^{s\times t}$ be given and $X_1 \in \mathbb{H}^{p_1\times q_1}$, $X_2 \in \mathbb{H}^{p_2\times q_2}$ to be determined. Denote*

$$A = A_3 L_{A_1},$$
$$B = R_{B_1} B_3,$$
$$C = A_4 L_{A_2},$$
$$D = R_{B_2} B_4,$$
$$N = D L_B, \tag{21}$$
$$M = R_A C,$$
$$S = C L_M,$$
$$E = C_c - A_3 A_1^{\dagger} C_1 B_3 - A C_2 B_1^{\dagger} B_3 - A_4 A_2^{\dagger} C_3 B_4$$
$$- C C_4 B_2^{\dagger} B_4.$$

Then the following conditions are tantamount:

(1) *System (6) is resolvable.*

(2) *The conditions in (19) are met and*

$$R_{A_2} C_3 = 0,$$
$$C_4 L_{B_2} = 0,$$
$$A_2 C_4 = C_3 B_2,$$
$$R_M R_A E = 0, \tag{22}$$
$$R_A E L_D = 0,$$
$$E L_B L_N = 0,$$
$$R_C E L_B = 0.$$

(3) *The equalities in (19) and (22) are satisfied and*

$$M M^{\dagger} R_A D^{\dagger} D = R_A E,$$
$$C C^{\dagger} E L_B N^{\dagger} N = E L_B. \tag{23}$$

In these conditions, the general solution to system (6) can be written as

$$X_1 = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + L_{A_1} A^{\dagger} E B^{\dagger} R_{B_1}$$
$$- L_{A_1} A^{\dagger} C M^{\dagger} E B^{\dagger} R_{B_1}$$
$$- L_{A_1} A^{\dagger} S C^{\dagger} E N^{\dagger} D B^{\dagger} R_{B_1} \tag{24}$$
$$- L_{A_1} A^{\dagger} S V_1 R_N D B^{\dagger} R_{B_1}$$
$$+ L_{A_1} \left(L_A U_1 + Z_1 R_B \right) R_{B_1},$$

$$X_2 = A_2^{\dagger} C_3 + L_{A_2} C_4 B_2^{\dagger} + L_{A_2} M^{\dagger} R_A E D^{\dagger} R_{B_2}$$
$$+ L_{A_2} L_{M_b} S^{\dagger} S C^{\dagger} E N^{\dagger} R_{B_2}$$
$$+ L_{A_2} L_M \left(V_1 - S^{\dagger} S V_1 N N^{\dagger} \right) R_{B_2} \tag{25}$$
$$+ L_{A_2} W_1 R_D R_{B_2},$$

where U_1, V_1, W_1 and Z_1 are free matrices over $\mathbb{H}$ with agreeable dimensions.

2.2. Determinantal Representations of Solutions to the Quaternion Matrix Equations. Due to noncommutativity of quaternions there is a problem of a determinant of matrices with noncommutative entries (which are also defined as noncommutative determinants). There are several versions of defining of noncommutative determinants (e.g., see [68–70]). But any of the previous noncommutative determinants has not fully retained those properties which it owned for matrices with real settings. Moreover, if functional properties of a noncommutative determinant over a ring are satisfied, then it takes on a value in its commutative subset. This dilemma can be avoided due to the theory of row-column determinants.

For $A \in \mathbb{H}^{n\times n}$, we define n row determinants and n column determinants. Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 9 (see [45]). *The ith row determinant* of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined for all $i = 1, \ldots, n$ by putting

$$\begin{aligned} \mathrm{rdet}_i A = & \sum_{\sigma\in S_n} (-1)^{n-r} \left(a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}\right) \\ & \ldots \left(a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}\right), \\ \sigma = & \left(i\ i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}\right) \\ & \ldots \left(i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}\right), \end{aligned} \tag{26}$$

where σ is the left-ordered permutation. It means that its first cycle from the left starts with i, other cycles start from the left with the minimal of all the integers which are contained in it,

$$i_{k_t} < i_{k_t+s} \quad \text{for all } t = 2, \ldots, r,\ s = 1, \ldots, l_t, \tag{27}$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from left to right of their first elements, $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$.

Definition 10 (see [45]). *The jth column determinant* of $A = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined for all $j = 1, \ldots, n$ by putting

$$\begin{aligned} \mathrm{cdet}_j A = & \sum_{\tau\in S_n} (-1)^{n-r} \left(a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}}\right) \\ & \ldots \left(a_{j j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} \tau = & \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right) \\ & \cdot \left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right), \end{aligned} \tag{29}$$

noindent where τ is the right-ordered permutation. It means that its first cycle from the right starts with j, other cycles start from the right with the minimal of all the integers which are contained in it,

$$j_{k_t} < j_{k_t+s} \quad \text{for all } t = 2, \ldots, r,\ s = 1, \ldots, l_t, \tag{30}$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from right to left of their first elements, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$.

Since [45] for Hermitian $\mathbf{A}$ we have

$$\mathrm{rdet}_1 \mathbf{A} = \cdots = \mathrm{rdet}_n \mathbf{A} = \mathrm{cdet}_1 \mathbf{A} = \cdots = \mathrm{cdet}_n \mathbf{A} \in \mathbb{R}, \tag{31}$$

the determinant of a Hermitian matrix is defined by putting

$$\det \mathbf{A} := \mathrm{rdet}_i \mathbf{A} = \mathrm{cdet}_i \mathbf{A} \quad \text{for all } i = 1, \ldots, n. \tag{32}$$

Its properties are similar to the properties of an usual (commutative) determinant and they have been completely explored in [46] by using row and column determinants that are so defined only by construction.

For determinantal representations of the Moore-Penrose inverse, we shall use the following notations. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \le k \le \min\{m, n\}$. Let A^{α}_{β} be a submatrix of A whose rows are indexed by α and the columns indexed by β. Similarly, let A^{α}_{α} be a principal submatrix of A whose rows and columns indexed by α. If $A \in \mathbb{H}^{n\times n}$ is Hermitian, then $|A|^{\alpha}_{\alpha}$ is the corresponding principal minor of $\det A$. For $1 \le k \le n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \le \alpha_1 < \cdots < \alpha_k \le n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$ denotes the collection of sequences of row indexes that contain the index i, and $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$ denotes the collection of sequences of column indexes that contain j.

Let $a_{.j}$ be the jth column and $a_{i.}$ be the ith row of A, respectively. Suppose $A_{.j}(b)$ denotes the matrix obtained from A by replacing its jth column with the column-vector b, and $A_{i.}(b)$ denotes the matrix obtained from A by replacing its ith row with the row-vector b. We denote the ith row and the jth column of A^* by $a^*_{i.}$ and $a^*_{.j}$, respectively.

Lemma 11 (see [47]). *If $A \in \mathbb{H}^{m\times n}_r$, then the Moore-Penrose inverse $A^{\dagger} = (a^{\dagger}_{ij}) \in \mathbb{H}^{n\times m}$ have the following determinantal representations,*

$$a^{\dagger}_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \mathrm{cdet}_i \left(\left(A^*A\right)_{.i}\left(a^*_{.j}\right)\right)^{\beta}_{\beta}}{\sum_{\beta\in J_{r,n}} \left|A^*A\right|^{\beta}_{\beta}}, \tag{33}$$

and

$$a^{\dagger}_{ij} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \mathrm{rdet}_j \left(\left(AA^*\right)_{j.}\left(a^*_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum_{\alpha\in I_{r,m}} \left|AA^*\right|^{\alpha}_{\alpha}}. \tag{34}$$

Remark 12. For an arbitrary full-rank matrix $A \in \mathbb{H}^{m\times n}_r$, a row-vector $b \in \mathbb{H}^{1\times m}$, and a column-vector $c \in \mathbb{H}^{n\times 1}$, we put

$$\begin{aligned} & (i)\ \ \mathrm{rdet}_i \left(\left(AA^*\right)_{i.}(b)\right) = \sum_{\alpha\in I_{m,m}\{i\}} \mathrm{rdet}_i \left(\left(AA^*\right)_{i.}(b)\right)^{\alpha}_{\alpha}, \\ & \det\left(AA^*\right) = \sum_{\alpha\in I_{m,m}} \left|AA^*\right|^{\alpha}_{\alpha}, \quad \text{when } r = m, \\ & (ii)\ \ \mathrm{cdet}_j \left(\left(A^*A\right)_{.j}(c)\right) \\ & \quad = \sum_{\beta\in J_{n,n}\{j\}} \mathrm{cdet}_j \left(\left(A^*A\right)_{.j}(c)\right)^{\beta}_{\beta}, \\ & \det\left(A^*A\right) = \sum_{\beta\in J_{n,n}} \left|A^*A\right|^{\beta}_{\beta}, \quad \text{when } r = n. \end{aligned} \tag{35}$$

Corollary 13. *If $A \in \mathbb{H}^{m\times n}_r$, then the projection matrix $A^{\dagger}A =: Q_A = (q_{ij})_{n\times n}$ has the determinantal representation*

$$q_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \mathrm{cdet}_i \left(\left(A^*A\right)_{.i}\left(\dot{a}_{.j}\right)\right)^{\beta}_{\beta}}{\sum_{\beta\in J_{r,n}} \left|A^*A\right|^{\beta}_{\beta}}, \tag{36}$$

where $\dot{a}_{.j}$ is the jth column of $A^*A \in \mathbb{H}^{n\times n}$.

Corollary 14. *If $A \in \mathbb{H}_r^{m\times n}$, then the projection matrix $AA^\dagger =: P_A = (p_{ij})_{m\times m}$ has the determinantal representation*

$$p_{ij} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j \left((AA^*)_{j.} (\ddot{a}_{i.}) \right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,m}} |AA^*|_\alpha^\alpha}, \quad (37)$$

where $\ddot{a}_{i.}$ is the ith row of $AA^* \in \mathbb{H}^{m\times m}$.

Lemma 15 (see [2]). *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{r\times s}$, $C \in \mathbb{H}^{m\times s}$ be known and $X \in \mathbb{H}^{n\times r}$ be unknown. Then the matrix equation*

$$AXB = C \quad (38)$$

is consistent if and only if $AA^\dagger CB^\dagger B = C$. In this case, its general solution can be expressed as

$$X = A^\dagger CB^\dagger + L_A V + WR_B, \quad (39)$$

where V, W are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

In [71], it's proved that (39) is the least squares solution to (38), and its minimum norm least squares solution is $X_{LS} = A^\dagger CB^\dagger$.

Lemma 16 (see [48]). *Let $A \in \mathbb{H}_{r_1}^{m\times n}$, $B \in \mathbb{H}_{r_2}^{r\times s}$. Then the minimum norm least squares solution $X = A^\dagger CB^\dagger = (x_{ij}) \in \mathbb{H}^{n\times r}$ to (38) have determinantal representations,*

$$x_{ij} = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A^*A)_{.i} \left(d_{.j}^B \right) \right)_\beta^\beta}{\sum_{\beta\in J_{r_1,n}} |A^*A|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} |BB^*|_\alpha^\alpha}, \quad (40)$$

or

$$x_{ij} = \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((BB^*)_{j.} \left(d_{i.}^A \right) \right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} |A^*A|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} |BB^*|_\alpha^\alpha}, \quad (41)$$

where

$$d_{.j}^B = \left[\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((BB^*)_{j.} (\tilde{c}_{k.}) \right)_\alpha^\alpha \right] \in \mathbb{H}^{n\times 1}, \quad k = 1, \ldots, n,$$
$$d_{i.}^A = \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A^*A)_{.i} (\tilde{c}_{.l}) \right)_\beta^\beta \right] \in \mathbb{H}^{1\times r}, \quad l = 1, \ldots, r, \quad (42)$$

are the column vector and the row vector, respectively. $\tilde{c}_{k.}$ and $\tilde{c}_{.l}$ are the kth row and the lth column of $\widetilde{C} = A^*CB^*$.

