Hindawi
Journal of Applied Mathematics
Volume 2019, Article ID 5072676, 13 pages
http://dx.doi.org/10.1155/2019/5072676

Research Article

Surfaces Modelling Using Isotropic Fractional-Rational Curves

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Received 30 April 2019; Accepted 11 July 2019; Published 1 August 2019

Academic Editor: Sazzad Hossien Chowdhury

The problem of building a smooth surface containing given points or curves is actual due to development of industry and computer technology. Previously used for those purposes, shells of zero Gaussian curvature and minimal surfaces based on isotropic analytic curves are restricted in their consumer properties. To expand the possibilities regarding the shaping of surfaces we propose the method of constructing surfaces based on isotropic fractional-rational curves. The surfaces are built using flat isothermal and orthogonal grids and on the basis of the Weierstrass method. In the latter case, the surfaces are minimal. Examples of surfaces that were built according to the proposed method are given.

1. Introduction

The intensive development of mechanical engineering, the construction industry, and computer technology brings to the fore the problem of building a smooth surface containing given points or curves [1, 2]. To solve this problem, shells of zero Gaussian curvature were used [3, 4], which are simple in design and manufacture. But it did not always give the best result when covering complex structures, since the carrying capacity of such shells significantly depends on small deviations of their overall contour from the ideal shape [5]. The elimination of these drawbacks in the most natural way is possible using minimal surfaces [6], whose theory has been successfully developed for a long time.

In the simulation of minimal surfaces, isotropic curves have been successfully applied [6]. There are many studies on the properties of isotropic curves [7–16]:

(i) studies of differential geometric invariants of curves of zero Cartan length in Minkowski space [11];

(ii) the study of minimal curves that can become Bertrand curves in space with a standard Euclidean metric [12] (Bertrand curves are two spatial curves that have common main normals at all their points, the main normals are constant along all the points of these curves);

(iii) study of minimal rational curves as Fano varieties [13];

(iv) in [14] the adaptation of the butterfly theorem, the classical planimetric theorems, in the isotropic plane is described.

For interactive control of the shape of surfaces, it is necessary to use parametric curves that are constructed using characteristic polygons, for example, Bezier curves [17]. To expand the possibilities regarding the shaping of surfaces, it is advisable to be able to control not only the coordinates of the points, but also the weights of the points, that is, to apply a fractional-rational representation of the curves.

The purpose of this work is to create mathematical models of surfaces with specific differential properties based on isotropic fractional-rational curves. To achieve this goal, it is necessary to construct isotropic fractional-rational curves in the plane and in space, to use flat isotropic fractional-rational curves to construct a network and use curves and grids to construct surfaces.

2. Methods of Analytic Isotropic Curves Simulation

If the third derivative of an arbitrary analytic function $f(t)$ does not vanish identically, then such a function corresponds to an isotropic line with a parametric equation of the form

$$\begin{aligned} x &= i\left(f - tf' - 0,5\left(1-t^2\right)f''\right), \\ y &= f - tf' + 0,5\left(1+t^2\right)f'', \\ z &= -i\left(f' - tf''\right). \end{aligned} \tag{1}$$

The curve defined by (1) will have a zero length of the form

$$x(t)'^2 + y(t)'^2 + z(t)'^2 = 0, \tag{2}$$

that is, being isotropic.

The modelling of a spatial isotropic curve using a flat parametric curve is considered in the works of Pilipaka, Chernysheva, and Korovina [7–9]. If a flat curve is given by a parametric equation of the form

$$\begin{aligned} x &= x(t), \\ y &= y(t), \end{aligned} \tag{3}$$

then from the isotropy condition (Equation (2)) we have the expression

$$z' = \sqrt{-\left(\left(x'\right)^2 + \left(y'\right)^2\right)} = i\sqrt{\left(x'\right)^2 + \left(y'\right)^2}, \tag{4}$$

After integrating Equation (4), we obtain the expression for the coordinate z in the form

$$z(t) = i\int \sqrt{\left(x'\right)^2 + \left(y'\right)^2}\,dt, \tag{5}$$

and the equations of the isotropic line have the form

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= i\int \sqrt{x'^2 + y'^2}\,dt. \end{aligned} \tag{6}$$

Curve (Equation (3)) for the resulting spatial isotropic line is the projection of line in Equation (6) on the xOy plane, and the isotropic curve itself is an imaginary curve. If an imaginary view curve is taken as a flat curve of a form

$$\begin{aligned} x &= x(t)\,i, \\ y &= y(t)\,i, \end{aligned} \tag{7}$$

then the isotropic line equation has the form

$$\begin{aligned} x &= x(t)\,i, \\ y &= y(t)\,i, \\ z &= \int \sqrt{x'^2 + y'^2}\,dt. \end{aligned} \tag{8}$$

The isotropic curve (Equation (8)) is an imaginary curve, and $z = z(t)$ is a straight line.

If in expressions Equation (3) a function of this variable $f(t)$ is substituted for the variable t, Equation (3) could be written as

$$\begin{aligned} x &= x\left(f(t)\right), \\ y &= y\left(f(t)\right). \end{aligned} \tag{9}$$

Thus, the equations of an isotropic curve have the form

$$\begin{aligned} x &= x\left(f(t)\right), \\ y &= y\left(f(t)\right), \\ z &= i\int f_t'\sqrt{\left(x_f'\right)^2 + \left(y_f'\right)^2}\,dt. \end{aligned} \tag{10}$$

Korovina [8] proposed two methods for finding isotropic spatial curves: on the basis of two analytic functions and on the basis of the equation of a spatial curve. For two analytic functions $f_1(t)$ and $f_2(t)$, the isotropic curve equations are

$$\begin{aligned} x &= f_1(t) + if_2(t), \\ y &= f_1(t) - if_2(t), \\ z &= \sqrt{2}\int \sqrt{\left(f_2'\right)^2 - \left(f_1'\right)^2}\,dt. \end{aligned} \tag{11}$$

For a spatial curve $x = x(t), y = y(t), z = z(t)$ equations are transformed so that the resulting curve is a zero-length curve. For this, the desired equations are written in the following form

$$\begin{aligned} x &= x(t) + u(t), \\ y &= y(t) + v(t), \\ z &= z(t), \end{aligned} \tag{12}$$

where $u(t)$ and $v(t)$ are some unknown functions.

