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Research Article

Partial Contraction Analysis of Coupled Fractional Order Systems

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Contraction theory regards the convergence between two arbitrary system trajectories. In this article we have introduced partial contraction theory as an extension of contraction theory to analyze coupled identical fractional order systems. It can, also, be applied to study the synchronization phenomenon in networks of various structures and with arbitrary number of systems. We have used partial contraction theory to derive exact and global results on synchronization and antisynchronization of fractional order systems.

1. Introduction

Since the Dutch researcher, Christiaan Huygens, initiated the study of synchronization phenomenon in the 17^{th} century [1], many researchers of various fields, such as mathematics [2–4], robotics [5], electronics [6], neuroscience, and biology [7, 8], have investigated coupled oscillators and the stability of synchrony between coupled systems. Due to the significance of stability in the control theory [9] and synchronization phenomena, many techniques have been proposed to examine this property [10–12]; these include the contraction theory which is a more recent tool for analyzing the stability and convergence behavior of nonlinear systems in state space form [13–15]. In his dissertation, Soon-Jo Chung focused on the synchronization of multiple dynamical systems using the contraction theory, with applications in the cooperative control of multiagent systems and synchronization of interconnected dynamics such as tethered formation flight. He used contraction theory to prove that a nonlinear control law stabilizing a single-tethered spacecraft can also stabilize arbitrarily large circular arrays of tethered spacecraft, as well as a three-spacecraft inline configuration [16].

Partial contraction method is employed to investigate the dynamics of coupled nonlinear systems, based on contraction analysis. Partial contraction theory extends contraction theory to include convergence of specific properties and it gives a general device for investigating the stability of systems. It is particularly suitable for the study of synchronization behaviors. In his thesis, using partial contraction theory, Wei Wang studied the spontaneous synchronization behavior of nonlinear networked systems [17].

In the last two decades, scientists have used fractional differential equations to model several physical phenomena. For the recent history of fractional calculus and state space representation, reference can be made to [18] and [19], respectively. Because of its significant role in engineering and modern sciences [20, 21], the study of the stability of fractional order systems (FOSs) [22] and synchronization of FOSs has attracted much attention [6, 8]. In this article, *partial contraction* method was developed to investigate the dynamics of coupled FOSs, with emphasis on the study of FOSs synchronization.

The remainder of this paper is organized as follows: in Section 2, after a summary of the contraction theory, the partial contraction theory was introduced for FOSs. In Section 3, the theory was clarified by studying the synchronization of FOSs and then the analysis was generalized to study synchronization in networks of arbitrary number of nonlinear FOSs. Examples are given to illustrate the concept.

2. Contraction and Partial Contraction Analysis of Fractional Order Systems

Basically, a nonlinear time-varying dynamic system is said to be *contracting*, if the initial conditions are exponentially forgotten, that is, if the dynamics of the system does not depend on the initial conditions and all trajectories converge to their nominal movement exponentially. The *partial contraction* theory extends the applications of contraction theory to a network of FOSs. In the extension, convergence to the specific properties of the systems is considered.

2.1. Contraction Theory. The basic definition of contraction theory for FOSs is summarized and the details can be found in [23].

First, consider the integer order system

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}, \mathrm{t}) \tag{1}$$

where x and f are $\mathrm{n} \times 1$ state vector and vector function, respectively. Suppose that $\mathrm{f}(\mathrm{x}, \mathrm{t})$ is continuously differentiable, (1) gives the relation

$$\delta\dot{\mathrm{x}} = \frac{\partial \mathrm{f}}{\partial \mathrm{x}}(\mathrm{x}, t)\,\delta \mathrm{x} \tag{2}$$

where $\delta\mathrm{x}$ is a virtual displacement between two neighboring trajectories of system (1). The rate of change of squared distance $(\delta\mathrm{x})^T\delta\mathrm{x}$ between two trajectories is defined as

$$\frac{d(\delta\mathrm{x})^T\delta\mathrm{x}}{dt} = 2(\delta\mathrm{x})^T\delta\dot{\mathrm{x}} = 2(\delta\mathrm{x})^T\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\delta\mathrm{x}. \tag{3}$$

Theorem 1. *If matrix $\partial\mathrm{f}/\partial\mathrm{x}$ is uniformly negative definite, all the solution trajectories of system (1) will converge exponentially to a single trajectory, irrespective of their initial conditions.*

Proof. See [13]. □

Definition 2. For the given system (1), a region $\Omega \subseteq \mathbb{R}^n$ of the state space is called a contraction (semicontraction) region, if the matrix $\partial\mathrm{f}/\partial\mathrm{x}$ is uniformly negative definite (negative semidefinite) in that region.

Note that, by matrix G, being uniformly negative definite means that the symmetric part of matrix G (that is, $(1/2)(G + G^T)$) is negative definite; in other words,

$$\frac{1}{2}\left(G + G^T\right) \leq -\beta\mathrm{I} < 0, \quad \exists\beta > 0,\ \forall\mathrm{x},\ \forall t \geq 0, \tag{4}$$

and we mean a *region*, an open connected set.

Consider now, an FOS:

$$D_t^{\alpha}\mathrm{x}(t) = \mathbf{f}(\mathrm{x}(t)) \tag{5}$$

where x and f are like that of the integer order system, and D_t^{α} indicates the fractional derivative of order $\alpha \in \mathbb{R}^+$. In this article, the Riemann-Liouville fractional operator was used as the main derivation tool which for order $\alpha > 0$, $\mathrm{m} - 1 < \alpha < m$, $m \in \mathbb{N}$, is defined as [23]

$$ {}_{a}^{RL}D^{\alpha}\mathrm{f}(x) = D^m J_a^{(m-\alpha)}\mathrm{f}(x), \tag{6}$$

where the operator J_a^{α} is defined on $L_1[a, b]$ by

$$J_a^{\alpha}\mathrm{f}(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\mathrm{f}(\tau)}{(x-\tau)^{1-\alpha}}d\tau, \tag{7}$$

for $a \leq x \leq b$ and is called the Riemann-Liouville fractional integral operator of order α. For simplicity, the left superscript RL and subscript a were omitted and D_t^{α} was assumed to be the Riemann-Liouville α-order fractional derivative operator which was assumed to exist and be continuous.