Corollary 17. *Let $A \in \mathbb{H}_k^{m\times n}$, $C \in \mathbb{H}^{m\times s}$ be known and $X \in \mathbb{H}^{n\times s}$ be unknown. Then the matrix equation $AX = C$ is consistent if and only if $AA^\dagger C = C$. In this case, its general solution can be expressed as $X = A^\dagger C + L_A V$, where V is an arbitrary matrix over $\mathbb{H}$ with appropriate dimensions. Its minimum norm least squares solution $X = A^\dagger C$ has the following determinantal representation,*

$$x_{ij} = \frac{\sum_{\beta\in J_{k,n}\{i\}} \operatorname{cdet}_i \left((A^*A)_{.i} (\hat{c}_{.j}) \right)_\beta^\beta}{\sum_{\beta\in J_{k,n}} |A^*A|_\beta^\beta}, \quad (43)$$

where $\hat{c}_{.j}$ is the jth column of $\widehat{C} = A^*C$.

Corollary 18. *Let $B \in \mathbb{H}_k^{r\times s}$, $C \in \mathbb{H}^{n\times s}$ be given, and $X \in \mathbb{H}^{n\times r}$ be unknown. Then the equation $XB = C$ is solvable if and only if $C = CB^\dagger B$ and its general solution is $X = CB^\dagger + WR_B$, where W is a any matrix with conformable dimension. Moreover, its minimum norm least squares solution $X = CB^\dagger$ has the determinantal representation,*

$$x_{ij} = \frac{\sum_{\alpha\in I_{k,r}\{j\}} \operatorname{rdet}_j \left((BB^*)_{j.} (\hat{c}_{i.}) \right)_\alpha^\alpha}{\sum_{\alpha\in I_{k,r}} |BB^*|_\alpha^\alpha}, \quad (44)$$

where $\hat{c}_{i.}$ is the ith row of $\widehat{C} = CB^*$.

3. A New Expression of the General Solution to System (6)

First, we show that Lemma 4 is true for the Moore-Penrose inverses.

Lemma 19. *Let Φ, Ω be matrices over $\mathbb{H}$ and*

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix}, \quad F = \Phi_2 L_{\Phi_1}, \quad T = R_{\Omega_1}\Omega_2. \quad (45)$$

Then

$$L_\Phi = L_{\Phi_1} L_F, \quad L_\Omega = \begin{bmatrix} L_{\Omega_1} & -\Omega_1^\dagger \Omega_2 L_T \\ 0 & L_T \end{bmatrix}, \quad (46)$$

$$R_\Omega = R_T R_{\Omega_1}, \quad R_\Phi = \begin{bmatrix} R_{\Phi_1} & 0 \\ -R_F \Phi_2 \Phi_1^\dagger & R_F \end{bmatrix}. \quad (47)$$

where $\Omega_1^\dagger$, $\Phi_1^\dagger$ are the Moore-Penrose inverses, and L_{Φ_1}, R_{Ω_1}, L_T, R_F, L_Ω, and R_Φ are projectors with respect to the corresponding Moore-Penrose inverses.

Proof. In ([65], Lemma 2.4), it is proved that for fixed reflexive inverses Ω_1^+ and T^+, the reflexive inverse Ω^+ can be expressed as follows,

$$\Omega^+ = \begin{bmatrix} \Omega_1^+ - \Omega_1^+ \Omega_2 T^+ R_{\Omega_1} \\ T^+ R_{\Omega_1} \end{bmatrix}. \quad (48)$$

We choose $\Omega_1^\dagger, T^\dagger$ as the Moore-Penrose inverses, and R_{Ω_1} as the projector with respect to the Moore-Penrose inverse $\Omega_1^\dagger$ and show that the obtained matrix

$$\Omega^\dagger = \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger R_{\Omega_1} \\ T^\dagger R_{\Omega_1} \end{bmatrix} \quad (49)$$

is the Moore-Penrose inverse of Ω. For this, it is enough to proof that $\Omega^\dagger$ satisfies the conditions (3) and (4) in Definition 1.

Since by Lemma 3, $T^\dagger R_{\Omega_1} = (R_{\Omega_1}\Omega_2)^\dagger R_{\Omega_1} = T^\dagger$, then $\Omega^\dagger$ can be expressed as

$$\Omega^\dagger = \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger \\ T^\dagger \end{bmatrix}. \quad (50)$$

So,

$$\begin{aligned} \Omega\Omega^\dagger &= [\Omega_1 \ \Omega_2] \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger \\ T^\dagger \end{bmatrix} \\ &= \left[\Omega_1\Omega_1^\dagger - \Omega_1\Omega_1^\dagger\Omega_2 T^\dagger + \Omega_2 T^\dagger\right] \\ &= \left[\Omega_1\Omega_1^\dagger + R_{\Omega_1}\Omega_2 T^\dagger\right] \\ &= \left[\Omega_1\Omega_1^\dagger + R_{\Omega_1}\Omega_2\left(R_{\Omega_1}\Omega_2\right)^\dagger\right] \end{aligned} \quad (51)$$

Since condition (3) is satisfied by components, namely,

$$\left(\Omega_1\Omega_1^\dagger\right)^* = \Omega_1\Omega_1^\dagger, \quad (52)$$

$$\left(R_{\Omega_1}\Omega_2\left(R_{\Omega_1}\Omega_2\right)^\dagger\right)^* = R_{\Omega_1}\Omega_2\left(R_{\Omega_1}\Omega_2\right)^\dagger \quad (53)$$

it follows that $\Omega^\dagger$ satisfies condition (3) as well; i.e., $(\Omega\Omega^\dagger)^* = \Omega\Omega^\dagger$.

Similar, it can be shown that $\Omega^\dagger$ satisfies condition (4). Hence, the Moore-Penrose inverse of Ω can be expressed by (49). From this (46) immediately follow.

The equations (47) can be proved similarly. □

Now we demonstrate the principal theorem of this section.

Theorem 20. *Assume that* $S_1 = [I_{p_1} \ 0]$, $S_2 = [0 \ I_{p_2}]$, $T_1 = \left[\begin{smallmatrix} I_{q_1} \\ 0 \end{smallmatrix}\right]$, $T_2 = \left[\begin{smallmatrix} I_{q_2} \\ 0 \end{smallmatrix}\right]$, $G = [A \ C]$, $H = \left[\begin{smallmatrix} B \\ -D \end{smallmatrix}\right]$, $H_1 = L_{A_1}L_A$, $H_2 = L_{A_1}S_1L_G$, $H_3 = R_HT_1R_{B_1}$, $H_4 = L_{A_2}L_C$, $H_5 = L_{A_2}S_2L_G$, $H_6 = R_HT_2R_{B_2}$ *and system (6) is solvable, then the general solution to our system can be formed as*

$$\begin{aligned} X_1 = {} & A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + L_{A_1}A^\dagger EB^\dagger R_{B_1} \\ & - L_{A_1}A^\dagger CM^\dagger EB^\dagger R_{B_1} \\ & - L_{A_1}A^\dagger SC^\dagger EN^\dagger DB^\dagger R_{B_1} + H_1V_1R_{B_1} \\ & + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}, \end{aligned} \quad (54)$$

$$\begin{aligned} X_2 = {} & A_2^\dagger C_3 + L_{A_2}C_4B_2^\dagger + L_{A_2}M^\dagger R_AED^\dagger R_{B_2} \\ & + L_{A_2}L_MS^\dagger SC^\dagger EN^\dagger R_{B_2} + H_4W_1R_{B_2} \\ & + H_5UH_6 + L_{A_2}W_2R_DR_{B_2}, \end{aligned} \quad (55)$$

where U, V_1, V_2, W_1, *and* W_2 *are free matrices over* $\mathbb{H}$ *with allowable dimensions.*

Proof. Our proof contains three parts. At the first step, we show that the matrices X_1 and X_2 have the forms of

$$X_1 = \phi_0 + H_1V_1R_{B_1} + L_{A_1}V_2R_BR_{B_1} + H_2UH_3, \quad (56)$$

$$X_2 = \psi_0 + H_4W_1R_{B_2} + L_{A_2}W_2R_DR_{B_2} + H_5UH_6, \quad (57)$$

where ϕ_0 and ψ_0 are any pair of particular solution to system (6), V_1, V_2, W_1, W_2 and U are free matrices of able shapes over $\mathbb{H}$, are solutions to system (6). At the second step, we display that any couple of solutions μ_0 and ν_0 to system (6) can be established as (56) and (57), respectively. At the end, we confirm that

$$\begin{aligned} \mu = {} & A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger \\ & - A^\dagger SC^\dagger EN^\dagger DB^\dagger \end{aligned} \quad (58)$$

and

$$\begin{aligned} \nu = {} & A_2^\dagger C_3 + L_{A_2}C_4B_2^\dagger + L_{A_2}M^\dagger R_AED^\dagger \\ & + L_{A_2}L_MS^\dagger SC^\dagger EN^\dagger R_{B_2} \end{aligned} \quad (59)$$

are a couple of particular solutions to system (6).

Now we prove that a couple of matrices X_1 and X_2 having the shape of (56) and (57), respectively, are solutions to system (6). Observe that

$$\begin{aligned} A_1^\dagger C_1B_1 + L_{A_1}C_2B_1^\dagger B_1 &= A_1^\dagger A_1C_2 + L_{A_1}C_2 = C_2, \\ A_2^\dagger C_3B_2 + L_{A_2}C_4B_2^\dagger B_2 &= A_2^\dagger A_2C_4 + L_{A_2}C_4 = C_4. \end{aligned} \quad (60)$$

It is evident that X_1 having the form (56) is a solution of $A_1X_1 = C_1$, and $X_1B_1 = C_2$ and X_2 having the form (57) is a solution to $A_2X_2 = C_3, X_2B_2 = C_4$. Now we are left to show that $A_3X_1B_3 + A_4X_2B_4 = C_c$ is satisfied by X_1 and X_2 given in (56) and (57). By Lemma 4, we have

$$\begin{aligned} AS_1L_G &= A\left[I_{p_1} \ 0\right]\begin{bmatrix} L_A & -A^\dagger CL_M \\ 0 & L_M \end{bmatrix} \\ &= A\left[L_A \ -A^\dagger CL_M\right] = \left[0 \ -AA^\dagger CL_M\right] \\ &= \left[0 \ -(C-M)L_M\right] = \left[0 \ -CL_M\right] \\ &= -\left[0 \ S\right] = -CS_2L_G, \end{aligned} \quad (61)$$

and

$$R_H T_1 B = \begin{bmatrix} R_B & 0 \\ R_N D B^{\dagger} & R_N \end{bmatrix} \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix} B = \begin{bmatrix} R_B \\ R_N D B^{\dagger} \end{bmatrix} B$$
$$= \begin{bmatrix} 0 \\ R_N D B^{\dagger} B \end{bmatrix} = \begin{bmatrix} 0 \\ R_N D (I - L_B) \end{bmatrix} \quad (62)$$
$$= \begin{bmatrix} 0 \\ R_N D \end{bmatrix} = R_H T_2 D.$$