The condition of equality of the length of the arc to zero is written in the form of the following system of equations:

$$\begin{aligned} x'^2 + y'^2 + z'^2 + u'^2 + v'^2 &= 0, \\ x'u' + y'v' &= 0. \end{aligned} \tag{13}$$

Cartan [10] considered the problem of contact and comparison of isotropic curves. The Frenet trihedron, which is defined by three unit vectors, which are plotted tangentially, principal normal and binormal, does not exist for a minimal curve, because any vector tangentially plotted has a length of zero. In addition, the normal and contiguous planes for such a curve coincide. Therefore, Cartan was engaged in joining the isotropic curve families of trihedrons of another type. He considered a trihedron created by the point O and three vectors $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$, which have a point O as its origin and whose components are equal, respectively

$$\begin{aligned} \mathbf{I}_1 &= \left[0,5 \quad 0,5i \quad 0\right], \\ \mathbf{I}_2 &= \left[0 \quad 0 \quad 1\right], \\ \mathbf{I}_3 &= \left[1 \quad -i \quad 0\right]. \end{aligned} \tag{14}$$

The trihedron, which is defined by the vectors in Equation (14), and all trihedrons equal to it, will be called the right cyclic trihedron. Trihedrons that are symmetrical to the right cyclic trihedrons are called left cyclic trihedrons. There is one and only one movement that turns one given cyclic trihedron into another cyclic trihedron of the same orientation. No offset translates the right cyclic trihedron to the left. The right and left cyclic trihedrons form two continuous families without common elements. Cartan introduces the concept of trihedron of the first and second order, the vertex of this trihedron belongs to a given isotropic curve, and the difference will be associated with Pfaffian form [10].

3. Simulation of Isotropic Fractional-Rational Curves

Let the fractional-rational curve of the nth order be given as

$$\mathbf{r}(\mathbf{t}) = \frac{\sum_{j=0}^{n} \mathbf{r_j} w_j J_{n,j}(t)}{\sum_{j=0}^{n} w_j J_{n,j}(t)}, \qquad J_{n,j}(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}, \tag{15}$$

where $\mathbf{r_j}$ – reference point vectors, given in a complex form, $\mathbf{r_j} = [x_j, y_j, z_j]$, w_j – weights of points.

We will build isotropic fractional-rational curves, that is, curves for which condition (Equation (2)) is fulfilled. We will separately consider modelling two-dimensional and three-dimensional curves. At the first stage, we consider the modelling of two-dimensional curves. We will use isotropic segments as the basic elements, which will then form the basis for isotropic characteristic polygons and isotropic chords. To construct isotropic polygons on the plane, we have conditions in form

$$\begin{gathered} (x_1 - x_0)^2 + (y_1 - y_0)^2 = 0, \\ \cdots, \\ (x_n - x_{n-1})^2 + (y_n - y_{n-1})^2 = 0, \\ (x_0 - x_n)^2 + (y_0 - y_n)^2 = 0, \end{gathered} \tag{16}$$

where n is the number of sides of the polygon.

Let us find out under what conditions Equations (16) are valid. To do this, we use the condition by which the segment of the polygon will be isotropic in form

$$y_{j+1} = \pm i\left(x_{j+1} - x_j\right) + y_j. \tag{17}$$

We will consistently find the ordinates of the points of the polygon and substitute into the equation of the system of Equations (16). We get

$$\begin{aligned} y_1 &= \pm i\left(x_1 - x_0\right) + y_0, \\ y_2 &= \pm i\left(x_2 - x_1\right) + y_1 = \pm i\left(x_2 - x_1\right) \pm i\left(x_1 - x_0\right) \\ &\quad + y_0, \\ &\cdots, \\ y_n &= i\sum_{j=0}^{n-1} \pm\left(x_{j+1} - x_j\right) + y_0, \\ y_0 &= i\sum_{j=0}^{n-1} \pm\left(x_{j+1} - x_j\right) \pm i\left(x_0 - x_n\right) + y_0. \end{aligned} \tag{18}$$

The last equality can be written as

$$\sum_{j=0}^{n-1} \pm\left(x_{j+1} - x_j\right) \pm \left(x_0 - x_n\right) = 0. \tag{19}$$

Equations (18) and (19) determine the isotropy conditions for an arbitrary polygon on the plane. These conditions can be used to model characteristic polygons for fractional-rational curves. In [18], it was proved that a fractional-rational curve of the nth order in the form of Equation (15) is isotropic if the isotropy conditions for the characteristic polygon and the condition of independence from the values of the weight of points are satisfied.

The derivation of the generalized equation for isotropic fractional-rational curves of the nth order will be a very cumbersome form; therefore, we give an example of obtaining such equations only for the third order [19].

Let the fractional-rational curve of the third order be given in the form

$$\mathbf{r}(t) = \frac{\mathbf{r}_0 w_0 (1-t)^3 + 3\mathbf{r}_1 w_1 (1-t)^2 t + 3\mathbf{r}_2 w_2 (1-t) t^2 + \mathbf{r}_3 w_3 t^3}{w_0 (1-t)^3 + 3w_1 (1-t)^2 t + 3w_2 (1-t) t^2 + w_3 t^3}, \tag{20}$$

where $\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are the reference points of the characteristic quadrilateral with the corresponding weights w_0, w_1, w_2, w_3.

Let us take the derivative of the expression (Equation (20)) in form

$$\mathbf{r}'(t) = \frac{p_1 p_2 - p_3 p_4}{\left(w_0 (1-t)^3 + 3w_1 (1-t)^2 t + 3w_2 (1-t) t^2 + w_3 t^3\right)^2}, \tag{21}$$

where

$$\begin{aligned}
p_1 &= \left(-\mathbf{r}_0 w_0 + 3\mathbf{r}_1 w_1 - 3\mathbf{r}_2 w_2 + \mathbf{r}_3 w_3\right) t^2 \\
&\quad + 2\left(\mathbf{r}_0 w_0 - 2\mathbf{r}_1 w_1 + \mathbf{r}_2 w_2\right) t \\
&\quad - \left(\mathbf{r}_0 w_0 - \mathbf{r}_1 w_1\right), \\
p_2 &= \left(-w_0 + 3w_1 - 3w_2 + w_3\right) t^3 \\
&\quad + \left(3w_0 - 6_1 w_1 + 3w_2\right) t^2 - 3\left(w_0 - w_1\right) t \\
&\quad + w_0, \\
p_3 &= \left(-\mathbf{r}_0 w_0 + 3\mathbf{r}_1 w_1 - 3\mathbf{r}_2 w_2 + \mathbf{r}_3 w_3\right) t^3 \\
&\quad + \left(3\mathbf{r}_0 w_0 - 6_1 \mathbf{r}_1 w_1 + 3\mathbf{r}_2 w_2\right) t^2 \\
&\quad + \left(-3\mathbf{r}_0 w_0 + 3\mathbf{r}_1 w_1\right) t + \mathbf{r}_0 w_0, \\
p_4 &= \left(-w_0 + 3w_1 - 3w_2 + w_3\right) t^2 \\
&\quad + 2\left(w_0 - 2w_1 + w_2\right) t + \left(-w_0 + w_1\right).
\end{aligned} \tag{22}$$

Let us substitute the derivative in the condition for curves of zero length (Equation (2)). This condition is satisfied and does not depend on the value of the parameter. If the coefficients at all powers of t are zero, the denominator should not be zero. Thus we get conditions in form

(i) for coefficients at t^0 and t^8:

$$\begin{aligned}
&\sum_{r=x,y,z} \left(r_1 - r_0\right)^2 = 0, \\
&\sum_{r=x,y,z} \left(r_3 - r_2\right)^2 = 0,
\end{aligned} \tag{23}$$