Theorem 3. *Assume that $\alpha_1, \alpha_2 \geq 0$. Moreover, let $\phi \in L_1[a, b]$ and $\mathrm{x}(t) = J_a^{\alpha_1+\alpha_2}\phi$. Then,*

$$D^{\alpha_1}D^{\alpha_2}\mathrm{x}(t) = D^{\alpha_1+\alpha_2}\mathrm{x}(t). \tag{8}$$

Proof. See [24]: Page 29. □

Note that the condition on $\mathrm{x}(t)$ implies a certain degree of smoothness and the fact that, as $t \longrightarrow a$, $\mathrm{x}(t) \longrightarrow 0$ sufficiently fast.

Suppose that the conditions of Theorem 3 hold. By applying $D_t^{1-\alpha}$ to both sides of (5), one can write

$$\dot{\mathrm{x}} = D_t^{1-\alpha}\mathbf{f}(\mathrm{x}(t)) \tag{9}$$

Therefore, the contraction condition for system (5) is stated as follows [23].

Theorem 4. *If matrix $G = D_x^{\alpha}D_t^{1-\alpha}\mathrm{f}(\mathrm{x}(t))(D_x^{\alpha}\mathrm{x})^{-1}$ is uniformly negative definite, all the solution trajectories of system (5) converge to a single trajectory, with exponential rate, irrespective of the initial conditions.*

Proof. See [23]. □

Definition 5. For the given FOS (5), a region $\Omega \subseteq \mathbb{R}^n$ of the state space is called contraction (semicontraction), if the matrix $D_x^{\alpha}D_t^{1-\alpha}\mathrm{f}(\mathrm{x}(t))(D_x^{\alpha}\mathrm{x})^{-1}$ is uniformly negative definite (negative semidefinite) in that region. By convention, $\mathrm{f}(\mathrm{x}(t))$ and system (5) are called contracting function and contracting system, respectively.

It is worth mentioning that, if $\Omega = \mathbb{R}^n$, then global exponential convergence is guaranteed.

Remark 6. Consider the linear time-invariant (LTI) FOS:

$$D_t^{\alpha}\mathrm{x} = \mathrm{A}\mathrm{x}. \tag{10}$$

Applying $D_t^{1-\alpha}$ to both sides of (10), one can write

$$\dot{\mathrm{x}} = D_t^{1-\alpha}\mathrm{A}\mathrm{x}. \tag{11}$$

Considering two neighboring trajectories of the above equation and the virtual displacement $\delta \mathrm{x}$ between them yields

$$\delta \dot{\mathrm{x}} = \frac{\partial D_t^{1-\alpha} \mathrm{Ax}}{\partial \mathrm{x}} \delta \mathrm{x}. \quad (12)$$

The rate of change of the squared distance $(\delta \mathrm{x})^T \delta \mathrm{x}$ between two neighboring trajectories of the system (10) is given by

$$\frac{d (\delta \mathrm{x})^T \delta \mathrm{x}}{dt} = 2 (\delta \mathrm{x})^T \delta \dot{\mathrm{x}} = 2 (\delta \mathrm{x})^T \left[\frac{\partial D_t^{1-\alpha} \mathrm{Ax}}{\partial \mathrm{x}} \right] \delta \mathrm{x}. \quad (13)$$

Considering

$$D_t^{1-\alpha} \mathrm{Ax} = \frac{1}{\Gamma (1 - (1 - \alpha))} \frac{d}{dt} \int_0^t \frac{\mathrm{Ax} (\tau)}{(t - \tau)^{1-\alpha}} d\tau, \quad (14)$$

we have

$$\frac{\partial D_t^{1-\alpha} \mathrm{Ax}}{\partial \mathrm{x}} = \frac{1}{\Gamma (\alpha)} \frac{d}{dt} \int_0^t \frac{\mathrm{A}}{(t - \tau)^{1-\alpha}} d\tau = \frac{\mathrm{A} t^{\alpha-1}}{\Gamma (\alpha)} \quad (15)$$

So, LTI system (10) is contracting (semicontracting) if $\mathrm{A}t^{\alpha-1}/\Gamma(\alpha)$ in $[t_0, \infty)$ is uniformly negative definite (negative semidefinite).

2.2. Partial Contraction Theory. The partial contraction theory was first introduced during the study of network synchronization [25]; since then, its applicability and flexibility have been proven in many fields. Now, the partial contraction theory for the FOSs, which is the base of this work, is introduced.

Theorem 7. *Consider an FOS*

$$D_t^{\alpha} \mathrm{x} = \mathrm{f} (\mathrm{x}, \mathrm{x}, t) \quad (16)$$

and its auxiliary system

$$D_t^{\alpha} \mathrm{y} = \mathrm{f} (\mathrm{y}, \mathrm{x}, t) \quad (17)$$

which is contracting with respect to y. Then trajectories of the original x-system verify each smooth specific property which auxiliary system verifies.

Proof. Two particular solutions of the virtual y-system are $\mathrm{y}(t) = \mathrm{x}(t)$ and the solution with the specific property. Given that the virtual system is contracting, the solution $\mathrm{x}(t)$ converges to the solution with the specific property. □

Definition 8. The original FOS (16) is said to be partially contracting.