Observe that $AL_A = 0$ and by using (61) and (62), we arrive that

$$A_3 X_1 B_3 + A_4 X_2 B_4 = C_c. \quad (63)$$

Conversely, assume that μ_0 and ν_0 are any couple of solutions to our system (6). By Lemma 7, we have

$$\begin{aligned} A_1 A_1^{\dagger} C_1 &= C_1, \\ C_2 B_1^{\dagger} B_1 &= C_2, \\ A_2 A_2^{\dagger} C_3 &= C_3, \\ C_4 B_2^{\dagger} B_2 &= C_4, \\ A_1 C_2 &= C_1 B_1, \\ A_2 C_4 &= C_3 B_2. \end{aligned} \quad (64)$$

Observe that

$$\begin{aligned} L_{A_1} \mu_0 R_{B_1} &= \left(I - A_1^{\dagger} A_1\right) \mu_0 \left(I - B_1 B_1^{\dagger}\right) \\ &= \mu_0 - \mu_0 B_1 B_1^{\dagger} - A_1^{\dagger} A_1 \mu_0 + A_1^{\dagger} A_1 \mu_0 B_1 B_1^{\dagger} \\ &= \mu_0 - C_2 B_1^{\dagger} - A_1^{\dagger} C_1 + A_1^{\dagger} A_1 C_2 B_1^{\dagger} \\ &= \mu_0 - L_{A_1} C_2 B_1^{\dagger} - A_1^{\dagger} C_1 \end{aligned} \quad (65)$$

produces

$$\mu_0 = L_{A_1} C_2 B_1^{\dagger} + A_1^{\dagger} C_1 + L_{A_1} \mu_0 R_{B_1}. \quad (66)$$

On the same lines, we can get

$$\nu_0 = L_{A_2} C_4 B_2^{\dagger} + A_2^{\dagger} C_3 + L_{A_2} \nu_0 R_{B_2}. \quad (67)$$

It is manifest that μ_0 and ν_0 defined in (66)-(67) are also solution pair of

$$A X_1 B + C X_2 D = E. \quad (68)$$

Since

$$\begin{aligned} A X_1 B + C X_2 D &= A_3 L_{A_1} \mu_0 R_{B_1} B_3 + A_4 L_{A_2} \nu_0 R_{B_2} B_4 \\ &= A_3 \left(\mu_0 - L_{A_1} C_2 B_1^{\dagger} - A_1^{\dagger} C_1\right) B_3 \\ &\quad + A_4 \left(\nu_0 - L_{A_2} C_4 B_2^{\dagger} - A_2^{\dagger} C_3\right) B_4 \\ &= A_3 \mu_0 B_3 - A_3 L_{A_1} C_2 B_1^{\dagger} B_3 \\ &\quad - A_1^{\dagger} C_1 B_3 + A_4 \nu_0 B_4 \\ &\quad - A_4 L_{A_2} C_4 B_2^{\dagger} B_4 - A_4 A_2^{\dagger} C_3 B_4 \\ &= A_3 \mu_0 B_3 + A_4 \nu_0 B_4 - A C_2 B_1^{\dagger} B_3 \\ &\quad - A_1^{\dagger} C_1 B_3 - C C_4 B_2^{\dagger} B_4 \\ &\quad - A_4 A_2^{\dagger} C_3 B_4 \\ &= C_c - A C_2 B_1^{\dagger} B_3 - A_1^{\dagger} C_1 B_3 \\ &\quad - C C_4 B_2^{\dagger} B_4 - A_4 A_2^{\dagger} C_3 B_4 = E. \end{aligned} \quad (69)$$

Hence by Lemma 6, μ_0 and ν_0 can be written as

$$\mu_0 = X_{01} + S_1 L_G U R_H T_1 + L_A V_1 + V_2 R_B, \quad (70)$$

$$\nu_0 = X_{02} + S_2 L_G U R_H T_2 + L_C W_1 + W_2 R_D, \quad (71)$$

where X_{01} and X_{02} are a couple of special solutions to (68) and U, V_1, V_2, W_1 and W_2 are free matrices with agreeable dimensions. Using (70) and (71) in (66) and (67), respectively, we get

$$\begin{aligned} \mu_0 &= X_{10} + H_2 U H_3 + H_1 V_1 R_{B_1} + L_{A_1} V_2 R_B R_{B_1}, \\ \nu_0 &= X_{20} + H_5 U H_6 + H_4 W_1 R_{B_2} + L_{A_2} W_2 R_D R_{B_2}, \end{aligned} \quad (72)$$

where $X_{10} = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + L_{A_1} X_{01} R_{B_1}$ and $X_{20} = A_2^{\dagger} C_3 + L_{A_2} C_4 B_2^{\dagger} + L_{A_2} X_{02} R_{B_2}$. It is evident that X_{10} and X_{20} are a couple of solutions to system (6). It is clear that μ_0 and ν_0 can be represented by (56) and (57), respectively. Lastly, by putting U_1, V_1, W_1, and Z_1 equal to zero in (24) and (25), we conclude that μ and ν are special solutions to system (6). Hence the expressions (54) and (55) represent the general solution to system (6) and the theorem is completed. □

Remark 21. Due to Lemma 3 and taking into account $L_{A_2} L_M = L_M L_{A_2}$, we have the following simplification of the solution pair to system (6) that is identical for (24)-(25) and (54)-(55) when $U, U_1, V_1, V_2, Z_1, W_1$, and W_2 disappear,

$$\begin{aligned} X_1 &= A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + A^{\dagger} E B^{\dagger} - A^{\dagger} A_4 M^{\dagger} E B^{\dagger} \\ &\quad - A^{\dagger} S C^{\dagger} E N^{\dagger} B_4 B^{\dagger}, \\ X_2 &= A_2^{\dagger} C_3 + L_{A_2} C_4 B_2^{\dagger} + M^{\dagger} E D^{\dagger} + S^{\dagger} S C^{\dagger} E N^{\dagger}. \end{aligned} \quad (73)$$

Comment 1. We have established a novel expression of the general solution to system (6) in Theorem 20 which is different from one created in [35]. With the help of this novel expression, we can explore the least-norm of the general solution which can not be studied with the help of the expression given in [35], which is one of the advantage of our new expression.

Now we discuss some special cases of our system.

If B_1, B_2, C_2 and C_4 disappear in Theorem 20, then we gain the following conclusion.

Corollary 22. *Denote* $S_1 = [I_{p_1} \;\; 0]$, $S_2 = [0 \;\; I_{p_2}]$, $T_1 = \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}$, $G = [A \;\; C]$, $H = \begin{bmatrix} B_3 \\ -B_4 \end{bmatrix}$, $H_1 = L_{A_1} L_A$, $H_2 = L_{A_1} S_1 L_G$, $H_3 = R_H T_1$, $H_4 = L_{A_2} L_C$, $H_5 = L_{A_2} S_2 L_G$, $H_6 = R_H T_2$ *and system (4) is solvable, then the general solution to system (4) can be formed as*

$$\begin{aligned} X_1 = {} & A_1^{\dagger} C_1 + A^{\dagger} E B_3^{\dagger} - A^{\dagger} A_4 M^{\dagger} E B_3^{\dagger} \\ & - A^{\dagger} S C^{\dagger} E N^{\dagger} B_4 B_3^{\dagger} - H_1 Y_1 + H_2 V H_3 \\ & + L_{A_1} Y_2 R_{B_3}, \\ X_2 = {} & A_2^{\dagger} C_3 + M^{\dagger} E B_4^{\dagger} + S^{\dagger} S C^{\dagger} E N^{\dagger} + H_4 Z_1 \\ & + H_5 V H_6 + L_{A_2} Z_2 R_{B_4}, \end{aligned} \tag{74}$$

where A, C, N, M, S *are the same as in Lemma 6,* $E = C_c - A_3 A_1^{\dagger} C_1 B_3 - A_4 A_2^{\dagger} C_3 B_4$, V, Y_1, Y_2, Z_1, *and* Z_2 *are free matrices over* $\mathbb{H}$ *obeying agreeable dimensions.*

Comment 2. The above consequence is a chief result of [64].

If A_2, B_2, C_3, A_4, B_4 and C_4 vanish in our system (6), then we get the following outcome.

Corollary 23. *Suppose that* $A_1, B_1, C_1, C_2, A_3, B_3$ *and* C_c *are given. Then the general solution to system (3) is established by*

$$\begin{aligned} X_1 = {} & A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + \left(A_3 L_{A_1}\right)^{\dagger} \\ & \cdot \left[C_c - A_3 A_1^{\dagger} C_1 B_3 - A_3 L_{A_1} C_2 B_1^{\dagger} B_3\right] \\ & \cdot \left(R_{B_1} B_3\right)^{\dagger} + L_{A_1} L_{A_3 L_{A_1}} W_1 R_{B_1} \\ & + L_{A_1} W_2 R_{R_{B_1} B_3} R_{B_1}, \end{aligned} \tag{75}$$

where W_1 *and* W_2 *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Comment 3. Corollary 23 is the rudimentary result of [32].

Comment 4. When A_1, B_1, A_4 and B_4 become zero in (8), then we will get the least-norm of the solution of (8) with the help of Theorem 20 quite smoothly. This is one of the advantage of the our expressions over the expressions given in [41].

An algorithm and numerical example is provided to obtain the general solution of (6) with the help of Theorem 20.

Algorithm 24. (1) Input $A_1, B_1, C_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, D_3, A_4, B_4, C_4$ with viable dimensions over $\mathbb{H}$.

(2) Evaluate X_1 and X_2 by (54)-(55).

Example 25. For given matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & -\mathbf{i} \\ \\ \mathbf{j} & \mathbf{k} \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -\mathbf{j} & \mathbf{k} & \mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{k} \end{bmatrix}, \\ C_1 &= \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ \mathbf{j} & -1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -\mathbf{i} & \mathbf{j} & 1 \\ -\mathbf{k} & -1 & \mathbf{j} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} \mathbf{i} \\ -\mathbf{k} \\ \mathbf{j} \end{bmatrix}, \\ C_3 &= \begin{bmatrix} \mathbf{i} & -\mathbf{k} \\ 1 & \mathbf{j} \\ \mathbf{j} & 1 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} \mathbf{i} & 1 \\ -1 & \mathbf{i} \\ \mathbf{k} & \mathbf{j} \end{bmatrix}, \\ A_4 &= \begin{bmatrix} -1 \\ 0 \\ \mathbf{j} \end{bmatrix}, \\ B_2 &= \begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}, \\ B_3 &= \begin{bmatrix} \mathbf{i} & 1 \\ \mathbf{k} & -\mathbf{j} \end{bmatrix}, \\ B_4 &= \begin{bmatrix} \mathbf{j} & -\mathbf{k} \\ \mathbf{k} & -\mathbf{j} \end{bmatrix}, \\ C_4 &= \left[-\mathbf{i} + \mathbf{j}\right], \\ C_c &= \begin{bmatrix} -1 - \mathbf{j} + \mathbf{k} & -2 + \mathbf{i} - \mathbf{j} + k \\ -\mathbf{i} & -1 \\ -1 + \mathbf{j} + \mathbf{k} & -i + \mathbf{j} + \mathbf{k} \end{bmatrix}. \end{aligned} \tag{76}$$