(ii) for coefficients at t^1 and t^7:

$$\begin{aligned}
&\sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_2 - r_0\right) = 0, \\
&\sum_{r=x,y,z} \left(r_3 - r_2\right)\left(r_3 - r_1\right) = 0,
\end{aligned} \tag{24}$$

(iii) for coefficients at t^2 and t^6:

$$\begin{aligned}
&2w_2^2 \sum_{r=x,y,z} \left(r_2 - r_0\right)^2 + w_1 w_3 \sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_3 - r_0\right) \\
&\quad = 0, \\
&2w_1^2 \sum_{r=x,y,z} \left(r_3 - r_1\right)^2 + w_0 w_2 \sum_{r=x,y,z} \left(r_3 - r_2\right)\left(r_3 - r_0\right) \\
&\quad = 0,
\end{aligned} \tag{25}$$

(iv) for coefficients at t^3 and t^5:

$$\begin{aligned}
&w_0 w_2 w_3 \sum_{r=x,y,z} \left(r_2 - r_0\right)\left(r_3 - r_0\right) \\
&\quad + w_1^2 w_3 \sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_3 - r_1\right) \\
&\quad + 3w_1 w_2^2 \sum_{r=x,y,z} \left(r_2 - r_0\right)\left(r_2 - r_1\right) = 0, \\
&w_0 w_1 w_3 \sum_{r=x,y,z} \left(r_3 - r_0\right)\left(r_3 - r_1\right) \\
&\quad + w_0 w_2^2 \sum_{r=x,y,z} \left(r_2 - r_0\right)\left(r_3 - r_2\right) \\
&\quad + 3w_1^2 w_2 \sum_{r=x,y,z} \left(r_2 - r_1\right)\left(r_3 - r_1\right) = 0,
\end{aligned} \tag{26}$$

(v) for coefficient at t^4:

$$\begin{aligned}
&w_0^2 w_3^2 \sum_{r=x,y,z} \left(r_3 - r_0\right)^2 + 9w_1^2 w_2^2 \sum_{r=x,y,z} \left(r_2 - r_1\right)^2 \\
&\quad + w_0 w_1 w_2 w_3 \left[6 \sum_{r=x,y,z} \left(r_3 - r_0\right)\left(r_2 - r_1\right)\right. \\
&\quad + 8 \sum_{r=x,y,z} \left(r_2 - r_0\right)\left(r_3 - r_1\right) \\
&\quad \left. + 2 \sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_3 - r_2\right)\right] = 0.
\end{aligned} \tag{27}$$

Let us turn to the values of the links of the characteristic polygon. We introduce in the conditions Equations (23)–(27) such replacements

$$\begin{aligned}
r_2 - r_0 &= \left[r_2 - r_1\right] + \left[r_1 - r_0\right], \\
r_3 - r_1 &= \left[r_3 - r_2\right] + \left[r_2 - r_1\right], \\
r_3 - r_0 &= \left[r_3 - r_2\right] + \left[r_2 - r_1\right] + \left[r_1 - r_0\right].
\end{aligned} \tag{28}$$

After simplification we get

(i) for coefficients at t^1 and t^7:

$$\begin{aligned}
&\sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_2 - r_1\right) = 0, \\
&\sum_{r=x,y,z} \left(r_3 - r_2\right)\left(r_2 - r_1\right) = 0,
\end{aligned} \tag{29}$$

(ii) for coefficients at t^2 and t^6:

$$\begin{aligned}
&2w_2^2 \sum_{r=x,y,z} \left(r_2 - r_1\right)^2 + w_1 w_3 \sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_3 - r_2\right) \\
&\quad = 0, \\
&2w_1^2 \sum_{r=x,y,z} \left(r_2 - r_1\right)^2 + w_0 w_2 \sum_{r=x,y,z} \left(r_1 - r_0\right)\left(r_3 - r_2\right) \\
&\quad = 0,
\end{aligned} \tag{30}$$

(iii) for coefficients at t^3 and t^5:

$$\begin{aligned}&\left(w_0w_2w_3+3w_1w_2^2\right)\sum_{r=x,y,z}\left(r_2-r_1\right)^2\\&\quad+\left(w_0w_2w_3+w_1^2w_3\right)\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)=0,\\&\left(w_0w_1w_3+3w_1^2w_2\right)\sum_{r=x,y,z}\left(r_2-r_1\right)^2\\&\quad+\left(w_0w_1w_3+w_0w_2^2\right)\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)=0,\end{aligned}\tag{31}$$

(iv) for coefficient at t^4:

$$\begin{aligned}&\left(w_0^2w_3^2+9w_1^2w_2^2+14w_0w_1w_2w_3\right)\sum_{r=x,y,z}\left(r_2-r_1\right)^2\\&\quad+\left(2w_0^2w_3^2+10w_0w_1w_2w_3\right)\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)\\&=0.\end{aligned}\tag{32}$$

From Equations (30) we can find dependencies for w_0 та w_3 in form

$$\begin{aligned}w_3&=\frac{-2w_2^2\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{w_1\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)},\\w_0&=\frac{-2w_1^2\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{w_2\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)}.\end{aligned}\tag{33}$$

After substituting the dependencies found (Equations (33)) into Equations (31) and (32), we obtain three identical equations that do not depend on the values of weight

$$2\sum_{r=x,y,z}\left(r_2-r_1\right)^2+\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)=0.\tag{34}$$

Let us rewrite dependencies (Equations (33)) in the following form:

$$\begin{aligned}\frac{w_3w_1}{-2w_2^2}&=\frac{\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)},\\\frac{w_0w_2}{-2w_1^2}&=\frac{\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)}.\end{aligned}\tag{35}$$

Equating in the obtained expressions (Equations (35)) the left parts, that is, getting rid of the values of the links of the characteristic polygon, we obtain the dependence between the values of the weight of points in form

$$w_0w_2^3=w_3w_1^3.\tag{36}$$

As a result of the permutations made and reductions, the isotropy conditions for fractional-rational curves of the third order will have the following form

$$\begin{aligned}&\sum_{r=x,y,z}\left(r_1-r_0\right)^2=0,\\&\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_2-r_1\right)=0,\\&w_3=\frac{-2w_2^2\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{w_1\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)},\\&2\sum_{r=x,y,z}\left(r_2-r_1\right)^2+\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)=0,\\&w_0=\frac{-2w_1^2\sum_{r=x,y,z}\left(r_2-r_1\right)^2}{w_2\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_3-r_2\right)},\\&\sum_{r=x,y,z}\left(r_1-r_0\right)\left(r_2-r_1\right)=0,\\&\sum_{r=x,y,z}\left(r_3-r_2\right)^2=0.\end{aligned}\tag{37}$$

Let us analyze the obtained conditions (Equations (37)). The first, second, fourth, sixth, and seventh conditions coincide with the conditions for isotropic cubic Bezier curves [20]. That is, we can formulate the statement that a fractional-rational curve is isotropic if the isotropy conditions for the Bezier curve constructed on the basis of reference points of the fractional-rational curve are satisfied.