Corollary 9. *A convex combination of* $D_t^{\alpha} \mathrm{x} = \mathbf{f}_i(\mathrm{x}, t)$, $i = 1, \dots, n$, *which are contracting with a common trajectory* $\mathrm{x}_0(t)$, *is contracting.*

Proof. Consider the convex combination

$$D_t^{\alpha} \mathrm{x} = \sum_i \alpha_i (\mathrm{x}, t) \mathbf{f}_i (\mathrm{x}, t) \quad (18)$$

which has a common trajectory $\mathrm{x}_0(t)$ (for instance, a common equilibrium), where $\alpha_i(\mathrm{x}, t) \geq 0$ and $\sum_i \alpha_i(\mathrm{x}, t) = 1$. Consider the contracting auxiliary system

$$D_t^{\alpha} \mathrm{y} = \sum_i \alpha_i (\mathrm{x}, t) \mathrm{f}_i (\mathrm{y}, t) \quad (19)$$

with two particular solutions $\mathrm{x}(t)$ and $\mathrm{x}_0(t)$. Therefore, all trajectories of the system converge to the trajectory $\mathrm{x}_0(t)$. □

Remark 10. The notion of a virtual contracting system can be applied to control problems. For example, consider a nonlinear fractional order control system:

$$D_t^{\alpha} \mathrm{x} = \mathrm{f} (\mathrm{x}, \mathrm{x}, \mathrm{u}, t) \quad (20)$$

and assume that it is desired to reach the state $\mathrm{x}_d(t)$, using the control input $\mathrm{u}(\mathrm{x}, \mathrm{x}_{\mathbf{d}}, t)$ such that

$$D_t^{\alpha} \mathrm{x}_d = \mathrm{f} (\mathrm{x}_d, \mathrm{x}, \mathrm{u}, t) . \quad (21)$$

Now, it is enough to have a contracting auxiliary system:

$$D_t^{\alpha} \mathrm{y} = \mathrm{f} (\mathrm{y}, \mathrm{x}, \mathrm{u}, t) \quad (22)$$

with two particular solutions $\mathrm{x}(t)$ and $\mathrm{x}_d(t)$ to guarantee the convergence of x to x_d.

3. Coupled Fractional Order Systems

Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$. Consider a pair of identical systems

$$D_t^{\alpha} \mathbf{x}_1 = \mathbf{f} (\mathbf{x}_1, t) \quad (23)$$

$$D_t^{\alpha} \mathbf{x}_2 = \mathbf{f} (\mathbf{x}_2, t) + \mathbf{u} (\mathbf{x}_1) - \mathbf{u} (\mathbf{x}_2) \quad (24)$$

which are paired together in a unidirectional (one-way) coupling way, with coupling force $\mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2)$.

Theorem 11. *If the function* $(\mathbf{f} - \mathbf{u})(\mathbf{x}_2)$ *in (24) is contracting, two systems (23) and (24) will be synchronized.*

Proof. A particular solution of second system is $\mathbf{x}_2(t) = \mathbf{x}_1(t)$, and the second system as virtual system is contracting. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ converge to each other and will be synchronized. □

Example 12. Consider two coupled identical fractional order financial systems [26]:

$$D_t^{\alpha} \mathbf{x}_1 = \mathbf{f} (\mathbf{x}_1) \quad (25)$$

and

$$D_t^{\alpha} \mathbf{x}_2 = \mathbf{f} (\mathbf{x}_2) + \mathbf{u} (\mathbf{x}_1) - \mathbf{u} (\mathbf{x}_2) \quad (26)$$

with

$$\mathbf{x_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$
$$\mathbf{x_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$
$$\mathbf{f}(\mathbf{x}) = \begin{cases} z + (y-a)x \\ 1 - by - x^2 \\ -x - cz, \end{cases}$$
$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} xy \\ -x^2 \\ -x \end{bmatrix}. \tag{27}$$

Therefore,

$$\mathbf{f}(\mathbf{x}_1) = \begin{cases} D_t^{\alpha} x_1 = z_1 + (y_1 - a) x_1 \\ D_t^{\alpha} y_1 = 1 - by_1 - x_1^2 \\ D_t^{\alpha} z_1 = -x_1 - cz_1 \end{cases} \tag{28}$$

and

$$\mathbf{f}(\mathbf{x}_2) + \mathbf{u}(\mathbf{x}_1) - \mathbf{u}(\mathbf{x}_2) = \begin{cases} D_t^{\alpha} x_2 = z_2 + (y_2 - a) x_2 + x_1 y_1 - x_2 y_2 \\ D_t^{\alpha} y_2 = 1 - by_2 - x_2^2 - x_1^2 + x_2^2 \\ D_t^{\alpha} z_2 = -x_2 - cz_2 - x_1 + x_2. \end{cases} \tag{29}$$

In this example, x, y, and z denote the interest rate, the investment demand, and the price index, respectively. The positive constants a, b, and c are the saving amount, the cost per investment, and the demand elasticity of commercial markets, respectively (see [26]).

The system

$$(\mathbf{f} - \mathbf{u})(\mathbf{x}) = \begin{cases} z - ax \\ 1 - by \\ -cz \end{cases} = \begin{bmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{30}$$

is linear. With parameters $a = 1$, $b = 0.1$, and $c = 1$, the coefficient matrix A in $\mathbf{f} - \mathbf{u}$ is uniformly negative definite, so $\mathbf{f} - \mathbf{u}$ is contracting and the states of two coupled fractional order financial systems will synchronize (Figure 1).

Remark 13. The extension of Theorem 11 for a network of n FOSs with an open chain structure

$$\begin{aligned} D_t^{\alpha} \mathrm{x}_1 &= \mathrm{f}(\mathrm{x}_1, t) \\ D_t^{\alpha} \mathrm{x}_2 &= \mathrm{f}(\mathrm{x}_2, t) + \mathrm{u}(\mathrm{x}_1) - \mathrm{u}(\mathrm{x}_2) \\ &\vdots \\ D_t^{\alpha} \mathrm{x}_n &= \mathrm{f}(\mathrm{x}_n, t) + \mathrm{u}(\mathrm{x}_{n-1}) - \mathrm{u}(\mathrm{x}_n) \end{aligned} \tag{31}$$

has the same synchronization condition as that for systems (23) and (24).

