Hindawi
International Journal of Differential Equations
Volume 2018, Article ID 1203031, 8 pages
http://dx.doi.org/10.1155/2018/1203031

Research Article

Existence of Weak Solutions for Fractional Integrodifferential Equations with Multipoint Boundary Conditions

Haide Gou and Baolin Li

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Baolin Li; ghdzxh@163.com

Received 5 September 2017; Revised 20 November 2017; Accepted 22 July 2018; Published 2 September 2018

Academic Editor: Yuji Liu

By combining the techniques of fractional calculus with measure of weak noncompactness and fixed point theorem, we establish the existence of weak solutions of multipoint boundary value problem for fractional integrodifferential equations.

1. Introduction

In recent years, fractional differential equations in Banach spaces have been studied and a few papers consider fractional differential equations in reflexive Banach spaces equipped with the weak topology. As long as the Banach space is reflexive, the weak compactness offers no problem since every bounded subset is relatively weakly compact and therefore the weak continuity suffices to prove nice existence results for differential and integral equations [1, 2]. De Blasi [3] introduced the concept of measure of weak noncompactness and proved the analogue of Sadovskiis fixed point theorem for the weak topology (see also [4]). As stressed in [5], in many applications, it is always not possible to show the weak continuity of the involved mappings, while the sequential weak continuity offers no problem. This is mainly due to the fact that Lebesgues dominated convergence theorem is valid for sequences but not for nets. Recall that a mapping between two Banach spaces is sequentially weakly continuous if it maps weakly convergent sequences into weakly convergent sequences.

The theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored. There are many papers dealing with multipoint boundary value problems both on resonance case and on nonresonance case; for more details see [6–11]. However, as far as we know, few results can be found in the literature concerning multipoint boundary value problems for fractional differential equations in Banach spaces and weak topologies. Zhou *et al.* [12] discuss the existence of solutions for nonlinear multipoint boundary value problem of integrodifferential equations of fractional order as follows:

$$\begin{gathered}{}^{c}D_{0+}^{\alpha}x(t)=f(t,x(t),(Hx)(t),(Kx)(t)),\\ \qquad t\in[0,1],\ \alpha\in(1,2],\\ a_1x(0)-b_1x'(0)=d_1x(\xi_1),\\ a_2x(1)+b_2x'(1)=d_2x(\xi_2),\end{gathered}\tag{1}$$

with respect to strong topology, where ${}^{c}D_{0+}^{\alpha}$ denotes the fractional Caputo derivative and the operators given by

$$\begin{aligned}(Hx)(t)&=\int_0^t g(t,s)\,x(s)\,ds,\\ (Kx)(t)&=\int_0^t h(t,s)\,x(s)\,ds.\end{aligned}\tag{2}$$

Moreover, theory for boundary value problem of integrodifferential equations of fractional order in Banach spaces endowed with its weak topology has been few studied until now. In [13], we discussed the existence theorem of weak solutions nonlinear fractional integrodifferential equations in nonreflexive Banach spaces E:

$$\begin{gathered} {}^{c}D_{0+}^{\alpha}x(t)=f(t,x(t),(Tx)(t),(Sx)(t)), \\ t\in[0,1],\ \alpha\in(1,2], \\ a_1x(0)-a_2x'(0)=\gamma_1, \\ b_1x(1)+b_2x'(1)=\gamma_2, \end{gathered} \tag{3}$$

and obtain a new result by using the techniques of measure of weak noncompactness and Henstock-Kurzweil-Pettis integrals, where ${}^{c}D_{0+}^{\alpha}$ denotes the fractional Caputo derivative and the operators given by

$$\begin{aligned} (Tx)(s)&=\int_0^s k_1(s,\tau)\,g(\tau,x(\tau))\,d\tau, \\ (Sx)(s)&=\int_0^1 k_2(s,\tau)\,h(\tau,x(\tau))\,d\tau. \end{aligned} \tag{4}$$

Our analysis relies on the Krasnoselskii fixed point theorem combined with the technique of measure of weak noncompactness.