In case when

$$\sum_{r=x,y,z}\left(r_2-r_1\right)^2=0\tag{38}$$

we obtain conditions under which the fractionally rational curve remains isotropic, regardless of the weight value. In addition, if we draw an analogy with Bezier curve, then such an isotropic fractional-rational curve lies in a plane which is arbitrarily located in three-dimensional space.

4. Surface Modelling Based on Isotropic Fractional-Rational Curves

We apply the constructed isotropic fractional-rational curves for modelling three-dimensional surfaces. The surface is modelled on the basis of a conformal change of the parameter in the equation of the curve and the selection of the real part. In this case, we obtain minimal surfaces.

Let us consider several options for creating third order isotropic curves using conditions (37) and determine the equation of minimal surfaces.

4.1. The Isotropic Guide Curve Created Using the Analytical Function. Third order isotropic curve can be represented in form

$$f(t)=\sum_{j=0}^{3}a_jt^j.\tag{39}$$

After substitution $f(t)$ in Equations (1) and then in Equation (20), and replacing $t=u+iv$ we get

$$\mathbf{r}(u+iv)=\frac{\mathbf{r}_0 w_0 (1-u-iv)^3+3\mathbf{r}_1 w_1 (1-u-iv)^2 (u+iv)+3\mathbf{r}_2 w_2 (1-u-iv)(u+iv)^2+\mathbf{r}_3 w_3 (u+iv)^3}{w_0 (1-u-iv)^3+3w_1 (1-u-iv)^2 (u+iv)+3w_2 (1-u-iv)(u+iv)^2+w_3 (u+iv)^3}, \tag{40}$$

where

$$\begin{aligned}
&\mathbf{r}_0 \left[(a_0-a_2)i \quad a_0+a_2 \quad -a_1 i\right],\\
&\mathbf{r}_1 \left[(a_0-a_2-a_3)i \quad a_0+a_2+a_3 \quad -a_1 i\right],\\
&\mathbf{r}_2 \left[(a_0-a_2-2a_3)i \quad a_0+a_2+2a_3 \quad (a_3-a_1)i\right],\\
&\mathbf{r}_3 \left[(a_0-a_2-2a_3)i \quad a_0+a_2+4a_3 \quad (3a_3-a_1)i\right].
\end{aligned} \tag{41}$$

After selecting the real part of Equation (40), we obtain the equation of the minimal surface.

4.2. The Guide Curve Constructed Using a Flat Curve Deformation. We calculate the coefficients of the first and second quadratic forms as usual in form

$$\begin{aligned}
F&=\sum_{r=x,y.z} r_{,u}(u,v)\, r_{,v}(u,v),\\
E&=\sum_{r=x,y,z} r_{,u}(u,v)^2,\\
G&=\sum_{r=x,y,z} r_{,v}(u,v)^2,\\
L&=\frac{1}{\sqrt{D}}\begin{vmatrix} x_{,uu} & y_{,uu} & z_{,uu}\\ x_{,u} & y_{,u} & z_{,u}\\ x_{,v} & y_{,v} & z_{,v}\end{vmatrix},\\
M&=\frac{1}{\sqrt{D}}\begin{vmatrix} x_{,uv} & y_{,uv} & z_{,uv}\\ x_{,u} & y_{,u} & z_{,u}\\ x_{,v} & y_{,v} & z_{,v}\end{vmatrix},\\
N&=\frac{1}{\sqrt{D}}\begin{vmatrix} x_{,vv} & y_{,vv} & z_{,vv}\\ x_{,u} & y_{,u} & z_{,u}\\ x_{,v} & y_{,v} & z_{,v}\end{vmatrix},
\end{aligned} \tag{42}$$

where $D = EG - F^2$, subscript after comma means differentiation according to the corresponding parameter. We calculate the average curvature H for the surface in the form

$$H=\frac{1}{2D^2}(LG-2MF+NE). \tag{43}$$

If the scales for a fractional-rational curve are equal to each other, that is

$$w_0=w_1=w_2=w_3, \tag{44}$$

then we obtain a surface with a coordinate grid of curvature lines for which $M = 0$. In other cases, we have an arbitrary orthogonal and isothermal grid [21].

4.3. Simulation of an Isotropic Fractional-Rational Curve of the Third Order Based on the Deformation of a Flat Curve. One of the approaches to the formation of spatial curves is the formation of the third coordinate on the basis of a given flat curve. This technique leads to the use of approximation methods. We construct a spatial isotropic curve based on a given flat isotropic curve without applying an approximation.

Let isotropic third-order fractional-rational curve be constructed using expressions (Equation (15)) with conditions Equations (18) and (19). We will deform a flat isotropic curve so that the length of the curve in the complex space remains unchanged, that is, isotropic. For a flat fractional-rational curve, all sides of the characteristic polygon and chord are zero, and for a spatial curve this condition should not be stored; therefore, we will develop a method for changing the points of the characteristic polygon while maintaining the isotropy condition of the curve length.

The number of isotropy conditions for a fractional-rational curve of the third order according to Equations (37) is 7. These conditions define 5 unknown coordinates and 2 weights of points. The number of coordinates for a spatial fractional-rational curve is 12. The number of coordinates that must be set is equal to 7. We set the known coordinates for a spatial curve: applicate z_0 and coordinates of the points of the flat curve $x_j, y_j, j=0\ldots 2$. Consider the first isotropy condition for a fractionally rational curve of the third order, namely, the isotropy of the side of the characteristic polygon, which coincides in direction with the tangent at the point

$$\mathbf{r_0}: \sum_{r=x,y,z}(r_1-r_0)^2=0 \tag{45}$$

According to the initial data, only the coordinate z_1 remains unknown in such an expression. Considering that for a flat curve we get

$$(x_1-x_0)^2+(y_1-y_0)^2=0, \tag{46}$$

the first condition is

$$z_1-z_0=0, \tag{47}$$

from where we have a mandatory value

$$z_1=z_0. \tag{48}$$

Substituting these relations into second condition in the form

$$\sum_{r=x,y,z}(r_1-r_0)(r_2-r_1)=0, \tag{49}$$

we obtain the identity equality to zero. So, the second condition allows to specify an arbitrary coordinate z_2, which is determined and depends on the order of the curve. The following unknown coordinates are determined by successively substituting the coordinates found in the previous equations. As a result, we obtain analytical expressions for each coordinate; that is, changing points interactively does not lead to the application of numerical methods.

4.4. Use of Isotropic Planar Fractional-Rational Curves for Modelling of Orthogonal and Isothermal Grids. We transform a flat mesh into a surface with the condition of orthogonality [18]. In order to form a surface from a flat orthogonal grid, which will be assigned to an orthogonal u-v-grid, we need to distribute one family of lines of a flat grid along the height with an arbitrary law [22, 23]. That is, the equation $z = z(u, v)$ should be dependent only on one variable:

$$z = z(u) \quad \text{or } z = z(v). \tag{50}$$

Then the partial derivative with respect to one of the variables for z is zero and the coefficient F does not change and is zero when moving from a flat orthogonal grid to a spatial grid. Consequently, the surface will be assigned to families of orthogonal coordinate lines.