Theorem 14. *Consider two identical systems which are paired together in a bidirectional (two-way) coupling method of the form*

$$\begin{aligned} D_t^{\alpha} \mathrm{x}_1 &= \mathrm{f}(\mathrm{x}_1, t) + \mathrm{u}(\mathrm{x}_2) - \mathrm{u}(\mathrm{x}_1) \\ D_t^{\alpha} \mathrm{x}_2 &= \mathrm{f}(\mathrm{x}_2, t) + \mathrm{u}(\mathrm{x}_1) - \mathrm{u}(\mathrm{x}_2) \end{aligned} \tag{32}$$

In such a system, if $\mathbf{f} - 2\mathbf{u}$ *is contracting, then* x_1 *and* x_2 *are synchronized.*

Proof. From the coupled system, we have

$$\begin{aligned} &D_t^{\alpha} \mathrm{x}_1 - \mathrm{f}(\mathrm{x}_1, t) - \mathrm{u}(\mathrm{x}_2) + \mathrm{u}(\mathrm{x}_1) \\ &\quad = D_t^{\alpha} \mathrm{x}_2 - \mathrm{f}(\mathrm{x}_2, t) - \mathrm{u}(\mathrm{x}_1) + \mathrm{u}(\mathrm{x}_2); \end{aligned} \tag{33}$$

define

$$\begin{aligned} \mathrm{g}(\mathrm{x}_1, \mathrm{x}_2, t) &= D_t^{\alpha} \mathrm{x}_1 - \mathrm{f}(\mathrm{x}_1, t) \\ &= D_t^{\alpha} \mathrm{x}_2 - \mathrm{f}(\mathrm{x}_2, t) - 2\mathrm{u}(\mathrm{x}_1) + 2\mathrm{u}(\mathrm{x}_2) \end{aligned} \tag{34}$$

and consider the following auxiliary system:

$$D_t^{\alpha} \mathrm{y} = \mathrm{f}(\mathrm{y}, t) + \mathrm{g}(\mathrm{x}_1, \mathrm{x}_2, t) + 2\mathrm{u}(\mathrm{x}_1) - 2\mathrm{u}(\mathrm{y}) \tag{35}$$

which has two particular solutions, $\mathrm{y} = \mathrm{x}_1(t)$ and $\mathrm{y} = \mathrm{x}_2(t)$. Given that $\mathbf{f} - 2\mathbf{u}$ is contracting, the auxiliary system is contracting. Therefore, $\mathrm{x}_1(t)$ and $\mathrm{x}_2(t)$, as two solutions of the auxiliary system, converge together. □

Researchers have given varying definitions to synchronization under different contexts. In this study, synchronization or *complete synchronization* is that the difference of states of synchronized systems converges to zero; that is, $\mathrm{x}_1 = \mathrm{x}_2$. In the case of *phase synchronization*, the difference between various states of synchronized systems converges to a constant vector or even a periodic state instead of the zero. Similarly, antisynchronization or *antiphase synchronization* was defined as $\mathrm{x}_1 = -\mathrm{x}_2$.

Now, one can state the following theorem which is more general than Theorems 11 and 14.

Theorem 15 (synchronization). *Consider two systems coupled in an arbitrary manner. If there is a contraction function,* $\mathrm{h}(\mathrm{x}, t)$*, such that*

$$D_t^{\alpha} \mathrm{x}_1 - \mathrm{h}(\mathrm{x}_1, t) = D_t^{\alpha} \mathrm{x}_2 - \mathrm{h}(\mathrm{x}_2, t), \tag{36}$$