By these given matrices, the consistency conditions of (6) from Lemma 3 are fulfilled. So, system (6) is resolvable. Now we compute the partial solution to system (6) when $U, U_1, V_1, V_2, Z_1, W_1$, and W_2 disappear. Using determinantal representations (33)-(34) for computing Moore-Penrose inverses, we find that

$$A_1^{\dagger} = \frac{1}{4} \begin{bmatrix} 1 & -\mathbf{j} \\ \mathbf{i} & -\mathbf{k} \end{bmatrix},$$

$$L_{A_1} = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{bmatrix},$$

$$B_1^{\dagger}=\frac{1}{6}\begin{bmatrix}\mathbf{j} & 1\\ -\mathbf{k} & -\mathbf{i}\\ -\mathbf{i} & \mathbf{k}\end{bmatrix},$$

$$R_{B_1}=\frac{1}{2}\begin{bmatrix}1 & \mathbf{j}\\ -\mathbf{j} & 1\end{bmatrix},$$

$$B_2^{\dagger}=\frac{1}{2}\begin{bmatrix}-\mathbf{j} & -\mathbf{k}\end{bmatrix},$$

$$A_2^{\dagger}=\frac{1}{3}\begin{bmatrix}-\mathbf{i} & \mathbf{k} & -\mathrm{j}\end{bmatrix},$$

$$A=\begin{bmatrix}0 & 0\\ 0 & 0\\ \mathbf{k} & \mathbf{j}\end{bmatrix},$$

$$B=\begin{bmatrix}\mathbf{i} & 1\\ \mathbf{k} & -\mathbf{j}\end{bmatrix},$$

$$A^{\dagger}=\frac{1}{2}\begin{bmatrix}0 & 0 & -\mathbf{k}\\ 0 & 0 & -\mathbf{j}\end{bmatrix},$$

$$B^{\dagger}=\frac{1}{4}\begin{bmatrix}-\mathbf{i} & -\mathbf{k}\\ 1 & \mathbf{j}\end{bmatrix},$$

$$E=\begin{bmatrix}\mathbf{k} & -\mathbf{j}\\ -1 & i\\ \mathbf{j} & \mathbf{k}\end{bmatrix},$$

$$R_{B_2}=\frac{1}{2}\begin{bmatrix}1 & \mathbf{i}\\ -\mathbf{i} & 1\end{bmatrix}. \tag{77}$$

Since $L_{A_2}=0$ and $D=0$, then C, S, M, N are zero-matrices. Hence the general solution to our system (6) is

$$\begin{aligned} X_1 &= A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+A^{\dagger}EB^{\dagger}\\ &= \frac{1}{12}\begin{bmatrix}5+\mathbf{i}-2\mathbf{j}-\mathbf{k} & -2-\mathbf{i}+7\mathbf{j}+5\mathbf{k}\\ -5+\mathbf{i}-\mathbf{j}+2\mathbf{k} & -1+2\mathbf{i}-\mathbf{j}-\mathbf{k}\end{bmatrix}, \end{aligned} \tag{78}$$

$$X_2=A_2^{\dagger}C_3=\frac{1}{3}\begin{bmatrix}2+\mathbf{k} & -\mathbf{i}-2\mathbf{j}\end{bmatrix}.$$

4. The Least-Norm of the General Solution to System (6)

We experience the least-norm to system (6) in this section. We first modify the description of quaternionic inner product space defined in [72] as follows:

A right $\mathbb{H}$-vector space $\mathscr{V}_r$ is a quaternionic inner product space if there is a mapping $\langle\cdot,\cdot\rangle:\mathscr{V}\times\mathscr{V}\longrightarrow\mathbb{H}$ such that for all $q_1,q_2\in\mathbb{H}$ and $\xi,\xi_1,\xi_2\in\mathscr{V}_r$:

(1) $\langle\xi,\xi_1q_1+\xi_2q_2\rangle=\overline{q_1}\langle\xi,\xi_1\rangle+\overline{q_2}\langle\xi,\xi_2\rangle$; $\langle\xi_1q_1+\xi_2q_2,\xi\rangle=\langle\xi_1,\xi\rangle q_1+\langle\xi_2,\xi\rangle q_2$;

(2) $\langle\xi_1,\xi_2\rangle=\overline{\langle\xi_2,\xi_1\rangle}$;

(3) $\langle\xi,\xi\rangle\geq 0$, *and* $\langle\xi,\xi\rangle=0$ *if and only if* $\xi=0$.

It can be achieved by putting $\langle\xi,\eta\rangle=\sum_i\overline{\eta_i}\xi_i$ for $\xi=(\xi_i)_{i=1}^n,\eta=(\eta_i)_{i=1}^n\in\mathscr{V}_r$. $\|\xi\|=\sqrt{\langle\xi,\xi\rangle}$ is referred as the norm of ξ. It is routine to verify that $\mathbb{H}$ is a quaternionic inner product space under the inner product defined by $\langle C,D\rangle=\mathrm{tr}(D^*C)$ where $C,D\in\mathbb{H}^{m\times n}$. $\|A\|=(\mathrm{tr}(A^*A))^{1/2}$ is the matrix norm defined by A. The real part of a quaternion q is denoted by $re[q]$.

By the definition and [73], we can get the following result easily.

Lemma 26. *Let $A\in\mathbb{H}^{m\times n}, B\in\mathbb{H}^{n\times m}$. Then we have*
(1) $\|A+B\|^2=\|A\|^2+\|B\|^2+2Re[\mathrm{tr}(B^*A)]$.
(2) $Re[\mathrm{tr}(AB)]=Re[\mathrm{tr}(BA)]$.

Theorem 27. *Assume that system (6) is solvable, then the least-norm of the solution pair X_1 and X_2 to system (6) can be extracted as follows:*

$$\begin{aligned}\|X_1\|_{min} &= A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+A^{\dagger}EB^{\dagger}\\ &\quad -A^{\dagger}A_4M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger},\end{aligned} \tag{79}$$

$$\|X_2\|_{min}=A_2^{\dagger}C_3+L_{A_2}C_4B_2^{\dagger}+M^{\dagger}ED^{\dagger}+S^{\dagger}SC^{\dagger}EN^{\dagger}. \tag{80}$$

Proof. With the help of Theorem 20 and Remark 21, the general solution to system (6) can be formed as

$$\begin{aligned} X_1 &= A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_4M^{\dagger}EB^{\dagger}\\ &\quad -A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger}-H_1V_1R_{B_1}+H_2UH_3\\ &\quad +L_{A_1}V_2R_BR_{B_1},\\ X_2 &= A_2^{\dagger}C_3+L_{A_2}C_4B_2^{\dagger}+M^{\dagger}ED^{\dagger}+S^{\dagger}SC^{\dagger}EN^{\dagger}\\ &\quad +H_4W_1R_{B_2}+H_5UH_6+L_{A_2}W_2R_DR_{B_2}, \end{aligned} \tag{81}$$

where U, V_1, V_2, W_1, and W_2 are free matrices over $\mathbb{H}$ having executable dimensions. By Lemma 26, the norm of X_1 can be established as

$$\begin{aligned} \|X_1\|^2 &= \Big\|A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+A^{\dagger}EB^{\dagger}-A^{\dagger}A_4M^{\dagger}EB^{\dagger}\\ &\quad -A^{\dagger}SC^{\dagger}EN^{\dagger}B_4\mathrm{B}^{\dagger}-H_1V_1R_{B_1}+H_2UH_3\\ &\quad +L_{A_1}V_2R_BR_{B_1}\Big\|^2=\Big\|A_1^{\dagger}C_1+L_{A_1}C_2B_1^{\dagger}+A^{\dagger}EB^{\dagger}\\ &\quad -A^{\dagger}A_4M^{\dagger}EB^{\dagger}-A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger}\Big\|^2+\Big\|H_1V_1R_{B_1}\\ &\quad +H_2UH_3+L_{A_1}V_2R_BR_{B_1}\Big\|^2+J, \end{aligned} \tag{82}$$

where

$$J = 2\,\mathrm{Re}\Big[\mathrm{tr}\Big(\big(H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}\big)^*\big(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\big)\Big)\Big]. \quad (83)$$

Now we want to show that $J = 0$. Applying Lemmas 3, 4, and 26, we have

$$\begin{aligned}
&\mathrm{Re}\left[\mathrm{tr}\left(\left(H_1V_1R_{B_1}\right)^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}V_1^*H_1^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}V_1^*L_AL_{A_1}\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}V_1^*L_AL_{A_1}\left(L_{A_1}C_2B_1^\dagger\right)\right)\right] = \mathrm{Re}\left[\mathrm{tr}\left(V_1^*L_AL_{A_1}\left(L_{A_1}C_2B_1^\dagger\right)R_{B_1}\right)\right] = 0,
\end{aligned} \quad (84)$$

$$\begin{aligned}
&\mathrm{Re}\left[\mathrm{tr}\left(\left(L_{A_1}V_2R_BR_{B_1}\right)^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}R_BV_2^*L_{A_1}^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}R_BV_2^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(V_2^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)R_{B_1}R_B\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(V_2^*L_{A_1}\left(A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)R_B\right)\right] = 0,
\end{aligned} \quad (85)$$

$$\begin{aligned}
&\mathrm{Re}\left[\mathrm{tr}\left(\left(H_2UH_3\right)^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(H_3^*U^*H_2^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(H_3^*U^*L_GS_1^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(H_3^*U^*\begin{bmatrix} L_A & -A^\dagger CL_M \\ 0 & L_M \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(H_3^*U^*L_A\left(A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(H_3^*U^*L_AL_{A_1}C_2B_1^\dagger\right)\right] = \mathrm{Re}\left[\mathrm{tr}\left(R_{B_1}T_1^*R_HU^*L_AL_{A_1}C_2B_1^\dagger\right)\right]\\
&= \mathrm{Re}\left[\mathrm{tr}\left(T_1^*R_HU^*L_AL_{A_1}C_2B_1^\dagger R_{B_1}\right)\right] = 0.
\end{aligned} \quad (86)$$

By using (84)-(86) in (83) produces $J = 0$. Since X_1 is arbitrary, we get (79) from (82). On the same way, we can prove that (80) hold. □

A special cases of our system (6) are given below.

If B_1, B_2, C_2 and C_4 become zero matrices in Theorem 27, then again we get the principal result of [30].

Corollary 28. *Assume that system (4) is solvable, then the least-norm of the solution pair X_1 and X_2 to system (4) can be furnished as*

$$\begin{aligned}
\|X_1\|_{min} &= A_1^\dagger C_1 + A^\dagger EB_3^\dagger - A^\dagger A_4M^\dagger EB_3^\dagger \\
&\quad - A^\dagger SC^\dagger EN^\dagger B_4B_3^\dagger,\\
\|X_2\|_{min} &= A_2^\dagger C_3 + M^\dagger EB_4^\dagger + S^\dagger SC^\dagger EN^\dagger.
\end{aligned} \quad (87)$$

If A_2, B_2, C_3, A_4, B_4, and C_4 vanish in our system, then we get the next consequence.

Corollary 29. *Suppose that $A_1, B_1, C_1, C_2, A_3, B_3$, and C_c are given. Then the least-norm of the least square solution to system (3) is launched by*

$$\begin{aligned}
\|X_1\|_{min} &= A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + \left(A_3L_{A_1}\right)^\dagger \\
&\quad \cdot \left[C_c - A_3A_1^\dagger C_1B_3 - A_3L_{A_1}C_2B_1^\dagger B_3\right]\left(R_{B_1}B_3\right)^\dagger.
\end{aligned} \quad (88)$$

Comment 5. Corollary 29 is the key result of [32].