Motivated by the above works, in this paper, we use the techniques of measure of weak noncompactness combine with the fixed point theorem to discuss the existence theorem of weak solutions for a class of nonlinear fractional integrodifferential equations of the form

$$\begin{gathered} {}^{c}D_{0+}^{\alpha}x(t)=f(t,x(t),(Tx)(t),(Sx)(t)), \\ t\in[0,1],\ \alpha\in(1,2], \\ a_1x(0)-b_1x'(0)=d_1x(\xi_1), \\ a_2x(1)+b_2x'(1)=d_2x(\xi_2), \end{gathered} \tag{5}$$

where T and S are two operators defined by

$$\begin{aligned} (Tu)(t)&=\int_0^t k_1(t,s)\,g(s,u(s))\,ds, \\ (Su)(t)&=\int_0^a k_2(t,s)\,h(s,u(s))\,ds, \end{aligned} \tag{6}$$

E is a nonreflexive Banach space, ${}^{c}D_{0+}^{\alpha}$ denotes the fractional Caputo derivative, $k_1\in C(D,R^+)$, $k_2\in C(D_0,R^+)$, $D=\{(t,s)\in R^2: 0\le s\le t\le 1\}$, $D_0=\{(t,s)\in R^2: 0\le t,s\le 1\}$, a_1,b_1,d_1,a_2,b_2,d_2 are real numbers, $0<\xi_1,\xi_2<1$, $f: I\times E^3\longrightarrow E, g,h: I\times E\longrightarrow E$ are given functions satisfying some assumptions that will be specified later, the integral is understood to be the Henstock-Kurzweil-Pettis, and solutions to (5) will be sought in $E=C(I,E_\omega)$.

The problems of our research are different between this paper and paper [13]. In paper [13], we studied two point boundary value problem by using the corresponding Green's function and fixed point theorems; moreover, we get some good results. In this paper, we use the techniques of measure of weak noncompactness and Henstock-Kurzweil-Pettis integrals to discuss the existence theorem of weak solutions for a class of the multipoint boundary value problem of fractional integrodifferential equations equipped with the weak topology. Our results generalized some classical results and improve the assumptions conditions, so our results improve the results in [13].

The paper is organized as follows: In Section 2 we recall some basic known results. In Section 3 we discuss the existence theorem of weak solutions for problem (5).

2. Preliminaries

Throughout this paper, we introduce notations, definitions, and preliminary results which will be used.

Let $I=[0,1]$ be the real interval, let E be a real Banach space with norm $\|\cdot\|$, its dual space E^* also $B(E^*)$ denotes the closed unit ball in E^*, and $E_w=(E,w)=(E,\sigma(E,E^*))$ denotes the space E with its weak topology. Denote by $C(I,E_\omega)=(C(I,E),\omega)$ the space of all continuous functions from I to E endowed with the weak topology and the usual supremum norm $\|x\|=\sup_{t\in I}|x(t)|$.

Let Ω_E be the collection of all nonempty bounded subsets of E, and let $\mathscr{W}_E$ be the subset of Ω_E consisting of all weakly compact subsets of E. Let B_r denote the closed ball in E centered at 0 with radius $r>0$. The De Blasi [14] measure of weak noncompactness is the map $\beta:\Omega_E\longrightarrow[0,\infty)$ defined by

$$\begin{aligned} \beta(A)=\inf\{&r>0: \text{there exists a set } W \\ &\in\mathscr{W}_E \text{ such that } A\subseteq W+B_r\} \end{aligned} \tag{7}$$

for all $A\in\Omega_E$. The fundamental tool in this paper is the measure of weak noncompactness; for some properties of $\beta(A)$ and more details see [3].

Now, for the convenience of the reader, we recall some useful definitions of integrals.

Definition 1 (see [15]). A function $u: I\longrightarrow E$ is said to be Henstock-Kurzweil integrable on I if there exists an $J\in E$ such that, for every $\varepsilon>0$, there exists $\delta(\xi): I\longrightarrow\mathbb{R}^+$ such that, for every δ-fine partition $D=\{(I_i,\xi_i)\}_{i=1}^n$, we have

$$\left\|\sum_{i=1}^{n}u(\xi_i)\,\mu(I_i)-J\right\|<\varepsilon, \tag{8}$$

and we denote the Henstock-Kurzweil integral J by (HK) $\int_a^b u(s)ds$.

Definition 2 (see [15]). A function $f: I\longrightarrow E$ is said to be Henstock-Kurzweil-Pettis integrable or simply HKP-integrable on I, if there exists a function $g: I\longrightarrow E$ with the following properties:

(i) $\forall x^*\in E^*$, x^*f is Henstock-Kurzweil integrable on I;

(ii) $\forall t\in I$, $\forall x^*\in E^*$, $x^*g(t)=(\mathrm{HK})\int_0^t x^*f(s)ds$.

This function g will be called a primitive of f and be denote by $g(t)=\int_0^t f(t)dt$ the Henstock-Kurzweil-Pettis integral of f on the interval I.

Definition 3 (see [16]). A family $\mathscr{M}$ of functions $f: S \longrightarrow E$ is called HK-equi-integrable if each $f \in \mathscr{M}$ is HK-integrable and for every $\varepsilon > 0$ there exists a gauge δ on S such that, for every δ-fine HK-partition π of S, we have

$$\left\| \sum_{(I,s)\in\pi} f(s)\lambda_m(I) - (\mathrm{HK})\int_S \right\| \leq \varepsilon, \tag{9}$$

for all $f \in \mathscr{M}$.