then x_1 *and* x_2 *will be synchronized.*

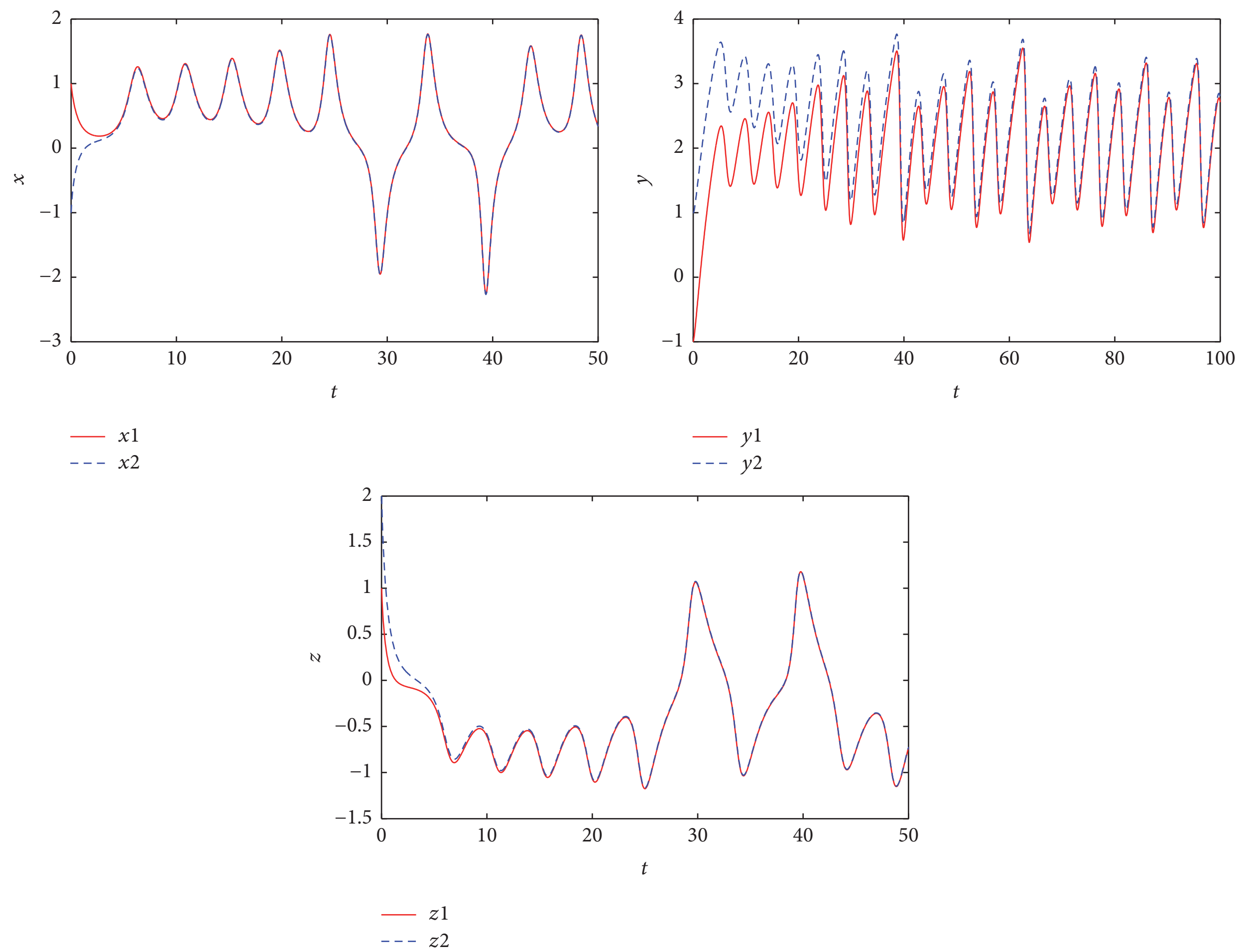

FIGURE 1: Synchronization of states of two financial FOSs with parameters $a = 1$, $b = 0.1$, $c = 1$, and $\alpha = 0.87$.

Proof. Let $x_1(t)$ and $x_2(t)$ be two trajectories of the coupled systems. Define

$$g(x_1, x_2, t) = D_t^{\alpha} x_1 - h(x_1, t) = D_t^{\alpha} x_2 - h(x_2, t), \quad (37)$$

and, therefore,

$$\begin{aligned} D_t^{\alpha} x_1 &= h(x_1, t) + g(x_1, x_2, t) \\ D_t^{\alpha} x_2 &= h(x_2, t) + g(x_1, x_2, t) \end{aligned} \quad (38)$$

Suppose the auxiliary system is as follows:

$$D_t^{\alpha} y = h(y, t) + g(x_1, x_2, t) \quad (39)$$

Due to the contraction of function h, the auxiliary system is contracting. Therefore, solutions of auxiliary system converge together exponentially, and it is true for the solutions, $y = x_1(t)$ and $y = x_2(t)$. □

Remark 16. (1) Theorems 11 and 14 are two special cases of Theorem 15. In fact, in Theorem 11, $h(x, t) = (f - u)(x, t)$, and in Theorem 14 $h(x, t) = (f - 2u)(x, t)$.

(2) For a network of n systems with a complete graph structure

$$\begin{aligned} D_t^{\alpha} x_1 &= f(x_1, t) + \sum_{i=2}^{n} u(x_i) - u(x_1) \\ D_t^{\alpha} x_2 &= f(x_2, t) + \sum_{\substack{i=1 \\ i \neq 2}}^{n} u(x_i) - u(x_2) \\ &\vdots \\ D_t^{\alpha} x_n &= f(x_n, t) + \sum_{i=1}^{n-1} u(x_i) - u(x_n) \end{aligned} \quad (40)$$

synchronization condition is like the condition of Theorem 14.

(3) For a network containing n oscillators with all-to-all symmetry, that is, a network in which each system is coupled to all the others,

$$D_t^{\alpha} x_i = f(x_i, t) + \sum_{j=1}^{n} \left(u(x_j) - u(x_i) \right) \quad i = 1, 2, \ldots, n \quad (41)$$

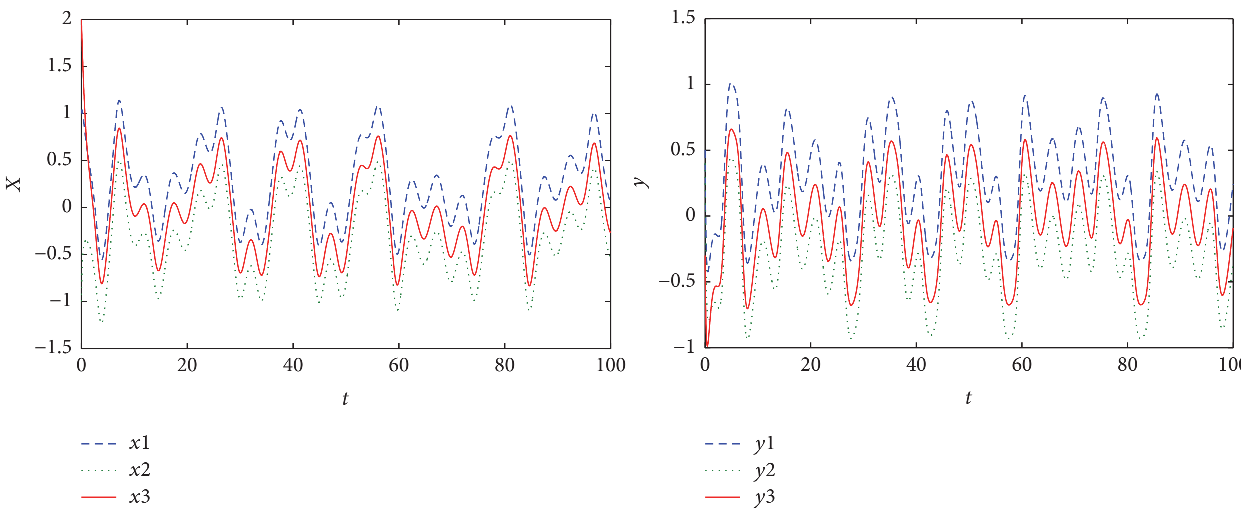

FIGURE 2: Phase synchronization of three Duffing systems for initial points $(x_{1,0}, y_{1,0}) = (1, 0.5)$, $(x_{2,0}, y_{2,0}) = (-1, 0.4)$, and $(x_{3,0}, y_{3,0}) = (2, -0.3)$ and parameters $a = b = 0.35$, $c = 0.3$, and $d = 0.2$.

if $\mathbf{f} - n\mathbf{u}$ is contracting, synchronization of the whole network is guaranteed.