If the flat grid is built on the basis of a fractional-rational curve, then, if we set the dependence $z = z(u)$ as a function in Equation (15), we have a surface with a spatial curve directing a fractional-rational curve for the cases $t = u + iv$ and $t = u + ikv$. For $t = ku + iv$ the spatial curve has a given projection on the grid. If we set the dependence $z = z(v)$; then, the surface is constructed on the basis of a flat curve.

Consider the construction of surfaces based on networks constructed on the basis of third-order isotropic fractional-rational curves. We set the third coordinate, depending on one parameter; for example, on the parameter u in form

$$z(u) = \frac{z_0 w_0 (1-u)^3 + 3z_1 w_1 (1-u)^2 u + 3z_2 w_2 (1-u) u^2 + z_3 w_3 u^3}{w_0 (1-u)^3 + 3w_1 (1-u)^2 u + 3w_2 (1-u) u^2 + w_3 u^3}. \tag{51}$$

Two coordinates for the conformal replacement may be found from [18] in form

$$x(u+iv) = \frac{\sum_{j=0}^{3} x_j w_j J_{n,j}(u+iv)}{\sum_{j=0}^{3} w_j J_{n,j}(u+iv)},$$
$$y(u+iv) = y_0 + \frac{i \sum_{j=1}^{3} \left(x_j - x_0\right) w_j J_{n,j}(u+iv)}{\sum_{j=0}^{3} w_j J_{n,j}(u+iv)}, \tag{52}$$

where

$$J_{3,j}(u+iv) = \frac{3!}{j!(3-j)!} (u+iv)^j (1-u-iv)^{3-j}. \tag{53}$$

For quasiconformal replacements, only the substitution parameter changes in this equation. If we select the real part, then we obtain the equation of the surface.

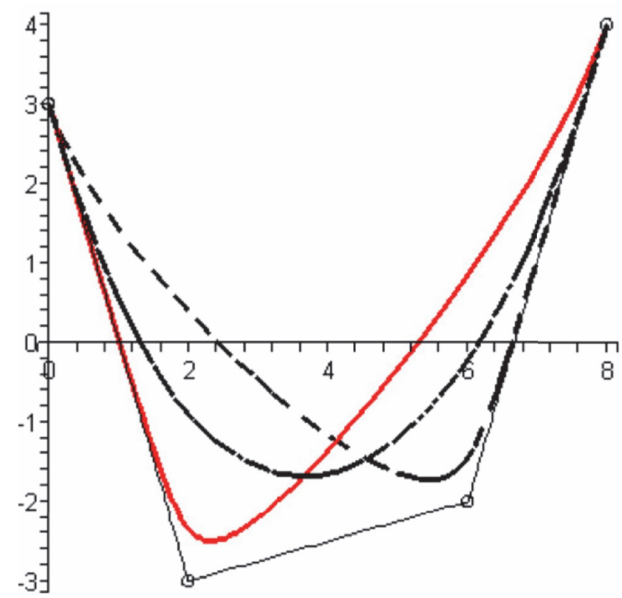


Figure 1: Cubic fractional-rational curves with different weights. Solid line – for $w_0 = 1$, $w_1 = 8$, $w_2 = 1$, $w_3 = 1$, dotted line – $w_0 = 1$, $w_1 = 1$, $w_2 = 8$, $w_3 = 1$, dash-dotted line – $w_0 = 1$, $w_1 = 2$, $w_2 = 2$, $w_3 = 1$. The reference points are indicated by circles and connected by a broken line.

5. Examples of Modelling Isotropic Curves and Surfaces Associated with Them

5.1. Plane Isotropic Third-Order Fractional-Rational Curves Based on an Isotropic Quadrangle with Different Weights. Let us set

$$x_0 = 0,$$
$$x_1 = 2 + 6i,$$
$$x_2 = 6 + 5i, \tag{54}$$
$$x_3 = 8 - 1i,$$
$$y_0 = 3 - 8i.$$

Let us find y_1, y_2, y_3 on the basis of Equations (18) and (19) with a positive sign:

$$y_1 = -3 - 6i,$$
$$y_2 = -2 - 2i, \tag{55}$$
$$y_3 = 4.$$

According to Equation (15) we get curves for different weights (see Figure 1). All three constructed curves have zero length and indefinite curvature.

5.2. Spatial Isotropic Fractional-Rational Curves of the Third Order. Let the three links of the characteristic polygon be zero and the following initial coordinates are given

$$\begin{aligned} x_0 &= 1 + i, \\ y_0 &= 1 - 2i, \\ z_0 &= 2 + i, \\ x_1 &= 3 - 3i, \\ x_2 &= 5 + 2i, \\ x_3 &= 6 - 6i, \\ y_3 &= 1 - i. \end{aligned} \tag{56}$$

To find the ordinates, we apply the condition for Bezier curve [15]

$$\begin{aligned} y_1 &= \frac{y_3 (x_1 - x_0) + y_0 (x_3 - x_1)}{x_3 - x_0}, \\ y_2 &= \frac{(y_1 - y_0)(x_2 - x_1)}{x_1 - x_0}. \end{aligned} \tag{57}$$

Based on these expressions, we get

$$\begin{aligned} y_1 &= 1,08 - 1,486i, \\ y_2 &= 0,55 - 1,82i. \end{aligned} \tag{58}$$

From the condition of isotropy of the links of the characteristic quadrilateral we obtain the z coordinates, which are determined depending on the chosen sign in form

$$z_{j+1} = \pm i \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2} + z_j. \tag{59}$$

If the sign in the expression is positive, then

$$\begin{aligned} z_1 &= 6,02 + 2,98i, \\ z_2 &= 1,02 + 5,02i, \\ z_3 &= 9,05 + 5,97i. \end{aligned} \tag{60}$$

If the sign in the expression is negative, then

$$\begin{aligned} z_1 &= -2,02 - 0,98i, \\ z_2 &= 2,98 - 3,02i, \\ z_3 &= -5,05 - 3,97i. \end{aligned} \tag{61}$$

Let us set weights

$$\begin{aligned} w_0 &= 1, \\ w_1 &= 4, \\ w_2 &= 15, \\ w_3 &= 1. \end{aligned} \tag{62}$$

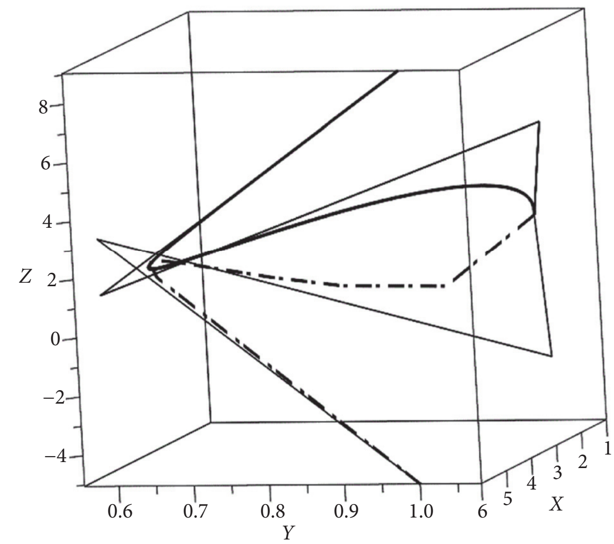


FIGURE 2: Real parts of cubic isotropic fractional-rational curves constructed on the basis of a flat curve deformation. Solid bold line corresponds to positive sign and dotted-dashed bold line to negative sign. The reference points are connected by broken thin lines.