Theorem 4 (see [16]). *Let (f_n) be a pointwise bounded sequence of HKP integrable functions $f_n : S \longrightarrow E$ and let $f: S \longrightarrow E$ be a function. Assume that,*

(i) *for every $x^* \in E^*$, $x^*(f_n(t)) \longrightarrow x^*(f(t))$ a.e. on S,*

(ii) *for every sequence $(x_k^*) \subset B(E^*)$, the sequence $(x_k^*(f_n))_{k,n}$ is HK-equi-integrable, then f is HKP-integrable and for every $I \in \mathscr{I}$, and we have*

$$\lim_{n\longrightarrow\infty} F_n(I) = F(I) \tag{10}$$

in the weak topology $\sigma(E, E^*)$, where F is the HKP-primitive of f and S is a fixed compact nondegenerate interval in $\mathbb{R}^n$. Denote by $\mathscr{I}$ the family of all closed nondegenerate subintervals of S.

Lemma 5 (see [17]). *If $B \subset C(I,E)$ is equicontinuous, $u_0 \in C(I,E)$, then $\overline{co}\{B, u_0\}$ is also equicontinuous in $C(I,E)$.*

Lemma 6 (see [17, 18]). *Let E be a Banach space, and let $B \subset C(I,E)$ be bounded and equicontinuous. Then $\beta(B(t))$ is continuous on I, and $\beta(B) = \max_{t\in I}\beta(B(t))$.*

Lemma 7 (see [14, 19]). *Let E be a Banach space and let $B \subset C(I,E)$ be bounded and equicontinuous. Then the map $t \longrightarrow \beta(B(t))$ is continuous on I and*

$$\beta(B) = \sup_{t\in I}\beta(B(t)) = \beta(B(I)), \tag{11}$$

where $B(t) = \{b(t) : b \in B\}$ and $B(I) = \bigcup_{t\in I}\{b(t) : h \in B\}$.

Lemma 8 (see [17]). *Let $B \subset C(I,E)$ be bounded and equicontinuous. Then $\beta(B(t))$ is continuous on I and*

$$\beta\left(\int_I B(s)\,ds\right) \leq \int_I \beta(B(s))\,ds. \tag{12}$$

We give the fixed point theorem, which play a key role in the proof of our main results.

Lemma 9 (see [20]). *Let E be a Banach space and β a regular and set additive measure of weak noncompactness on E. Let C be a nonempty closed convex subset of E, $x_0 \in C$, and n_0 a positive integer. Suppose $F: C \longrightarrow C$ is β-convex power condensing about x_0 and n_0. If F is weakly sequentially continuous and $F(C)$ is bounded, then F has a fixed point in C.*

The following we recall the definition of the Caputo derivative of fractional order.

Definition 10. Let $x: I \longrightarrow E$ be a function. The fractional HKP-integral of the function x of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_{0+}^{\alpha}x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\,ds. \tag{13}$$

In the above definition the sign "$\int$" denotes the HKP-integral integral.

Definition 11. The Riemann-Liouville derivative of order α with the lower limit zero for a function $f: [0,\infty) \longrightarrow R$ can be written as

$$D_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds, \qquad t > 0,\ n-1 < \alpha < n. \tag{14}$$

Definition 12. The Caputo fractional derivative of order α for a function $f: [0,\infty) \longrightarrow E$ can be written as

$${}^cD_{0+}^{\alpha}f(t) = D_{0^+}^{\alpha}\left[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right], \qquad t > 0,\ n-1 < \alpha < n, \tag{15}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

3. Main Results

In this section, we present the existence of solutions to problem (5) in the space $C(I, E_\omega)$.

Definition 13. A function $x \in C(I, E_w)$ is said to be a solution of problem (5) if x satisfies the equation ${}^cD_{0+}^{\alpha}x(t) = f(t, x(t), (Tx)(t), (Sx)(t))$ on I and satisfies the conditions $a_1x(0) - b_1x'(0) = d_1x(\xi_1)$, $a_2x(1) + b_2x'(1) = d_2x(\xi_2)$.

Lemma 14 (see [21]). *Let $\alpha > 0$. If one assumes $u \in C(0,1) \cap L(0,1)$, then the differential equation*

$${}^cD_{0+}^{\alpha}u(t) = 0 \tag{16}$$

has solution $u(t) = c_0 + c_1t + c_2t^2 + \cdots + c_nt^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

From the lemma above, we deduce the following statement.

Lemma 15 (see [21]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha}\left({}^cD_{0+}^{\alpha}u(t)\right) = u(t) + c_0 + c_1t + c_2t^2 + \cdots + c_nt^{n-1} \tag{17}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

The following we give the corresponding Greens function for problem (5).