Example 17. The forced Duffing FOS [27] is as follows:

$$\begin{aligned} D_t^\alpha x &= y \\ D_t^\alpha y &= ax - by - x^3 + c\cos(2\pi dt) \end{aligned} \tag{42}$$

which is chaotic for the parameter values $a = b = 0.35$, $c = 0.3$, and $d = 0.2$. Using a coupling function to make a network of three identical Duffing FOSs yields

$$\begin{aligned} D_t^\alpha x_1 &= y_1 + x_1 - x_1 + s(x_2 + x_3 - x_1) \\ D_t^\alpha y_1 &= ax_1 - by_1 - x_1^3 + c\cos(2\pi dt) \\ &\quad + s(-x_2^3 - x_3^3 + x_1^3), \\ D_t^\alpha x_2 &= y_2 + x_2 - x_2 + s(x_1 + x_3 - x_2) \\ D_t^\alpha y_2 &= ax_2 - by_2 - x_2^3 + c\cos(2\pi dt) \\ &\quad + s(-x_1^3 - x_3^3 + x_2^3), \end{aligned} \tag{43}$$

and

$$\begin{aligned} D_t^\alpha x_3 &= y_3 + x_3 - x_3 + s(x_1 + x_2 - x_3) \\ D_t^\alpha y_3 &= ax_3 - by_3 - x_3^3 + c\cos(2\pi dt) \\ &\quad + s(-x_1^3 - x_2^3 + x_3^3), \end{aligned} \tag{44}$$

where s is the coupling strength; the values $s = 0.5$ and $\alpha = 0.85$ were chosen.

Therefore,

$$\mathbf{f} - 2\mathbf{u} = \begin{bmatrix} y - x \\ ax - by \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ a & -b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{45}$$

Choosing $a = b = 0.35$, $\alpha = 0.85$, the coefficient matrix $A = \left[\begin{smallmatrix} -1 & 1 \\ a & -b \end{smallmatrix}\right]$ is negative semidefinite; therefore, $\mathbf{f} - 2\mathbf{u}$ is semicontracting and phase synchronization occurs. As we can see in Figure 2 trajectories of the given systems have been synchronized with constant difference. If $a = 0.2$, $b = 0.35$, the coefficient matrix A is uniformly negative definite and $\mathbf{f} - 2\mathbf{u}$ is contracting, so, complete synchronization occurs. As we can see in Figure 3 trajectories of the given systems converge to each other and complete synchronization occurs. But, choosing $a = 0.4$ and $b = 0.35$, the coefficient matrix A is positive definite and $\mathbf{f} - 2\mathbf{u}$ is not contracting and, as shown in Figure 4, two coupled systems diverge.

Theorem 18. *If the vector function* $\mathrm{h}(\mathrm{x}, t)$ *in*

$$\begin{aligned} D_t^\alpha \mathrm{x}_1 &= \mathrm{h}(\mathrm{x}_1, t) + \mathrm{u}(\mathrm{x}_2, t) - \mathrm{u}(\mathrm{x}_1, t) \\ D_t^\alpha \mathrm{x}_2 &= \mathrm{h}(\mathrm{x}_2, t) + \mathrm{u}(\mathrm{x}_1, t) - \mathrm{u}(\mathrm{x}_2, t) \end{aligned} \tag{46}$$

is contracting and $\mathrm{h}(-\mathrm{x}, t) = -\mathrm{h}(\mathrm{x}, t)$*, then* $\mathrm{x}_1 + \mathrm{x}_2$ *will converge to zero. Moreover, for each initial condition, other than zero, there will be antisynchrony between* x_1 *and* x_2*, if the system*

$$D_t^\alpha \mathrm{z} = \mathrm{h}(\mathrm{z}, t) - 2\mathrm{u}(\mathrm{z}, t) \tag{47}$$

has a stable limit-cycle.

Proof. From (46) and oddness of h, we have

$$\begin{aligned} D_t^\alpha \mathrm{x}_1 - \mathrm{h}(\mathrm{x}_1, t) &= \mathrm{u}(\mathrm{x}_2, t) - \mathrm{u}(\mathrm{x}_1, t) \\ &= -D_t^\alpha \mathrm{x}_2 + \mathrm{h}(\mathrm{x}_2, t) \\ &= D_t^\alpha(-\mathrm{x}_2) - \mathrm{h}(-\mathrm{x}_2, t) \end{aligned} \tag{48}$$

and, therefore, from Theorem 15, it is concluded that x_1 and $-\mathrm{x}_2$ converge to each other exponentially; in other words, x_1 and x_2 reach antisynchrony. □