5. Determinantal Representations of the Least-Norm Solution to System (6)

In this section, we give determinantal representations of the least-norm solution to system (6). Let $A_1 \in \mathbb{H}_{r_1}^{m\times n}$, $B_1 \in \mathbb{H}_{r_2}^{r\times s}$, $A_2 \in \mathbb{H}_{r_3}^{k\times p}$, $B_2 \in \mathbb{H}_{r_4}^{q\times l}$, $A_3 \in \mathbb{H}_{r_5}^{t\times n}$, $B_3 \in \mathbb{H}_{r_6}^{r\times h}$, $A_4 \in \mathbb{H}_{r_7}^{t\times p} B_4 \in \mathbb{H}_{r_8}^{q\times h}$, $r(A) = r_9$, $r(B) = r_{10}$, $r(C) = r_{11}$, $r(D) = r_{12}$, $r(M) = r_{13}$, $r(N) = r_{14}$, and $r(S) = r_{15}$.

First, consider each term of (79) separately.

(i) Denote $C_{11} \coloneqq A_1^* C_1$. Due to Corollary 17 for the first term of (79), $X_{11} = A_1^\dagger C_1 = (x_{ij}^{(11)})$, we have

$$x_{ij}^{(11)} = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A_1^* A_1)_{.i} \left(c_{.j}^{(11)} \right) \right)_\beta^\beta}{\sum_{\beta\in J_{r_1,n}} \left| A_1^* A_1 \right|_\beta^\beta}, \tag{89}$$

where $c_{.j}^{(11)}$ is the jth column of C_{11}.

(ii) For the second term of (79) we have, $X_{12} = (x_{ij}^{(12)}) \coloneqq L_{A_1} C_2 B_1^\dagger = C_2 B_1^\dagger - Q_{A_1} C_2 B_1^\dagger$. So, due to Corollaries 18 and 13,

$$x_{ij}^{(12)} = \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(c_{i.}^{(12)} \right) \right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha} - \frac{\sum_f \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A_1^* A_1)_{.i} \left(\dot{a}_{.f}^{(1)} \right) \right)_\beta^\beta \sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(c_{f.}^{(12)} \right) \right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} \left| A_1^* A_1 \right|_\alpha^\alpha \sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha}, \tag{90}$$

where $c_{i.}^{(12)}$ is the ith row of $C_{12} \coloneqq C_2 B_1^*$ and $\dot{a}_{.f}^{(1)}$ is the fth column of $A_1^* A_1$.

Construct the matrix $\Psi_1 = (\psi_{if}^{(1)})$, where

$$\psi_{if}^{(1)} = \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A_1^* A_1)_{.i} \left(\dot{a}_{.f}^{(1)} \right) \right)_\beta^\beta, \tag{91}$$

and denote $\widetilde{\Psi}_1 = \Psi_1 C_2 B_1^*$. Then, from (90), it follows that

$$\begin{aligned} x_{ij}^{(12)} &= \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(c_{i.}^{(12)} \right) \right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha} \\ &\quad - \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(\widetilde{\psi}_{i.}^{(1)} \right) \right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} \left| A_1^* A_1 \right|_\alpha^\alpha \sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha}, \end{aligned} \tag{92}$$

where $\widetilde{\psi}_{i.}^{(1)}$ is the ith row of the matrix $\widetilde{\Psi}_1$.

If we construct the matrix $\Psi_2 = (\psi_{if}^{(2)})$, where

$$\psi_{fj}^{(2)} = \sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(c_{f.}^{(12)} \right) \right)_\alpha^\alpha, \tag{93}$$

and denote $\widetilde{\Psi}_2 \coloneqq A_1^* A_1 \Psi_2$, and then, from (90), we obtain

$$\begin{aligned} x_{ij}^{(12)} &= \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{rdet}_j \left((B_1 B_1^*)_{j.} \left(c_{i.}^{(12)} \right) \right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha} \\ &\quad - \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i \left((A_1^* A_1)_{.i} \left(\widetilde{\psi}_{.j}^{(2)} \right) \right)_\beta^\beta}{\sum_{\beta\in J_{r_1,n}} \left| A_1^* A_1 \right|_\alpha^\alpha \sum_{\alpha\in I_{r_2,r}} \left| B_1 B_1^* \right|_\alpha^\alpha}, \end{aligned} \tag{94}$$

where $\widetilde{\psi}_{.j}^{(2)}$ is the jth column of the matrix $\widetilde{\Psi}_2$.

(iii) Due to Theorem 2.15 for the third term $A^\dagger E B^\dagger =: X_{13} = (x_{ij}^{(13)})$, we obtain

$$x_{ij}^{(13)} = \frac{\sum_{\beta\in J_{r_9,m}\{i\}} \operatorname{cdet}_i \left((A^* A)_{.i} \left(d_{.j}^{B} \right) \right)_\beta^\beta}{\sum_{\beta\in J_{r_9,m}} \left| A^* A \right|_\beta^\beta \sum_{\alpha\in I_{r_{10},r}} \left| B B^* \right|_\alpha^\alpha}, \tag{95}$$

or

$$x_{ij}^{(13)} = \frac{\sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j \left((B B^*)_{j.} \left(d_{i.}^{A} \right) \right)_\alpha^\alpha}{\sum_{\beta\in J_{r_9,m}} \left| A^* A \right|_\beta^\beta \sum_{\alpha\in I_{r_{10},r}} \left| B B^* \right|_\alpha^\alpha}, \tag{96}$$

where

$$\begin{aligned} d_{.j}^{B} &= \left[\sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j \left((B B^*)_{j.} \left(e_{u.}^{(1)} \right) \right)_\alpha^\alpha \right] \in \mathbb{H}^{p\times 1}, \\ &\qquad u = 1, \ldots, m, \\ d_{i.}^{A} &= \left[\sum_{\beta\in J_{r_9,m}\{i\}} \operatorname{cdet}_i \left((A^* A)_{.i} \left(e_{.v}^{(1)} \right) \right)_\beta^\beta \right] \in \mathbb{H}^{1\times r}, \\ &\qquad v = 1, \ldots, r, \end{aligned} \tag{97}$$

are the column vector and the row vector, respectively. $e_{u.}^{(1)}$ and $e_{.v}^{(1)}$ are the uth row and the vth column of $E_1 \coloneqq A^* E B^*$.

(iv) For the fourth term of (79), $A^\dagger A_4 M^\dagger E B^\dagger \coloneqq X_{14} = (x_{ij}^{(14)})$, using the determinantal representation (33) for $A^\dagger$ and by Theorem 2.15, we have

$$\begin{aligned} x_{ij}^{(14)} \\ = \frac{\sum_f \sum_{\beta\in J_{r_9,m}\{i\}} \operatorname{cdet}_i \left((A^* A)_{.i} \left(a_{.f}^{(14)} \right) \right)_\beta^\beta \phi_{fj}}{\sum_{\beta\in J_{r_9,m}} \left| A^* A \right|_\beta^\beta \sum_{\beta\in J_{r_{13},p}} \left| M^* M \right|_\beta^\beta \sum_{\alpha\in I_{r_{10},r}} \left| B B^* \right|_\alpha^\alpha}, \end{aligned} \tag{98}$$

where

$$\begin{aligned} \phi_{fj} &= \sum_{\beta\in J_{r_{13},p}\{f\}} \operatorname{cdet}_f \left((M^* M)_{.f} \left(\varphi_{.j}^{B} \right) \right)_\beta^\beta \\ &= \sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j \left((B B^*)_{j.} \left(\varphi_{f.}^{M} \right) \right)_\alpha^\alpha, \end{aligned} \tag{99}$$

and

$$\varphi_{.j}^{B}=\left[\sum_{\alpha\in I_{r_{10},r}\{j\}}\operatorname{rdet}_j\left((BB^*)_{j.}\left(e_{u.}^{(2)}\right)\right)_{\alpha}^{\alpha}\right]\in\mathbb{H}^{p\times 1},\quad u=1,\ldots,p,$$
$$\varphi_{f.}^{M}=\left[\sum_{\beta\in J_{r_{13},p}\{f\}}\operatorname{cdet}_f\left((M^*M)_{.f}\left(e_{.v}^{(2)}\right)\right)_{\beta}^{\beta}\right]\in\mathbb{H}^{1\times r},\quad v=1,\ldots,r, \tag{100}$$

are the column vector and the row vector, respectively. $a_{.f}^{(14)}$ is the fth column of $A_{14}:=A^*A_4$, $e_{u.}{}^{(2)}$, and $e_{.v}{}^{(2)}$ are the uth row and the vth column of $E_2:=M^*EB^*$, respectively.

Construct the matrix $\Phi=(\phi_{ij})$, where ϕ_{ij} is given by (99) and denote $A^*A\Phi=:\widetilde{\Phi}=(\widetilde{\phi}_{ij})$. Then, from (98), we get the following final determinantal representation of the fourth term of (79),

$$x_{ij}^{(14)}=\frac{\sum_{\beta\in J_{r_9,m}\{i\}}\operatorname{cdet}_i\left((A^*A)_{.i}\left(\widetilde{\phi}_{.j}\right)\right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_9,m}}|A^*A|_{\beta}^{\beta}\sum_{\beta\in J_{r_{13},p}}|M^*M|_{\beta}^{\beta}\sum_{\alpha\in I_{r_{10},r}}|BB^*|_{\alpha}^{\alpha}}, \tag{101}$$

where $\widetilde{\phi}_{.j}$ is the jth column of $\widetilde{\Phi}$.

(v) For the fifth term of (79), $A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger}:=X_{15}=(x_{ij}^{(15)})$, due to Corollary 17 to $A^{\dagger}S$, by Theorem 2.15 to $C^{\dagger}EN^{\dagger}$, and Corollary 18 to $B_4B^{\dagger}$, we obtain

$$x_{ij}^{(15)}=\frac{\sum_f\sum_l\sum_{\beta\in J_{r_9,n}\{i\}}\operatorname{cdet}_i\left((A^*A)_{.i}\left(s_{.l}^{(1)}\right)\right)_{\beta}^{\beta}\omega_{lf}\sum_{\alpha\in I_{r_{10},r}\{j\}}\operatorname{rdet}_j\left((BB^*)_{j.}\left(b_{f.}^{(15)}\right)\right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_9,n}}|A^*A|_{\beta}^{\beta}\sum_{\beta\in J_{r_{11},p}}|C^*C|_{\beta}^{\beta}\sum_{\beta\in I_{r_{14},q}}|NN^*|_{\alpha}^{\alpha}\sum_{\beta\in I_{r_{10},r}}|BB^*|_{\alpha}^{\alpha}}, \tag{102}$$

where $s_{.l}^{(1)}$ is the lth column of $S_1:=A^*S$, $b_{f.}^{(15)}$ is the fth row of $B_{15}:=B_4B^*$,

$$\omega_{lf}=\sum_{\beta\in J_{r_{11},p}\{l\}}\operatorname{cdet}_l\left((C^*C)_{.l}\left(\zeta_{.f}^{N}\right)\right)_{\beta}^{\beta}=\sum_{\alpha\in I_{r_{14},q}\{f\}}\operatorname{rdet}_f\left((NN^*)_{f.}\left(\zeta_{l.}^{C}\right)\right)_{\alpha}^{\alpha}, \tag{103}$$