Lemma 16. *Let $\Delta \neq 0, \rho \in C(I, E_w)$ and $\alpha \in (1,2]$, then the unique solution of*

$$\begin{aligned} {}^{c}D_{0+}^{\alpha} x(t) &= \rho(t), \quad t \in I, \\ a_1 x(0) - b_1 x'(0) &= d_1 x(\xi_1), \\ a_2 x(1) + b_2 x'(1) &= d_2 x(\xi_2) \end{aligned} \tag{18}$$

is given by

$$x(t) = \int_0^1 G(t,s)\,\rho(s)\,ds, \tag{19}$$

where the Green function G is given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{d_1[a_2(1-t)+b_2+d_2(t-\xi_2)](\xi_1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{d_2[(b_1+d_1\xi_1)+t(a_1-d_1)](\xi_2-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \le \xi_1,\ s \le t; \\ \dfrac{d_1[a_2(1-t)+b_2+d_2(t-\xi_2)](\xi_1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{d_2[(b_1+d_1\xi_1)+t(a_1-d_1)](\xi_2-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \le \xi_1,\ t \le s; \\ \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{d_2[(b_1+d_1\xi_1)+t(a_1-d_1)](\xi_2-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi_1 \le s \le \xi_2,\ s \le t; \\ -\dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \dfrac{d_2[(b_1+d_1\xi_1)+t(a_1-d_1)](\xi_2-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi_1 \le s \le \xi_2,\ t \le s; \\ \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \xi_2 \le s,\ s \le t; \\ -\dfrac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)](1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \xi_2 \le s,\ t \le s. \end{cases} \tag{20}$$

Proof. Based on the idea of paper [7], assuming that $x(t)$ satisfies (18), by Lemma 15, we formally put

$$\begin{aligned} x(t) &= I_{0+}^{\alpha}\rho(t) - c_1 - c_2 t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(t)\,ds - c_1 - c_2 t \end{aligned} \tag{21}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

On the other hand, by the relations $D_{0+}^{\alpha} I_{0+}^{\alpha} x(t) = x(t)$ and $I_{0+}^{\alpha} I_{0+}^{\beta} x(t) = I_{0+}^{\alpha+\beta} x(t)$, for $\alpha, \beta > 0,\ x \in C(I, E_w)$, we get

$$x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \rho(s)\,ds - c_2. \tag{22}$$

By the boundary conditions of (18), we have

$$\begin{aligned} &(d_1 - a_1) c_1 + (b_1 + d_1\xi_1) c_2 \\ &\quad = d_1 I_{0+}^{\alpha}\rho(\xi_1) + b_1 I_{0+}^{\alpha-1}\rho(0) - a_1 I_{0+}^{\alpha}\rho(0), \\ &(d_2 - a_2) c_1 + (-a_2 - b_2 + d_2\xi_2) c_2 \\ &\quad = d_2 I_{0+}^{\alpha}\rho(\xi_2) - b_2 I_{0+}^{\alpha-1}\rho(1) - a_2 I_{0+}^{\alpha}\rho(1), \end{aligned} \tag{23}$$

By the proof of paper [12], we get

$$\begin{aligned} c_1 &= -\frac{d_1(a_2+b_2-d_2\xi_2)}{\Delta\Gamma(\alpha)} \int_0^{\xi_1} (\xi_1-s)^{\alpha-1} \rho(s)\,ds \\ &\quad + \frac{(b_1+d_1\xi_1)}{\Delta} \left[a_2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s)\,ds \right. \\ &\quad + b_2 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \rho(s)\,ds \\ &\quad \left. - d_2 \int_0^{\xi_2} \frac{(\xi_2-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s)\,ds \right], \\ c_2 &= \frac{d_1(a_2-d_2)}{\Delta\Gamma(\alpha)} \int_0^{\xi_1} (\xi_1-s)^{\alpha-1} \rho(s)\,ds \\ &\quad + \frac{(a_1-d_1)}{\Delta} \left[a_2 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s)\,ds \right. \\ &\quad + b_2 \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \rho(s)\,ds \\ &\quad \left. - d_2 \int_0^{\xi_2} \frac{(\xi_2-s)^{\alpha-1}}{\Gamma(\alpha)} \rho(s)\,ds \right], \end{aligned} \tag{24}$$

where $\Delta = [(b_1+d_1\xi_1)(a_2-d_2)+(a_2+b_2-d_2\xi_2)(a_1-d_1)] \neq 0$. Substituting the values of c_1 and c_2 in (21), we get