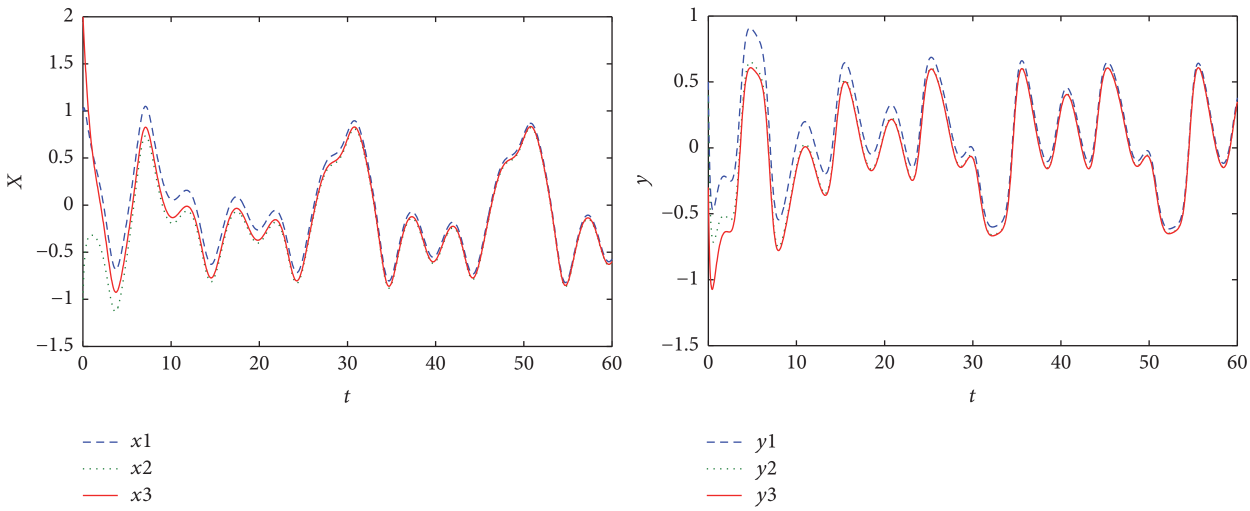

Figure 3: Complete synchronization of three Duffing systems for initial points $(x_{1,0}, y_{1,0}) = (1, 0.5)$, $(x_{2,0}, y_{2,0}) = (-1, 0.4)$, and $(x_{3,0}, y_{3,0}) = (2, -0.3)$ and parameters $a = 0.2$, $b = 0.35$, $c = 0.3$, and $d = 0.2$.

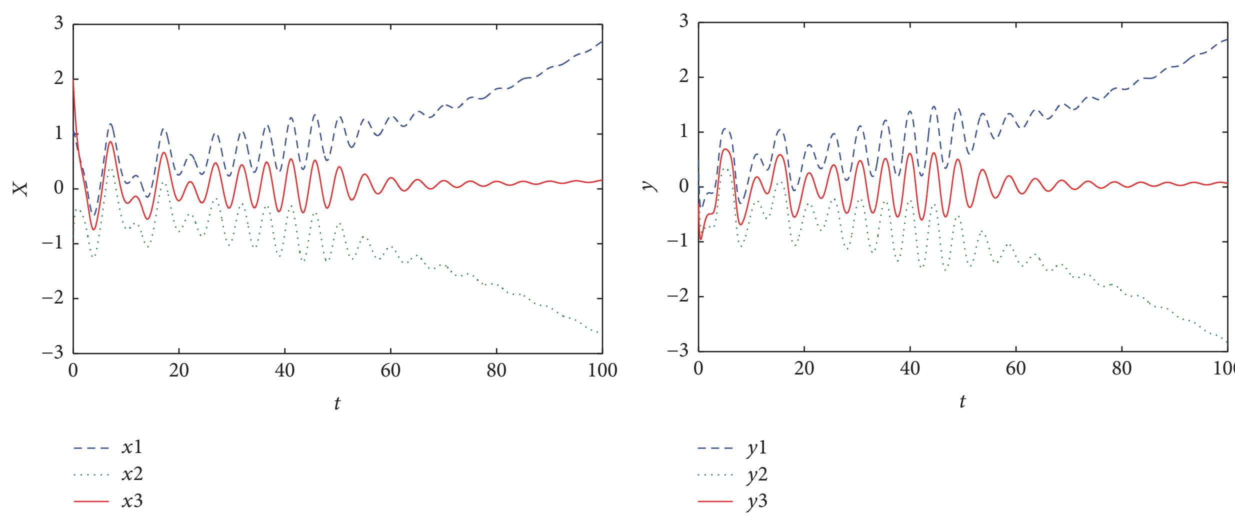

Figure 4: Divergence of three coupled Duffing systems for $a = 0.4$, $b = 0.35$, $c = 0.3$, and $d = 0.2$.

Example 19. Two unforced Duffing systems

$$\begin{aligned} D_t^{\alpha} x_1 &= y_1 + x_1 - x_1 + s\left(x_2 - x_1\right) \\ D_t^{\alpha} y_1 &= a x_1 - b y_1 - x_1^3 + s\left(-x_2^3 + x_1^3\right) \end{aligned} \tag{49}$$

and

$$\begin{aligned} D_t^{\alpha} x_2 &= y_2 + x_2 - x_2 + s\left(x_1 - x_2\right) \\ D_t^{\alpha} y_2 &= a x_2 - b y_2 - x_2^3 + s\left(-x_1^3 + x_2^3\right), \end{aligned} \tag{50}$$

with parameters $a = -0.25$, $b = 0.35$, negative coupling strength $s = -0.5$ (inhibitory coupling instead of excitatory coupling), and fractional order $\alpha = 0.85$, are antisynchronized, because

$$\mathrm{h} = \begin{bmatrix} y \\ ax - by - x^3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -0.25 & -0.35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ -x^3 \end{bmatrix} \tag{51}$$

is odd in $\mathrm{x} = (x, y)$; moreover the system $\mathrm{h} - 2\mathrm{u}$ yields a stable limit-cycle; therefore, for nonzero initial conditions, $\mathrm{x}_1 = (x_1, y_1)$ and $\mathrm{x}_2 = (x_2, y_2)$ will oscillate and reach antisynchrony. As shown in Figure 5, the unforced Duffing systems with given parameters and two arbitrary initial points, $\mathrm{x}_{1,0} = [-0.2, 0.5]$ and $\mathrm{x}_{2,0} = [-1, 0.4]$, reach antisynchrony and summation of states (error of antisynchronization) converges to zero. Figure 6 represents the stable limit-cycle of $\mathrm{h} - 2\mathrm{u}$ for unforced Duffing system with three arbitrary initial points $\mathrm{x}_{1,0} = [1, 0.5]$, $\mathrm{x}_{2,0} = [-1, -0.5]$, and $\mathrm{x}_{3,0} = [2, 3]$.