In Figure 2 the real parts of two isotropic fractional-rational curves of the third order are reflected. The lengths of the curves in the complex space are zero.

Let us construct a spatial isotropic fractional-rational curve of the third order, if the two links of the characteristic polygon are equal to zero. Let us set

$$\begin{aligned} x_0 &= 0, \\ y_0 &= 3 - 8i, \\ z_0 &= 0, \\ x_1 &= 2 + i, \\ x_2 &= 6 - 3i, \\ z_2 &= 3 - 4i. \end{aligned} \tag{63}$$

The points of the characteristic quadrilateral are calculated on the basis of the deformation of a flat isotropic curve [16]. That is, we get

$$\begin{aligned} y_1 &= 2 - 6i, \\ y_2 &= -2i, \\ z_1 &= z_0 = 0, \\ z_3 &= 15 - 20i, \\ x_3 &= 21,6 + 15,2i, \\ y_3 &= 4,2 - 1,6i. \end{aligned} \tag{64}$$

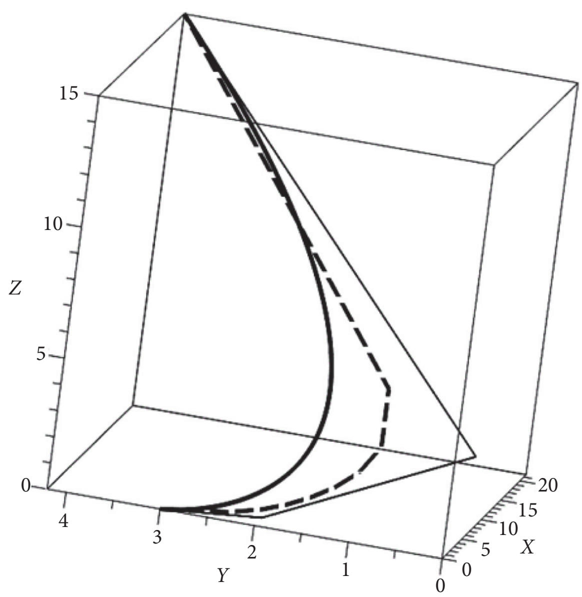


Figure 3: Real parts of cubic isotropic fractional-rational curves of the third order. Solid bold line corresponds to the isotropy condition and dashed bold line to their absence. The reference points are connected by broken thin line.

Let us check the isotropy condition (Equations (37)). We build two curves on the basis of one characteristic quadrilateral with different weights. For the first curve, we define

$$\begin{aligned} w_1 &= 4, \\ w_2 &= 8, \end{aligned} \tag{65}$$

and calculate the weight value based on the isotropy condition, so we have

$$\begin{aligned} w_0 &= 2, \\ w_3 &= 16. \end{aligned} \tag{66}$$

For the second curve, we set weights arbitrarily, for example, as

$$\begin{aligned} w_0 &= 0,5, \\ w_1 &= 2, \\ w_2 &= 1, \\ w_3 &= 0,1. \end{aligned} \tag{67}$$

For the first curve, the length is zero and for the second it is not. In Figure 3 the curves are plotted in Figure 4 reflected graphs of their curvature and torsion.

5.3. Minimal Surfaces Based on Different Functions. Firstly we construct a minimal surface based on the analytic function of the form

$$f(t) = 1 + 2i + (3 + 1i)t + (2 + 4i)t^2 + (3 + 6i)t^3. \tag{68}$$

The coordinates of the points of the characteristic quadrilateral will be

$$\begin{aligned} &\mathbf{r}_0 \left[2 - 1i \quad 3 + 6i \quad 1 - 3i\right], \\ &\mathbf{r}_1 \left[8 - 4i \quad 6 + 12i \quad 1 - 3i\right], \\ &\mathbf{r}_2 \left[14 - 7i \quad 9 + 18i \quad -5\right], \\ &\mathbf{r}_3 \left[14 - 7i \quad 15 + 30i \quad -17 + 6i\right]. \end{aligned} \tag{69}$$

Let us substitute the obtained values into (40). We construct and compare surfaces for different weights. For the first case we set

$$\begin{aligned} w_1 &= 4, \\ w_2 &= 8, \end{aligned} \tag{70}$$

and for the second one we set

$$\begin{aligned} w_1 &= 0,5, \\ w_2 &= 3. \end{aligned} \tag{71}$$

From condition (Equations (37)), we find the weight value at other points:

for the first surface

$$\begin{aligned} w_0 &= 2, \\ w_3 &= 16, \end{aligned} \tag{72}$$

for the second surface

$$\begin{aligned} w_0 &= 0,08, \\ w_3 &= 18. \end{aligned} \tag{73}$$

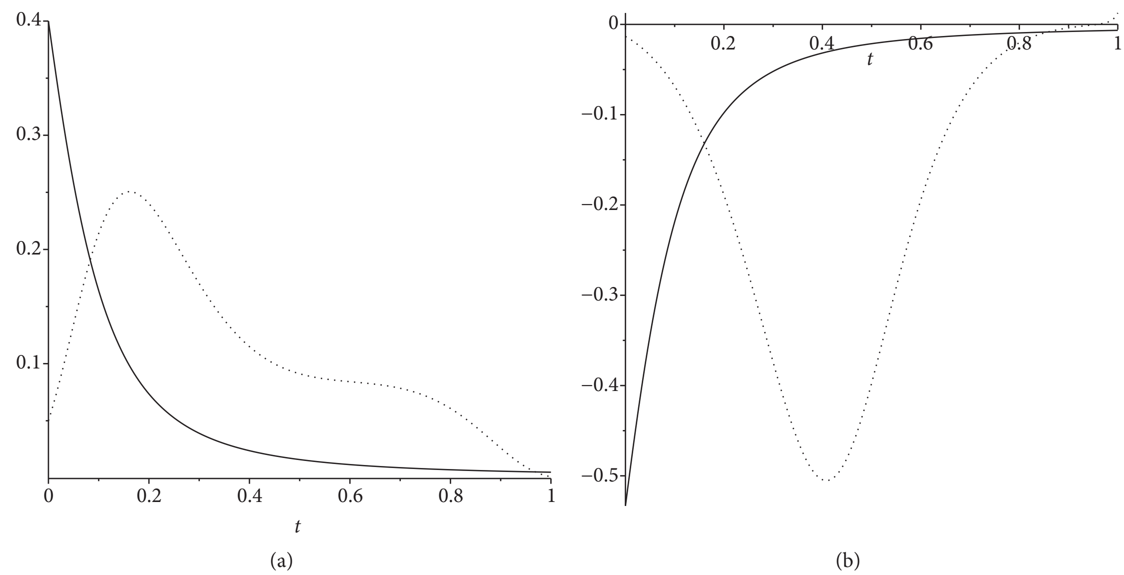


FIGURE 4: Parameters for real parts of cubic isotropic fractional-rational curves, constructed on the basis of the deformation of a flat curve. Solid line corresponds to the isotropy condition and dotted line to its absence. (a) Curvature. (b) Torsion.