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s)\,ds \\ &\quad + \frac{d_1[a_2(1-t)+b_2+d_2(t-\xi_2)]}{\Delta\Gamma(\alpha)} \\ &\quad \cdot \int_0^{\xi_1} (\xi_1-s)^{\alpha-1} \rho(s)\,ds \\ &\quad - \frac{a_2[(b_1+d_1\xi_1)+t(a_1-d_1)]}{\Delta\Gamma(\alpha)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-1} \rho(s)\,ds \\ &\quad - \frac{b_2[(b_1+d_1\xi_1)+t(a_1-d_1)]}{\Delta\Gamma(\alpha-1)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-2} \rho(s)\,ds \\ &\quad + \frac{d_2[(b_1+d_1\xi_1)+t(a_1-d_1)]}{\Delta\Gamma(\alpha)} \\ &\quad \cdot \int_0^{\xi_2} (\xi_2-s)^{\alpha-1} \rho(s)\,ds = \int_0^1 G(t,s)\,\rho(s)\,ds. \end{aligned} \tag{25}$$

This completes the proof. □

Let $D_r = \{z \in C(I, E_w), \|z\| \le r\}$, $BV(I, \mathbb{R})$ denote the space of real bounded variation functions with its classical norm $\|\cdot\|_{BV}$.

Problem (5) will be studied under the following assumptions:

(1) For each weakly continuous function $x : I \longrightarrow E$, the functions $k_1(t,\cdot)g(\cdot, x(\cdot)), k_2(t,\cdot)h(\cdot, x(\cdot))$, $f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot))$ are HKP-integrable, $f : I \times E^3 \longrightarrow E, g, h : I \times E \longrightarrow E$ are weakly-weakly continuous function, and $\int_0^t g(s, x(s))ds$, $\int_0^1 h(s, x(s))ds$ are bounded.

(2)

(i) For any $r > 0$, there exist a HK-integrable function $m : I \longrightarrow \mathbb{R}^+$ and nondecreasing continuous functions $\psi_1 : [0, +\infty) \longrightarrow (0, \infty), \psi_2 : [0, +\infty) \longrightarrow [0, +\infty), \psi_3 : [0, +\infty) \longrightarrow [0, +\infty)$, ψ_2, ψ_3 satisfying $\psi_2(\lambda x) \le \lambda\psi_2(x)$, $\psi_3(\lambda x) \le \lambda\psi_3(x)$ for $\lambda > 0$ such that

$$\begin{aligned} &\|f(s,x,y,z)\| \\ &\quad \le m(s)\left[\psi_1(\|x\|) + \psi_2(|y|) + \psi_3(|z|)\right], \\ &\psi_2(|g(s,x)|) \le \psi_2(|x|), \\ &\psi_3(|h(s,x)|) \le \psi_3(|x|) \end{aligned} \tag{26}$$

for all $s \in I, (x, y, z) \in D_r \times D_r \times D_r$ with

$$\int_0^1 M(s)\,ds < \int_0^\infty \frac{dr}{\sum_{i=1}^3 \psi_i(s)}. \tag{27}$$

(ii) For each bounded set $X, Y, Z \subset D_r$, and each for each closed interval $J \subset I, t \in I$, there exists positive constant $l \ge 0$ such that

$$\begin{aligned} &\beta(k_1(J,J)\,g(J,Y) \le k_1^*\beta(Y(J)), \\ &\beta(k_2(J,J)\,h(J,Z) \le k_2^*\beta(Z(J)) \\ &\beta(f(t,X,Y,Z)) \le l\max\{\beta(X), \beta(Y), \beta(Z)\}, \end{aligned} \tag{28}$$

where $M(s) = G^* m(s)\max\{1, ak_1^*, ak_2^*\}, k_1^* = \sup_{t\in I}\|k_1(t,\cdot)\|_{BV}, k_2^* = \sup_{t\in I}\|k_2(t,\cdot)\|_{BV}$.

(3) For each $t \in I, G(t,.), k_i(t,\cdot) \in BV(I, \mathbb{R})$, $i = 1, 2$ are continuous; i.e., the maps $t \longmapsto G(t,.)$ and $t \longmapsto k_i(t,.)$ are $\|.\|_{BV}$-continuous.

(4) The family $\{x^* f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot)) : x^* \in E^*, \|x^*\| \le 1\}$ is uniformly HK-integrable over I for every $x \in D_r$.

Remark 17. From assumption (3) and the expression of function $G(t,s)$, it is obvious that it is bounded and let $G^* = \sup_{t\in I}\|G(t,\cdot)\|_{BV}$.

Now, we present the existence theorem for problem (5).