and

$$\zeta_{.f}^{N}=\left[\sum_{\alpha\in I_{r_{14},q}\{f\}}\operatorname{rdet}_f\left((NN^*)_{f.}\left(e_{u.}^{(3)}\right)\right)_{\alpha}^{\alpha}\right]\in\mathbb{H}^{m\times 1},\quad u=1,\ldots,p,$$
$$\zeta_{l.}^{C}=\left[\sum_{\beta\in J_{r_{11},p}\{i\}}\operatorname{cdet}_l\left((C^*C)_{.l}\left(e_{.v}^{(3)}\right)\right)_{\beta}^{\beta}\right]\in\mathbb{H}^{1\times t},\quad v=1,\ldots,q, \tag{104}$$

are the column vector and the row vector, respectively. $e_{u.}^{(3)}$ and $e_{.v}^{(3)}$ are the uth row and the vth column of $E_3=C^*EN^*$. Construct the matrix $\Omega=(\omega_{lf})$, where ω_{lf} is determined by (103), and denote $\widehat{\Omega}:=A^*S\Omega B_4B^*$. Then, from (102), it follows that

$$x_{ij}^{(15)}=\frac{\sum_f\sum_l\sum_{\beta\in J_{r_9,n}\{i\}}\operatorname{cdet}_i\left((A^*A)_{.i}\left(e_{.l}\right)\right)_{\beta}^{\beta}\widehat{\omega}_{lf}\sum_{\alpha\in I_{r_{10},r}\{j\}}\operatorname{rdet}_j\left((BB^*)_{j.}\left(e_{f.}\right)\right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_9,n}}|A^*A|_{\beta}^{\beta}\sum_{\beta\in I_{r_{11},p}}|C^*C|_{\beta}^{\beta}\sum_{\alpha\in I_{r_{14},q}}|NN^*|_{\alpha}^{\alpha}\sum_{\beta\in I_{r_{10},r}}|BB^*|_{\alpha}^{\alpha}}, \tag{105}$$

where $e_{f.}$ and $e_{.l}$ are the unit row-vector and the unit column-vector whose components are 0 except the fth or lth components which are 1, respectively.

If we denote

$$\omega_{if}^{(1)}:=\sum_l\sum_{\beta\in J_{r_9,n}\{i\}}\operatorname{cdet}_i\left((A^*A)_{.i}\left(e_{.l}\right)\right)_{\beta}^{\beta}\widehat{\omega}_{lf}=\sum_{\beta\in J_{r_9,n}\{i\}}\operatorname{cdet}_i\left((A^*A)_{.i}\left(\widehat{\omega}_{.f}\right)\right)_{\beta}^{\beta}, \tag{106}$$

then, from (102), it follows the determinantal representation

$$x_{ij}^{(15)}=\frac{\sum_{\alpha\in I_{r_{10},r}\{j\}}\operatorname{rdet}_j\left((BB^*)_{j.}\left(\omega_{i.}^{(1)}\right)\right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_9,n}}|A^*A|_{\beta}^{\beta}\sum_{\beta\in I_{r_{11},p}}|C^*C|_{\beta}^{\beta}\sum_{\alpha\in I_{r_{14},q}}|NN^*|_{\alpha}^{\alpha}\sum_{\beta\in I_{r_{10},r}}|BB^*|_{\alpha}^{\alpha}}, \tag{107}$$

where $\omega_{i.}^{(1)}$ is the ith row of the matrix $\Omega^{(1)} = (\omega_{if}^{(1)})$ that is determined by (106).

If we denote

$$\omega_{lj}^{(2)} := \sum_f \widehat{\omega}_{lf} \sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j\left(\left(BB^*\right)_{j.}\left(e_{f.}\right)\right)_\alpha^\alpha = \sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j\left(\left(BB^*\right)_{j.}\left(\widehat{\omega}_{l.}\right)\right)_\alpha^\alpha, \tag{108}$$

then, from (102), it follows the determinantal representation

$$x_{ij}^{(15)} = \frac{\sum_{\beta\in J_{r_9,n}\{i\}} \operatorname{cdet}_i\left(\left(A^*A\right)_{.i}\left(\omega_{.j}^{(2)}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r_9,n}} \left|A^*A\right|_\beta^\beta \sum_{\beta\in J_{r_{11},p}} \left|C^*C\right|_\beta^\beta \sum_{\alpha\in I_{r_{14},q}} \left|NN^*\right|_\alpha^\alpha \sum_{\beta\in I_{r_{10},r}} \left|BB^*\right|_\alpha^\alpha}, \tag{109}$$

where $\omega_{.j}^{(2)}$ is the jth column of the matrix $\Omega^{(2)} = (\omega_{lj}^{(2)})$ that is determined by (108).

Similarly, consider each term of (80) separately.

(i) Denote $C_{21} := A_2^* C_3$. Due to Corollary 17 for the first term of (80), $X_{21} = A_2^\dagger C_3 = (x_{gf}^{(21)})$, we have

$$x_{gf}^{(21)} = \frac{\sum_{\beta\in J_{r_3,p}\{g\}} \operatorname{cdet}_g\left(\left(A_2^*A_2\right)_{.i}\left(c_{.f}^{(21)}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r_3,p}} \left|A_2^*A_2\right|_\beta^\beta}, \tag{110}$$

where $c_{.f}^{(21)}$ is the fth column of C_{21}.

(ii) For the second term of (80) we have, $X_{22} = (x_{ij}^{(22)}) := L_{A_2} C_4 B_2^\dagger = C_4 B_2^\dagger - P_{A_2} C_4 B_2^\dagger$. So, due to Corollaries 18 and 13,

$$\begin{aligned} x_{gf}^{(22)} = {} & \frac{\sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(c_{g.}^{(22)}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha} \\ & - \frac{\sum_j \sum_{\beta\in J_{r_3,p}\{g\}} \operatorname{cdet}_g\left(\left(A_2^*A_2\right)_{.g}\left(\dot{a}_{.j}^{(2)}\right)\right)_\beta^\beta \sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(c_{j.}^{(22)}\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_3,p}} \left|A_2^*A_2\right|_\alpha^\alpha \sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha}, \end{aligned} \tag{111}$$

where $c_{g.}^{(22)}$ is the gth row of $C_{22} := C_4 B_2^*$ and $\dot{a}_{.j}^{(2)}$ is the jth column of $A_2^* A_2$.

Construct the matrix $\Upsilon_1 = (\upsilon_{gj}^{(1)})$, where

$$\upsilon_{gj}^{(1)} = \sum_{\beta\in J_{r_3,p}\{g\}} \operatorname{cdet}_g\left(\left(A_2^*A_2\right)_{.g}\left(\dot{a}_{.j}^{(2)}\right)\right)_\beta^\beta, \tag{112}$$

and denote $\widetilde{\Upsilon}_1 = \Upsilon_1 C_4 B_2^*$. Then, from (111), it follows that

$$\begin{aligned} x_{gf}^{(22)} = {} & \frac{\sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(c_{g.}^{(22)}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha} \\ & - \frac{\sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(\widetilde{\upsilon}_{g.}^{(1)}\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_3,p}} \left|A_2^*A_2\right|_\alpha^\alpha \sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha}, \end{aligned} \tag{113}$$

where $\widetilde{\upsilon}_{g.}^{(1)}$ is the gth row of $\widetilde{\Upsilon}_1$.

If we construct the matrix $\Upsilon_2 = (\upsilon_{if}^{(2)})$, where

$$\upsilon_{jf}^{(2)} = \sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(c_{j.}^{(22)}\right)\right)_\alpha^\alpha, \tag{114}$$

and denote $\widetilde{\Upsilon}_2 = A_2^* A_2 \Upsilon_2$, then, from (111), we obtain

$$\begin{aligned} x_{gf}^{(22)} = {} & \frac{\sum_{\alpha\in I_{r_4,q}\{f\}} \operatorname{rdet}_f\left(\left(B_2B_2^*\right)_{f.}\left(c_{g.}^{(22)}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha} \\ & - \frac{\sum_{\beta\in J_{r_3,p}\{g\}} \operatorname{cdet}_g\left(\left(A_2^*A_2\right)_{.g}\left(\widetilde{\upsilon}_{.f}^{(2)}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r_3,p}} \left|A_2^*A_2\right|_\alpha^\alpha \sum_{\alpha\in I_{r_4,q}} \left|B_2B_2^*\right|_\alpha^\alpha}, \end{aligned} \tag{115}$$

where $\widetilde{\upsilon}_{.f}^{(2)}$ is the gth row of $\widetilde{\Upsilon}_2$.

(iii) Due to Theorem 2.15 for the third term $M^{\dagger}ED^{\dagger} =: X_{23} = (x_{gf}^{(23)})$, we obtain

$$x_{gf}^{(23)} = \frac{\sum_{\beta\in J_{r_{13},p}\{g\}} \operatorname{cdet}_g \left((M^*M)_{.g} \left(d_{.f}^{D} \right) \right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_{13},p}} |M^*M|_{\beta}^{\beta} \sum_{\alpha\in I_{r_{10},r}} |BB^*|_{\alpha}^{\alpha}}, \tag{116}$$

or

$$x_{ij}^{(13)} = \frac{\sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j \left((BB^*)_{j.} \left(d_{i.}^{A} \right) \right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_9,m}} |A^*A|_{\beta}^{\beta} \sum_{\alpha\in I_{r_{10},r}} |BB^*|_{\alpha}^{\alpha}}, \tag{117}$$

where

$$d_{.j}^{B} = \left[\sum_{\alpha\in I_{r_{10},r}\{j\}} \operatorname{rdet}_j \left((BB^*)_{j.} \left(e_{u.}^{(4)} \right) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{p\times 1}, \quad u = 1, \ldots, m,$$
$$d_{i.}^{A} = \left[\sum_{\beta\in J_{r_9,m}\{i\}} \operatorname{cdet}_i \left((A^*A)_{.i} \left(e_{.v}^{(4)} \right) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1\times r}, \quad v = 1, \ldots, r, \tag{118}$$

are the column vector and the row vector, respectively. $e_{u.}^{(1)}$ and $\mathrm{e}_{.v}^{(1)}$ are the uth row and the vth column of $E_4 := M^*ED^*$.

(iv) Using Corollary 13 to $S^{\dagger}S$ and by Theorem 2.15 to $C^{\dagger}EN^{\dagger}$, we obtain the the following representation of the fourth term, $X_{24} = (x_{gf}^{(24)}) := S^{\dagger}SC^{\dagger}EN^{\dagger}$, of (80)

$$x_{gf}^{(24)} = \frac{\sum_l \sum_{\beta\in J_{r_{15},p}\{g\}} \operatorname{cdet}_g \left((S^*S)_{.g} \left(\dot{s}_{.l} \right) \right)_{\beta}^{\beta} \omega_{lf}}{\sum_{\beta\in J_{r_{15},p}} |S^*S|_{\beta}^{\beta} \sum_{\alpha\in I_{r_{11},t}} |CC^*|_{\alpha}^{\alpha} \sum_{\beta\in I_{r_{14},q}} |NN^*|_{\alpha}^{\alpha}}, \tag{119}$$

where ω_{lf} is determined by (103). Construct the matrix $\Omega = (\omega_{lf})$ and denote $\widetilde{\Omega} = S^*S\Omega$. Then, from (119) finally, we have

$$x_{gf}^{(24)} = \frac{\sum_{\beta\in J_{r_{15},p}\{g\}} \operatorname{cdet}_g \left((S^*S)_{.g} \left(\widetilde{\omega}_{.f} \right) \right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_{15},p}} |S^*S|_{\beta}^{\beta} \sum_{\alpha\in I_{r_{11},t}} |CC^*|_{\alpha}^{\alpha} \sum_{\beta\in I_{r_{14},q}} |NN^*|_{\alpha}^{\alpha}}, \tag{120}$$

where $\widetilde{\omega}_{.f}$ is the fth column of $\widetilde{\Omega}$.