4. Conclusion

We used partial contraction method to express and prove the conditions to reach synchronization and antisynchronization of two FOSs. In comparison with previous methods which were based on linearization, the results here are exact and

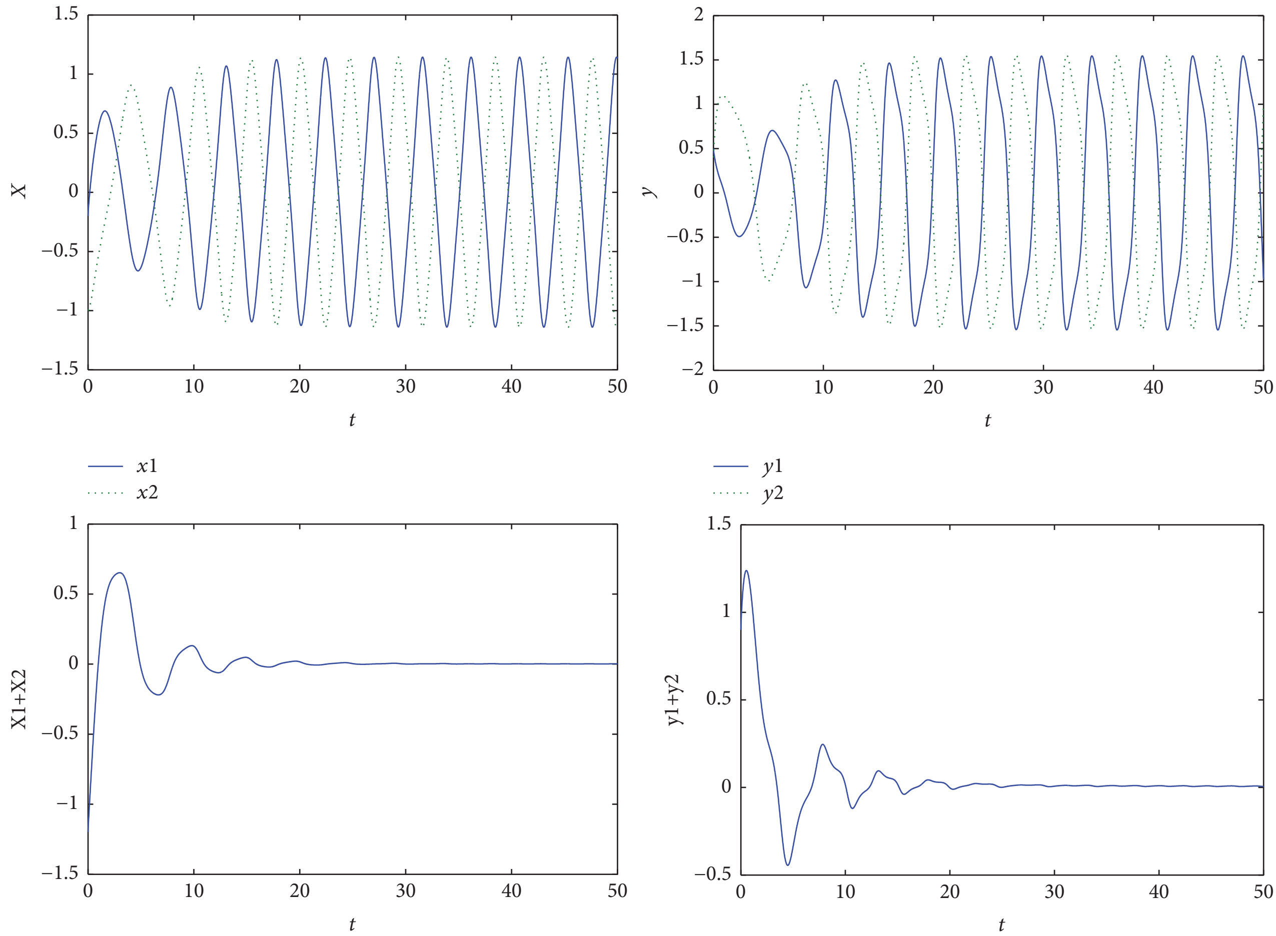

FIGURE 5: Antisynchronization of two unforced Duffing systems and the plot of errors $x_1 + x_2$ and $y_1 + y_2$.

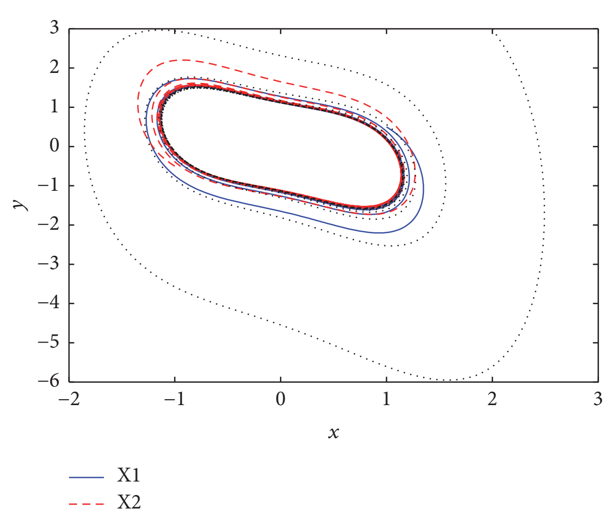

FIGURE 6: The stable limit-cycle of h − 2u for unforced Duffing system with three arbitrary initial points $x_{1,0} = [1, 0.5]$, $x_{2,0} = [-1, -0.5]$, and $x_{3,0} = [2, 3]$.

global. We also used the partial contraction method to study networks with various structure and arbitrary number of systems.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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