Theorem 18. *Assume that conditions (5)-(20). Then problem (5) has a solution $x \in C(I, E_w)$.*

Proof. Let $m = \max\{\sup_{t\in I}\|k_i(t,\cdot)\|_{BV}, i = 1, 2\}$ and $k_0 = \max\{\sup_{t\in I}|\int_0^t g(s, x(s))ds|, \sup_{t\in I}|\int_0^1 h(s, x(s))ds|\}$. Let $0 < k_0 < \min(r_0, r_0/m)$, for $x \in D_{r_0}$ and $x^* \in E^*$ such that $\|x\|^* \le 1$; we have

$$\begin{aligned} |x^*(Tx(s))| &= \left|(\mathrm{HK})\int_0^t x^*(k_1(t,s)\,g(s,x(s)))\,ds\right| \\ &\le \|x^*\| \sup_{t\in I}\|k_1(t,\cdot)\|_{BV}\int_0^1 \|g(s,x(s))\|\,ds \\ &\le m\cdot k_0 \le r_0, \end{aligned} \tag{29}$$

and also

$$\sup\{|x^*Tx| : x \in E^*, \|x^*\| \le 1\} \le r_0. \tag{30}$$

So $Tx \in D_{r_0}$. Similarly, we prove $Sx \in D_{r_0}$.

Defining the set

$$\begin{aligned} Q := \Big\{x \in D_{r_0} &: \|x(\cdot)\| \le r_0,\ \|x(t) - x(s)\| \\ &\le \frac{r_0}{G^*}\|G(t_2,\cdot) - G(t_1,\cdot)\|_{BV},\ t_1, t_2 \in I\Big\}, \end{aligned} \tag{31}$$

it is clear that the convex closed and equicontinuous subset $Q \subset D_{r_0} \subset C(I, E_w)$, where

$$\begin{aligned} b(t) &= I^{-1}\left(\int_0^t M(s)\,ds\right) \text{ and} \\ I(z) &= \int_0^z \frac{ds}{\sum_{i=1}^3 \psi_i(s)}. \end{aligned} \tag{32}$$

Clearly,

$$\begin{aligned} b'(t) &= M(t)\left(\sum_{i=1}^3 \psi_i(b(t))\right), \text{ and} \\ b(0) &= 0 \end{aligned} \tag{33}$$

for all $t \in I$. Also notice that Q is a closed, convex, bounded, and equicontinuous subset of $C(I, E_w)$. We define the operator $F : C(I, E_w) \longrightarrow C(I, E_w)$ by

$$Fx(t) = \int_0^1 G(t,s)\,f(s, x(s), (Tx)(s), (Sx)(s))\,ds, \quad t \in I, \tag{34}$$

where $G(\cdot,\cdot)$ is Green's function defined by (20). Clearly the fixed points of the operator F are solutions of problem (5). Since for $t \in I$ the function $s \longmapsto G(t,s)$ is of bounded variation, then by the proof of Theorem 3.1 in [13] and assumption (4), the function $G(t,\cdot)f(\cdot, x(\cdot), T(x)(\cdot), S(x)(\cdot))$ is HKP-integrable on I and thus the operator F makes sense.

We will show that F satisfies the assumptions of Lemma 8; the proof will be given in three steps.

Step 1. We shall show that the operator F maps into itself. To see this, let $x \in Q, t \in I$. Without loss of generality, assume that $Fx(t) \neq 0$. By Hahn-Banach theorem, there exists $x^* \in E^*$ with $\|x^*\| = 1$ and $\|Fx(t)\| = |x^*(Fx(t))|$. Thus

$$\begin{aligned}
\|Fx(t)\| &= \left|x^*(Fx(t))\right| \\
&\leq x^*\left(\int_0^1 G(t,s)\, f(s, x(s), (Tx)(s), (Sx)(s))\, ds\right) \\
&\leq \sup_{t\in I} \|G(t,\cdot)\|_{BV} \int_0^1 m(s) \\
&\quad \cdot \left[\psi_1(b(s)) + ak_1^*\psi_2(b(s)) + ak_2^*\psi_3(b(s))\right] ds \\
&\leq \int_0^1 b'(s)\, ds \leq I^{-1}\left(\int_0^1 M(s)\, ds\right) = r_0.
\end{aligned} \tag{35}$$

Then $\|Fx\| = \sup_{t\in I}|Fx(t)| \leq r_0$. Hence $F : Q \longrightarrow Q$.