So, we prove the following theorem.

Theorem 30. *Let $A_1 \in \mathbb{H}_{r_1}^{m\times n}$, $B_1 \in \mathbb{H}_{r_2}^{r\times s}$, $A_2 \in \mathbb{H}_{r_3}^{k\times p}$, $B_2 \in \mathbb{H}_{r_4}^{q\times l}$, $A_3 \in \mathbb{H}_{r_5}^{t\times n}$, $B_3 \in \mathbb{H}_{r_6}^{r\times h}$, $A_4 \in \mathbb{H}_{r_7}^{t\times p} B_4 \in \mathbb{H}_{r_8}^{q\times h}$, $r(A) = r_9$, $r(B) = r_{10}$, $r(C) = r_{11}$, $r(D) = r_{12}$, $r(M) = r_{13}$, $r(N) = r_{14}$, and $r(S) = r_{15}$. The least-norm solution (79)-(80) to system (8), $X_1 = (x_{ij}^{(1)}) \in \mathbb{H}^{n\times r}$, $X_2 = (x_{gf}^{(2)}) \in \mathbb{H}^{p\times q}$, by components*

$$\begin{aligned} x_{ij}^{(1)} &= x_{ij}^{(11)} + x_{ij}^{(12)} + x_{ij}^{(13)} - x_{ij}^{(14)} - x_{ij}^{(15)}, \\ x_{gf}^{(2)} &= \mathrm{x}_{gf}^{(21)} + x_{gf}^{(22)} + x_{gf}^{(23)} + x_{gf}^{(24)}, \end{aligned} \tag{121}$$

has determinantal representations, where the term $x_{ij}^{(11)}$ is (89), $x_{ij}^{(12)}$ is (92) or (94), $x_{ij}^{(13)}$ is (95) or (96), $x_{ij}^{(14)}$ is (101), and $x_{ij}^{(15)}$ is (107) or (109); similarly, $x_{gf}^{(21)}$ is (110), $x_{gf}^{(22)}$ is (113) or (115), $x_{gf}^{(23)}$ is (116) or (117), and $x_{gf}^{(24)}$ is (120).

A numerical example is provided to obtain the least norm of the general solution of (6) with the help of Theorem 30.

Example 31. We use the given matrices from the Example 25. Since $r(A_1) = 1$ and

$$C_{11} = A_1^*C_1 = \begin{bmatrix} 1+\mathbf{i} & \mathbf{j}+\mathbf{k} \\ -1+\mathbf{i} & -\mathbf{j}+\mathbf{k} \end{bmatrix}, \quad A_1^*A_1 = \begin{bmatrix} 2 & -2\mathbf{i} \\ 2\mathbf{i} & 2 \end{bmatrix}, \tag{122}$$

and then, by (89),

$$\begin{aligned} x_{11}^{(11)} &= \frac{1}{4} + \frac{1}{4}\mathbf{i}, \\ x_{12}^{(11)} &= \frac{1}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}, \\ x_{21}^{(11)} &= -\frac{1}{4} + \frac{1}{4}\mathbf{i}, \\ x_{22}^{(11)} &= -\frac{1}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}. \end{aligned} \tag{123}$$

Now, by (92), we find $x_{ij}^{(12)}$ for all $i, j = 1, 2$. So,

$$C_{12} = C_2B_1^* = \begin{bmatrix} -2\mathbf{i}-\mathbf{k} & -\mathbf{i}+2\mathbf{k} \\ \mathbf{i}+2\mathbf{k} & 2\mathbf{i}-\mathbf{k} \end{bmatrix}, \quad B_1B_1^* = \begin{bmatrix} 3 & -3\mathbf{j} \\ 3\mathbf{j} & 3 \end{bmatrix}, \tag{124}$$

Similarly, by (91), $\Psi_1 = A_1^*A_1$. So

$$\widetilde{\Psi}_1 = \Psi_1C_2B_1^* = \begin{bmatrix} 2-4\mathbf{i}+4\mathbf{j}-2\mathbf{k} & 4-2\mathbf{i}-2\mathbf{j}+4\mathbf{k} \\ 4+2\mathbf{i}+2\mathbf{j}+4\mathbf{k} & 2+4\mathbf{i}-4\mathbf{j}-2\mathbf{k} \end{bmatrix} \tag{125}$$

Since $r(B_1) = 1$, then by (92),

$$\begin{aligned} x_{11}^{(12)} &= \frac{1}{6}(-2\mathbf{i}-\mathbf{k}) - \frac{1}{24}(2-4\mathbf{i}+4\mathbf{j}-2\mathbf{k}) \\ &= -\frac{1}{12} - \frac{1}{6}\mathbf{i} - \frac{1}{6}\mathbf{j} - \frac{1}{12}\mathbf{k}, \\ x_{12}^{(12)} &= \frac{1}{6}(-\mathbf{i}+2\mathbf{k}) - \frac{1}{24}(4-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}) \\ &= -\frac{1}{6} - \frac{1}{12}\mathbf{i} + \frac{1}{12}\mathbf{j} + \frac{1}{6}\mathbf{k}, \end{aligned}$$

$$x_{21}^{(12)} = \frac{1}{6}(\mathbf{i}+2\mathbf{k}) - \frac{1}{24}(4+2\mathbf{i}+2\mathbf{j}+4\mathbf{k}) = -\frac{1}{6} + \frac{1}{12}\mathbf{i} - \frac{1}{12}\mathbf{j} + \frac{1}{6}\mathbf{k},$$
$$x_{22}^{(12)} = \frac{1}{6}(2\mathbf{i}-\mathbf{k}) - \frac{1}{24}(2+4\mathbf{i}-4\mathbf{j}-2\mathbf{k}) = -\frac{1}{12} + \frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} - \frac{1}{12}\mathbf{k}. \quad (126)$$

Since $r(A) = r(B) = 1$ and

$$E_1 = A^* E B^* = \begin{bmatrix} 2 & 2\mathbf{j} \\ -2\mathbf{i} & -2\mathbf{k} \end{bmatrix},$$
$$A^* A = \begin{bmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{bmatrix}, \quad (127)$$
$$BB^* = \begin{bmatrix} 2 & 2\mathbf{j} \\ -2\mathbf{j} & 2 \end{bmatrix},$$

and then, by (95),

$$d_{.1}^{B} = \begin{bmatrix} 2 \\ -2\mathbf{i} \end{bmatrix},$$
$$d_{.2}^{B} = \begin{bmatrix} 2\mathbf{j} \\ -2k \end{bmatrix}, \quad (128)$$

and

$$x_{11}^{(13)} = \frac{1}{4},$$
$$x_{12}^{(13)} = \frac{1}{4}\mathbf{j},$$
$$x_{21}^{(13)} = -\frac{1}{4}\mathbf{i}, \quad (129)$$
$$x_{22}^{(13)} = -\frac{1}{4}\mathbf{k}.$$

Further, due to Example 25, $x_{ij}^{(14)} = x_{ij}^{(15)} = 0$ for all $i, j = 1, 2$. So,

$$x_{11}^{(1)} = \frac{5}{12} + \frac{1}{12}\mathbf{i} - \frac{1}{6}\mathbf{j} - \frac{1}{12}\mathbf{k},$$
$$x_{12}^{(1)} = -\frac{1}{6} - \frac{1}{12}\mathbf{i} + \frac{7}{12}\mathbf{j} + \frac{5}{12}\mathbf{k},$$
$$x_{21}^{(1)} = -\frac{5}{12} + \frac{1}{12}\mathbf{i} - \frac{1}{12}\mathbf{j} + \frac{1}{6}\mathbf{k}, \quad (130)$$
$$x_{22}^{(1)} = -\frac{1}{12} + \frac{1}{6}\mathbf{i} - \frac{1}{12}\mathbf{j} - \frac{1}{12}\mathbf{k}.$$

Since, $r(A_2) = 1$ and

$$C_{21} = A_2^* C_3 = \left[2+\mathbf{k} \quad -\mathbf{i}-2\mathbf{j}\right],$$
$$A_2^* A_2 = [3], \quad (131)$$

and, due to Example 25, $x_{1j}^{(22)} = x_{1j}^{(23)} = x_{1j}^{(24)} = 0$ for all $j = 1, 2$; then by (110) and (121)

$$x_{11}^{(2)} = x_{11}^{(21)} = \frac{2}{3} + \frac{1}{3}\mathbf{k},$$
$$x_{12}^{(2)} = x_{12}^{(21)} = -\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j}. \quad (132)$$

Hence, the least norm solution of (6) obtained by Cramer's Rule and the matrix method in Example 25 are the same as expected.

Note that we used Maple with the package CLIFFORD in the calculations.

6. Conclusion

We have constructed a novel expression of the general solution to system (6) over $\mathbb{H}$ and used this result to explore the least-norm of the general solution to this system when it is solvable. Some particular cases of our system are also discussed. Our results carry the principal results of [32, 64]. Finally, we give determinantal representations (analogous of Cramer's Rule) of the least norm solutions to the systems using row-column noncommutative determinants. Numerical examples are also provided to interpret the results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] T. Hungerford, *Algebra*, Springer, New York, NY, USA, 1980.

[2] Q. Wang, "A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity," *Linear Algebra and its Applications*, vol. 384, pp. 43–54, 2004.

[3] F. Zhang, "Quaternions and matrices of quaternions," *Linear Algebra and its Applications*, vol. 251, pp. 21–57, 1997.

[4] J. H. Conway and D. A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, A K Peters, Ltd., Natick, MA, USA, 2003.

[5] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Field*, Oxford University Press, New York, NY, USA, 1995.

[6] S. de Leo and G. Scolarici, "Right eigenvalue equation in quaternionic quantum mechanics," *Journal of Physics A: Mathematical and Theoretical*, vol. 33, no. 15, pp. 2971–2995, 2000.

[7] C. C. Took and D. P. Mandic, "Augmented second-order statistics of quaternion random signals," *Signal Processing*, vol. 91, no. 2, pp. 214–224, 2011.

[8] C. C. Took and D. P. Mandic, "The quaternion LMS algorithm for adaptive filtering of hypercomplex processes," *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1316–1327, 2009.

[9] C. C. Took and D. P. Mandic, "A quaternion widely linear adaptive filter," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4427–4431, 2010.

[10] C. C. Took, D. P. Mandic, and F. Zhang, "On the unitary diagonalisation of a special class of quaternion matrices," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1806–1809, 2011.

[11] A. Shahzad, B. L. Jones, E. C. Kerrigan, and G. A. Constantinides, "An efficient algorithm for the solution of a coupled Sylvester equation appearing in descriptor systems," *Automatica*, vol. 47, no. 1, pp. 244–248, 2011.

[12] V. L. Syrmos and F. L. Lewis, "Coupled and constrained sylvester equations in system design," *Circuits, Systems and Signal Processing*, vol. 13, no. 6, pp. 663–694, 1994.

[13] A. Varga, "Robust pole assignment via Sylvester equation based state feedback parametrization," in *Proceedings of the CACSD. Conference Proceedings. IEEE International Symposium on Computer-Aided Control Systems Design*, pp. 13–18, Anchorage, AK, USA.

[14] V. Syrmos and F. Lewis, "Output feedback eigenstructure assignment using two Sylvester equations," *IEEE Transactions on Automatic Control*, vol. 38, no. 3, pp. 495–499, 1993.

[15] R. Li, "A bound on the solution to a structured sylvester equation with an application to relative perturbation theory," *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 2, pp. 440–445, 2000.