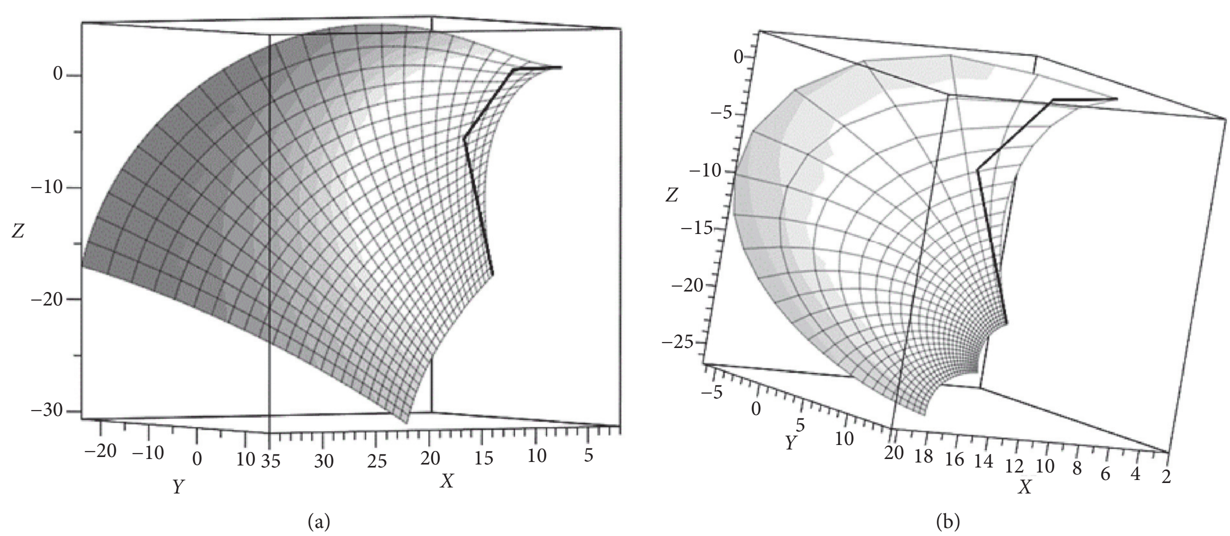


FIGURE 5: Minimal surfaces built with different weights. (a) The first case. (b) The second case. The reference points are connected by broken bold line.

The simulated surfaces are shown in Figure 5. The coefficients of the first quadratic form for given values for two surfaces satisfy the equations

$$\begin{aligned} E &= G, \\ F &= 0, \end{aligned} \tag{74}$$

that is, the surfaces are minimal.

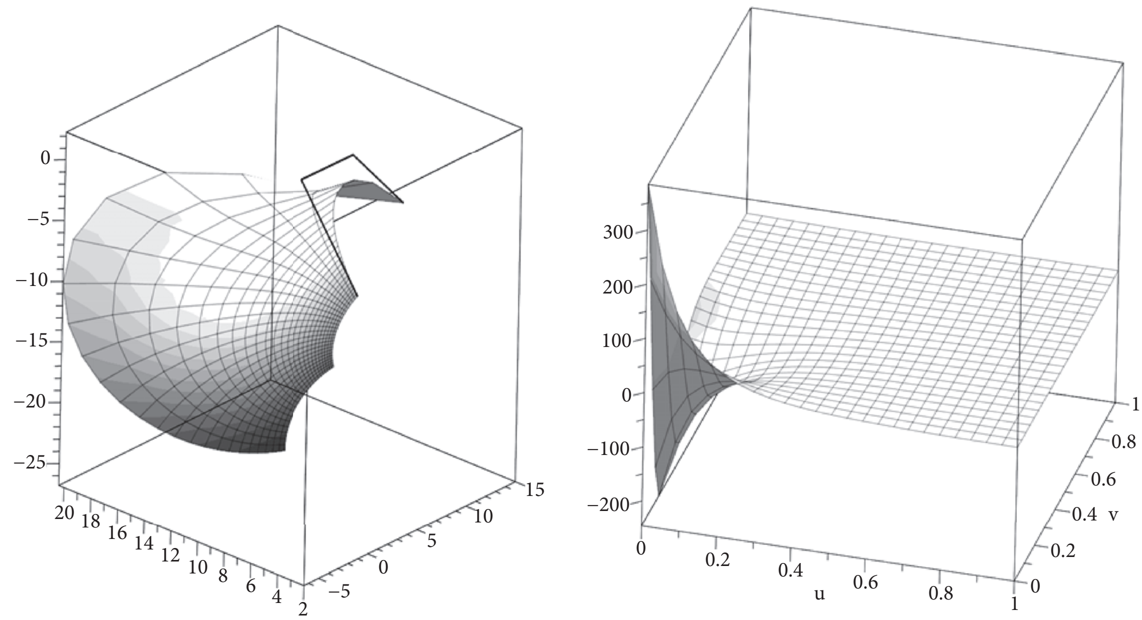
In Figure 6 a minimal surface is constructed on the basis of a fractional-rational curve with different weights and a graph of the function M for such a surface is shown.

5.4. Surface Modelling on the Basis of Grids Constructed on the Basis of Third-Order Isotropic Fractional-Rational Curves. In Figure 7 is shown a constructed surface on the basis of a flat grid, created using a fractional-rational curve of the third order, and a graph of the function F, which indicates the grid orthogonality.

Figure 8 shows grids based on fractional-rational curves of the second order based on real and complex values of weight and conformal and quasiconformal replacements.