Let $0 < t_1 < t_2 \leq 1$, without loss of generality; assume that $Fx(t_2) - Fx(t_1) \neq 0$. By Hahn-Banach theorem, there exists $x^* \in E^*$ with $\|x^*\| = 1$ and

$$\begin{aligned}
\|Fx(t_2) - Fx(t_1)\| &= x^*(Fx(t_2) - Fx(t_1)) \\
&\leq \int_0^1 \left|G(t_2, s) - G(t_1, s)\right| \\
&\quad \cdot \left|x^*(f(s, x(s), (Tx)(s), (Sx)(s)))\right| ds \\
&\leq \left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_{BV} \int_0^1 m(s) \\
&\quad \cdot \left[\psi_1(b(s)) + ak_1^*\psi_2(|b(s)) + ak_2^*\psi_3(b(s))\right] ds \\
&\leq \frac{1}{G^*}\left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_{BV} \int_0^1 b'(s)\, ds \\
&\leq \frac{1}{G^*}\left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_{BV} I^{-1}\left(\int_0^1 M(s)\, ds\right) \\
&= \frac{r_0}{G^*}\left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_{BV},
\end{aligned} \tag{36}$$

and this estimation shows that F maps Q into itself.

Step 2. We will show that the operator F is weakly sequentially continuous. In order to be simple, we denote $Tx(t) = \phi(x)(t) = \int_0^1 k_1(t,s)g(s,x(s))ds$, $Sx(t) = \varphi(x)(t) = \int_0^1 k_2(t,s)h(s,x(s))ds$. To see this, by Lemma 9 of [22], a sequence $x_n(\cdot)$ weakly convergent to $x(\cdot) \in Q$ if and only if $x_n(\cdot)$ tends weakly to $x(t)$ for each $t \in I$. From Dinculeanu ([23, p. 380]) $(C(I,E))^* = M(I,E^*)$, $M(I,E^*)$ is the set of all bounded regular vector measures from I to E^* which are of bounded variation). Let $x^* \in E^*$, $t \in I$. Put $P_t = x^*\delta_t$, where δ_t is the Dirac measure concentrated at the point t. Then $P_t \in M(I,E^*)$. Since x_n converges weakly to $x \in Q$, then we have

$$\lim_{n\longrightarrow\infty} \langle p_t, x_n - x\rangle = 0 \tag{37}$$

which means that

$$\lim_{n\longrightarrow\infty} \langle x^*, x_n - x\rangle = 0. \tag{38}$$

Thus, for each $t \in I$, $x_n(t)$ converges weakly to $x(t) \in E$. Since $g(s,\cdot)$, $h(s,\cdot)$ are weakly-weakly sequentially continuous, then $g(s, x_n(s))$ and $h(s, x_n(s))$ converge weakly to $g(s, x(s))$ and $h(s, x(s))$, respectively. Hence, and by Theorem 4 and assumptions (1), we have

$$\lim_{n\longrightarrow\infty} \int_0^1 \left(\int_0^t (k_1(t,s)\, g(s, x_n(s)) - k_1(t,s)\, g(s, x(s)))\, ds\right) dm(s) = 0, \quad \forall m \in M(I,E^*). \tag{39}$$

This relation is equivalent to

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} (m, \phi(x_n) - \phi(x)) \\
&\quad = \lim_{n\longrightarrow\infty} \int_0^1 (\phi(x_n)(t) - \phi(x)(t))\, dm(t) = 0, \quad \forall m \in M(I,E^*).
\end{aligned} \tag{40}$$

Similarly, we have

$$\lim_{n\longrightarrow\infty} \int_0^1 \left(\int_0^1 (k_2(t,s)\, h(s, x_n(s)) - k_2(t,s)\, h(s, x(s)))\, ds\right) dm(s) = 0, \quad \forall m \in M(I,E^*). \tag{41}$$

This relation is equivalent to

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} (m, \varphi(x_n) - \varphi(x)) \\
&\quad = \lim_{n\longrightarrow\infty} \int_0^1 (\varphi(x_n)(t) - \varphi(x)(t))\, dm(t) = 0, \quad \forall m \in M(I,E^*).
\end{aligned} \tag{42}$$

Therefore, the operators T, S are weakly sequentially continuous in Q.

Moreover, because f is weakly-weakly sequentially continuous, we have that $f(s, x_n(s), (Tx_n)(s), (Sx_n)(s))$ converges weakly to $f(s, x(s), (Tx)(s), (Sx)(s))$ in E. By assumption (4), for every weakly convergent $(x_n)_n \subset D_{r_0}$, the set

$$\{x^* f(\cdot, x_n(\cdot), T(x_n)(\cdot), S(x_n)(\cdot)) : n \in N, x^* \in B(E^*)\} \tag{43}$$

is HK-equi-integrable. Since for $t \in I$ the function $s \longmapsto G(t, s)$ is of bounded variation, and by the proof of Theorem 3.1 in [13], the function $G(t, \cdot) f(\cdot, x_n(\cdot), (Tx_n)(\cdot), (Sx_n)(\cdot))$ is HKP-integrable on I for every $n \geq 1$, and by Theorem 4, we have that $\int_0^1 G(t, s) f(s, x_n(s), (Tx_n)(s), (Sx_n)(s)) ds$ converges weakly to $\int_0^1 G(t, s) f(s, x(s), (Tx)(s), (Sx)(s)) ds$ in E which means that