[16] M. Darouach, "Solution to Sylvester equation associated to linear descriptor systems," *Systems & Control Letters*, vol. 55, no. 10, pp. 835–838, 2006.

[17] Y. Zhang, D. Jiang, and J. Wang, "A recurrent neural network for solving sylvester equation with time-varying coefficients," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 13, no. 5, pp. 1053–1063, 2002.

[18] F. De Terán and F. M. Dopico, "The solution of the equation $XA+AX^T = 0$ and its application to the theory of orbits," *Linear Algebra and its Applications*, vol. 434, no. 1, pp. 44–67, 2011.

[19] M. Dehghan and M. Hajarian, "Efficient iterative method for solving the second-order Sylvester matrix equation $EVF^2 - AVF - CV = BW$," *IET Control Theory & Applications*, vol. 3, no. 10, pp. 1401–1408, 2009.

[20] F. Ding and T. Chen, "Gradient based iterative algorithms for solving a class of matrix equations," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1216–1221, 2005.

[21] A. Dmytryshyn, V. Futorny, T. Klymchuk, and V. V. Sergeichuk, "Generalization of Roth's solvability criteria to systems of matrix equations," *Linear Algebra and its Applications*, vol. 527, pp. 294–302, 2017.

[22] Z. He and Q. Wang, "A real quaternion matrix equation with applications," *Linear and Multilinear Algebra*, vol. 61, no. 6, pp. 725–740, 2013.

[23] Z.-H. He and Q.-W. Wang, "The η-bihermitian solution to a system of real quaternion matrix equations," *Linear and Multilinear Algebra*, vol. 62, no. 11, pp. 1509–1528, 2014.

[24] I. Kyrchei, "Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations," *Linear Algebra and its Applications*, vol. 438, no. 1, pp. 136–152, 2013.

[25] I. Kyrchei, "Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations," *Applied Mathematics and Computation*, vol. 219, no. 14, pp. 7632–7644, 2013.

[26] A. Rehman and Q. Wang, "A system of matrix equations with five variables," *Applied Mathematics and Computation*, vol. 271, pp. 805–819, 2015.

[27] A. Rehman, Q.-W. Wang, I. Ali, M. Akram, and M. O. Ahmad, "A constraint system of generalized Sylvester quaternion matrix equations," *Advances in Applied Clifford Algebras (AACA)*, vol. 27, no. 4, pp. 3183–3196, 2017.

[28] A. Rehman, Q.-W. Wang, and Z.-H. He, "Solution to a system of real quaternion matrix equations encompassing η-Hermicity," *Applied Mathematics and Computation*, vol. 265, pp. 945–957, 2015.

[29] A. Rehman and M. Akram, "Optimization of a nonlinear Hermitian matrix expression with application," *Filomat*, vol. 31, no. 9, pp. 2805–2819, 2017.

[30] Q.-W. Wang, F. Qin, and C.-Y. Lin, "The common solution to matrix equations over a regular ring with applications," *Indian Journal of Pure and Applied Mathematics*, vol. 36, no. 12, pp. 655–672, 2005.

[31] Q.-W. Wang, A. Rehman, Z.-H. He, and Y. Zhang, "Constraint generalized Sylvester matrix equations," *Automatica*, vol. 69, pp. 60–64, 2016.

[32] Y. Bao, "Least-norm and extremal ranks of the least square solution to the quaternion matrix equation AXB=C subject to two equations," *Algebra Colloquium*, vol. 21, no. 03, pp. 449–460, 2014.

[33] Q. Wang, H. Chang, and C. Lin, "P-(skew)symmetric common solutions to a pair of quaternion matrix equations," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 721–732, 2008.

[34] H. Li, Z. Gao, and D. Zhao, "Least squares solutions of the matrix equation $AXB + CYD = E$ with the least norm for symmetric arrowhead matrices," *Applied Mathematics and Computation*, vol. 226, pp. 719–724, 2014.

[35] Q.-W. Wang, J. W. van der Woude, and H.-X. Chang, "A system of real quaternion matrix equations with applications," *Linear Algebra and its Applications*, vol. 431, no. 12, pp. 2291–2303, 2009.

[36] Y.-g. Peng and X. Wang, "A finite iterative algorithm for solving the least-norm generalized (P, Q) reflexive solution of the matrix equations $A_iXB_i = C_i$," *Journal of Computational Analysis and Applications*, vol. 17, no. 3, pp. 547–561, 2014.

[37] S. Yuan and A. Liao, "Least squares Hermitian solution of the complex matrix equation AXB+CXD=E with the least norm," *Journal of The Franklin Institute*, vol. 351, no. 11, pp. 4978–4997, 2014.

[38] W. F. Trench, "Minimization problems for (R, S)-symmetric and (R, S)-skew symmetric matrices," *Linear Algebra and its Applications*, vol. 389, pp. 23–31, 2004.

[39] W. F. Trench, "Characterization and properties of matrices with generalized symmetry or skew symmetry," *Linear Algebra and its Applications*, vol. 377, pp. 207–218, 2004.

[40] W. F. Trench, "Characterization and properties of (R,S)-symmetric, (R,S)-skew symmetric, and (R,S)-conjugate matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 26, no. 3, pp. 748–757, 2005.

[41] X. Zhang, "Characterization for the general solution to a system of matrix equations with quadruple variables," *Applied Mathematics and Computation*, vol. 226, pp. 274–287, 2014.

[42] R. B. Bapat, K. P. S. Rao, and K. M. Prasad, "Generalized inverses over integral domains," *Linear Algebra and its Applications*, vol. 140, pp. 181–196, 1990.

[43] P. Stanimirovic, "General determinantal representation of pseudoinverses of matrices," *Matematički Vesnik*, vol. 48, pp. 1–9, 1996.

[44] I. I. Kyrchei, "Cramer's rule for generalized inverse solutions," in *Advances in Linear Algebra Research*, I. I. Kyrchei, Ed., pp. 79–132, Nova Science Publishers, New York, NY, USA, 2015.

[45] I. I. Kyrchei, "Cramer's rule for quaternion systems of linear equations," *Fundamentalnaya i Prikladnaya Matematika*, vol. 13, no. 4, pp. 67–94, 2007.

[46] I. Kyrchei, "The theory of the column and row determinants in a quaternion linear algebra," in *Advances in Mathematics Research*, A. R. Baswell, Ed., vol. 15, pp. 301–359, Nova Science Publ, New York, NY, USA, 2012.

[47] I. I. Kyrchei, "Determinantal representations of the Moore-Penrose inverse over the quaternion skew field and corresponding Cramer's rules," *Linear and Multilinear Algebra*, vol. 59, no. 4, pp. 413–431, 2011.

[48] I. I. Kyrchei, "Explicit determinantal representation formulas of W-weighted drazin inverse solutions of some matrix equations over the quaternion skew field," *Mathematical Problems in Engineering*, vol. 2016, Article ID 8673809, 13 pages, 2016.

[49] I. Kyrchei, "Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations," *Applied Mathematics and Computation*, vol. 238, pp. 193–207, 2014.

[50] I. Kyrchei, "Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field," *Applied Mathematics and Computation*, vol. 264, Article ID 21118, pp. 453–465, 2015.

[51] I. I. Kyrchei, "Explicit determinantal representation formulas for the solution of the two-sided restricted quaternionic matrix equation," *Applied Mathematics and Computation*, vol. 58, no. 1-2, pp. 335–365, 2018.

[52] I. Kyrchei, "Determinantal representations of the Drazin and w-weighted Drazin inverses over the quaternion skew field with applications," in *Quaternions: Theory and Applications*, S. Griffin, Ed., pp. 201–275, Nova Science Publishers, New York, NY, USA, 2017.

[53] I. Kyrchei, "Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse," *Applied Mathematics and Computation*, vol. 309, pp. 1–16, 2017.

[54] I. I. Kyrchei, "Determinantal representations of the quaternion weighted moore-penrose inverse and its applications," in *Advances in Mathematics Research*, A. R. Baswell, Ed., vol. 23, pp. 35–96, Nova Science Publishers, New York, NY, USA, 2017.

[55] G.-J. Song, Q.-W. Wang, and H.-X. Chang, "Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field," *Computers & Mathematics with Applications*, vol. 61, no. 6, pp. 1576–1589, 2011.

[56] G.-J. Song and C.-Z. Dong, "New results on condensed Cramer's rule for the general solution to some restricted quaternion matrix equations," *Applied Mathematics and Computation*, vol. 53, no. 1-2, pp. 321–341, 2017.

[57] G.-J. Song and Q.-W. Wang, "Condensed Cramer rule for some restricted quaternion linear equations," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3110–3121, 2011.

[58] G.-J. Song, Q.-W. Wang, and S.-W. Yu, "Cramer's rule for a system of quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 336, pp. 490–499, 2018.

[59] I. Kyrchei, "Determinantal representations of solutions to systems of quaternion matrix equations," *Advances in Applied Clifford Algebras (AACA)*, vol. 28, no. 1, article 23, 16 pages, 2018.

[60] I. I. Kyrchei, "Determinantal representations of solutions and hermitian solutions to some system of two-sided quaternion matrix equations," *Journal of Mathematics*, vol. 2018, 12 pages, 2018.

[61] I. I. Kyrchei, "Cramer's rules for Sylvester quaternion matrix equation and its special cases," *Advances in Applied Clifford Algebras (AACA)*, vol. 28, no. 5, p. 90, 2018.

[62] I. I. Kyrchei, "Determinantal representations of general and (Skew-)Hermitian solutions to the generalized sylvester-type quaternion matrix equation," *Abstract and Applied Analysis*, vol. 2019, Article ID 5926832, 14 pages, 2019.

[63] G. Marsaglia and G. P. H. Styan, "Equalities and inequalities for ranks of matrices," *Linear and Multilinear Algebra*, vol. 2, pp. 269–292, 1974.

[64] Q. Wang and C. Li, "Ranks and the least-norm of the general solution to a system of quaternion matrix equations," *Linear Algebra and its Applications*, vol. 430, no. 5-6, pp. 1626–1640, 2009.

[65] Q. W. Wang, H. X. Chang, and Q. Ning, "The common solution to six quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 209–226, 2008.

[66] Y. Tian, "The solvability of two linear matrix equations," *Linear and Multilinear Algebra*, vol. 48, no. 2, pp. 123–147, 2000.

[67] Q. Wang, Z. Wu, and C. Lin, "Extremal ranks of a quaternion matrix expression subject to consistent systems of quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1755–1764, 2006.

[68] J. Dieudonne, "Les determinantes sur un corps non-commutatif," *Bulletin de la Société Mathématique de France*, vol. 71, pp. 27–45, 1943.

[69] F. J. Dyson, "Quaternion determinants Helvetica," *Physica Acta*, vol. 45, pp. 289–302, 1972.

[70] I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, "Quasideterminants," *Advances in Mathematics*, vol. 193, no. 1, pp. 56–141, 2005.

[71] W. Cao, "Solvability of a quaternion matrix equation," *Applied Mathematics-A Journal of Chinese Universities*, vol. 17, no. 4, pp. 490–498, 2002.

[72] D. R. Farenick and B. A. Pidkowich, "The spectral theorem in quaternions," *Linear Algebra and its Applications*, vol. 371, pp. 75–102, 2003.

[73] Y. Tian, "Equalities and inequalities for traces of quaternionic matrices," *Algebras, Groups and Geometries*, vol. 19, no. 2, pp. 181–193, 2002.