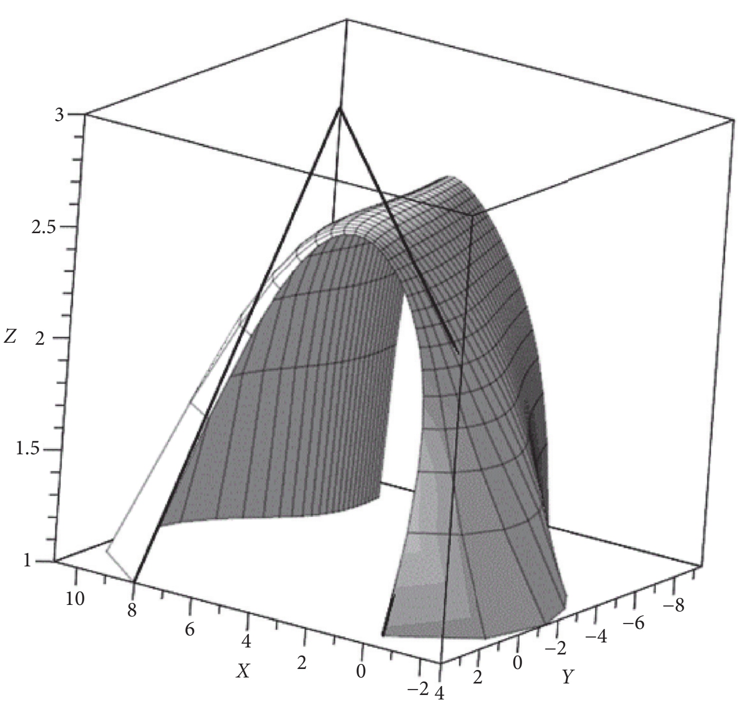
6. Conclusions

As a result of the studies, surfaces were constructed on the basis of isotropic fractional-rational curves, which allow controlling the shape of curves and grids in user mode without



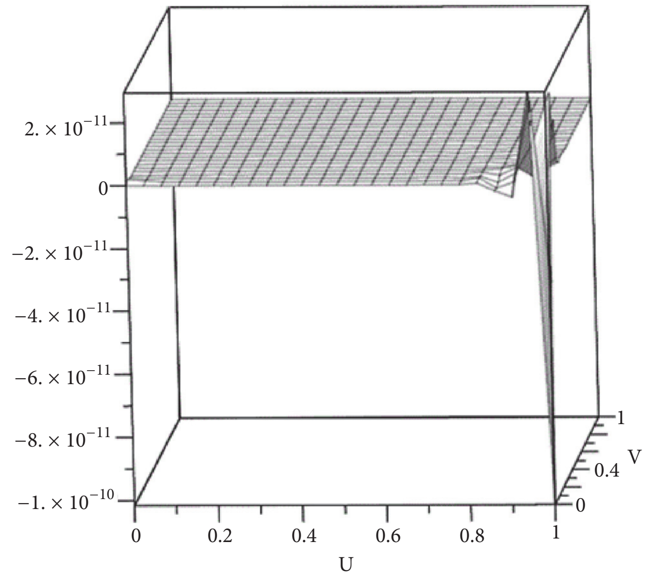
(a) (b)

Figure 6: Minimal surfaces built on the basis of a fractional-rational curve. (a) The surface. The reference points are connected by broken bold line. (b) Graph of the function M.



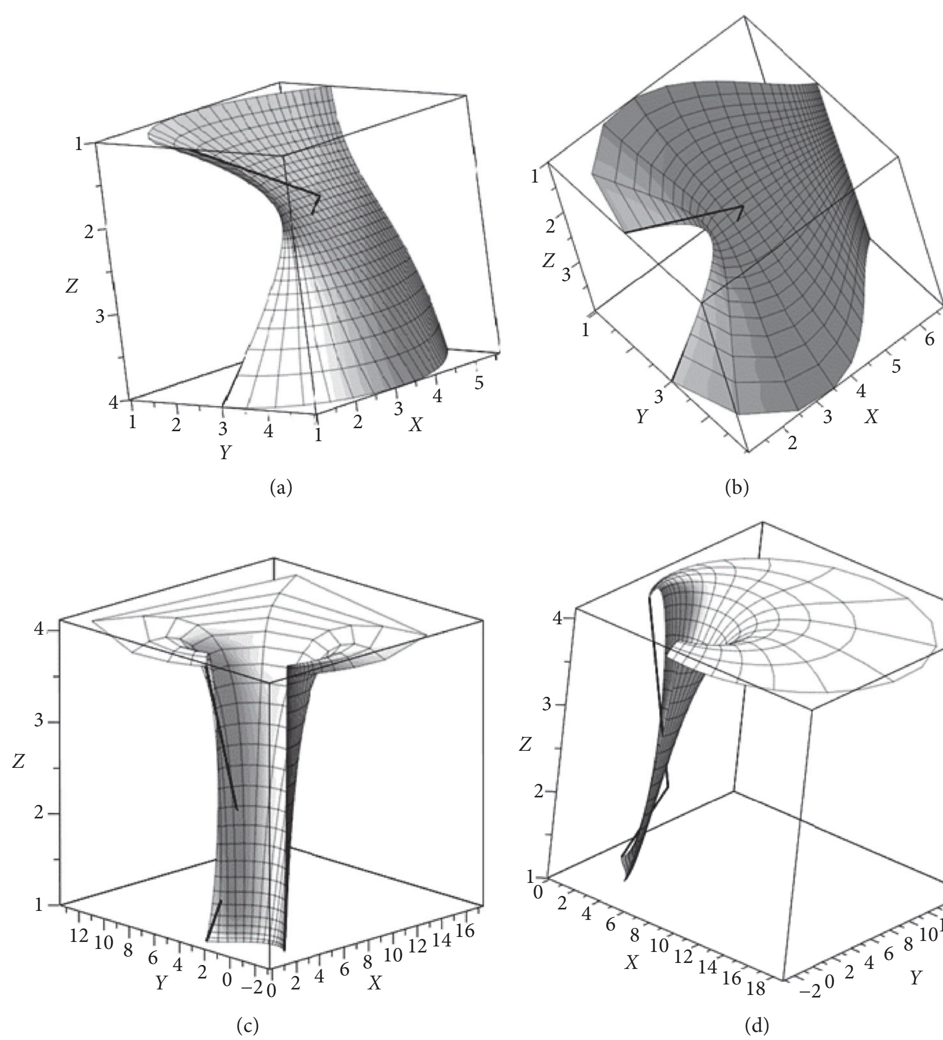
(a)

Figure 7: Continued.



(b)

Figure 7: Minimal surfaces built on the basis of a fractional-rational curve. (a) The surface. The reference points are connected by broken bold line. (b) Graph of the function M.



(a) (b) (c) (d)

Figure 8: Surfaces based on a fractional-rational curve of the second order and a flat isotropic grid. The reference points are connected by broken bold line. (a) Real values of weights, $t = u + iv$. (b) Real values of weights, $t = u + i3v$. (c) Complex values of weights, $t = u + iv$. (d) Complex values of weights, $t = u + i3v$.

recalculating of expressions for surface parameters determination. This makes it possible to speed up the processing of images when using computer graphics while maintaining important surface properties, in particular, isotropism and orthogonality of grid lines. Closed formulas for calculating the parameters of such surfaces are proposed and the isotropy conditions of the reference curves are established.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The research and publication of this article were funded by the authors themselves.

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