$$\lim_{n \longrightarrow \infty} \int_0^1 \left(\int_0^1 (G(t, s) [f(s, x_n(s), (Tx_n)(s), (Sx_n)(s)) - f(s, x(s), (Tx)(s), (Sx)(s))] ds \right) dm(s) = 0, \tag{44}$$

for all $m \in M(I, E^*)$. This relation is equivalent to

$$\lim_{n \longrightarrow \infty} (m, F(x_n) - F(x)) = \lim_{n \longrightarrow \infty} \int_0^1 (F(x_n)(t) - F(x)(t)) dm(t) = 0, \quad \forall m \in M(I, E^*). \tag{45}$$

Therefore F is weakly-weakly sequentially continuous.

Step 3. We show that there is an integer n_0 such that the operator F is β-power-convex condensing about 0 and n_0. To see this, notice that, for each bounded set $H \subseteq Q$ and for each $t \in I$,

$$\begin{aligned} \beta\left(F^{(1,0)}(H)(t)\right) &= \beta(F(H)(t)) = \beta\left(\left\{\int_0^t G(t, s) \cdot f(s, x(s), (Tx)(s), (Sx)(s)) ds : x \in H\right\}\right) \\ &\leq \beta(G^* t \overline{co}\{f(s, x(s), (Tx)(s), (Sx)(s)) : x \in H, s \in I\}) = G^* t \beta(\overline{co}\{f(s, x(s), (Tx)(s), (Sx)(s)) : x \in H, s \in I\}) \leq G^* t \beta(f(I \times H(I) \times T(H)(I) \times S(H)(I))) \leq G^* t \cdot \max\{1, k_1^*, k_2^*\} \cdot l \beta(H(I)). \end{aligned} \tag{46}$$

Let $\tau = G^* \cdot \max\{1, k_1^*, k_2^*\} \cdot l > 0$. Lemma 7 implies (since H is equicontinuous) that

$$\beta\left(F^{(1,0)}(H)(t)\right) \leq t\tau\beta(H). \tag{47}$$

Since $F^{(1,0)}(H)$ is equicontinuous, it follows from Lemma 5 that $F^{(2,0)}(H)$ is equicontinuous. Using (47), we get

$$\begin{aligned} \beta\left(F^{(2,0)}(H)(t)\right) &= \beta\left(\left\{\int_0^t G(t, s) \cdot f(s, x(s), (Tx)(s), (Sx)(s)) ds : x \in \overline{co}\left(F^{(1,0)}(H) \cup \{0\}\right)\right\}\right) \\ &\leq \beta\left(\left\{\int_0^t G(t, s) \cdot f(s, x(s), (Tx)(s), (Sx)(s)) ds : x \in V\right\}\right), \end{aligned} \tag{48}$$

where $V = \overline{co}(F^{(1,0)}(H) \cup \{0\})$; it is clear that V is equicontinuous set. By Lemma 8, we get

$$\beta(V(s)) = \beta\left(F^{(1,0)}(H)(s)\right) \leq s\tau\beta(H), \tag{49}$$

and therefore,

$$\int_0^t \beta(V(s)) ds \leq s\tau \frac{t^2}{2} \beta(H). \tag{50}$$

Thus,

$$\beta\left(F^{(2,0)}(H)(t)\right) \leq \frac{(\tau t)^2}{2} \beta(H). \tag{51}$$

By induction, we get

$$\beta\left(F^{(n,0)}(H)(t)\right) \leq \frac{(\tau t)^n}{n!} \beta(H). \tag{52}$$

And by Lemma 7, we have

$$\beta\left(F^{(n,0)}(H)\right) \leq \frac{(\tau T)^n}{n!} \beta(H). \tag{53}$$

Since $\lim_{n \longrightarrow \infty}((\tau t)^n / n!) = 0$, then there exist an n_0 with $(\tau t)^{n_0} / n_0! < 1$, and we have

$$\beta\left(F^{(n_0,0)}(H)\right) \leq \beta(H). \tag{54}$$

Consequently, F is β-power-convex condensing about 0 and n_0, by Lemma 8, then problem (5) has a solution $x \in C(I, E_w)$. □

4. Conclusions

In this paper, we use the techniques of measure of weak noncompactness and Henstock-Kurzweil-Pettis integrals to discuss the existence theorem of weak solutions for a class of the multipoint boundary value problem of fractional integrodifferential equations equipped with the weak topology. Our results generalized some classical results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Grant no. 11061031).

References

[1] D. O'Regan, "Existence results for nonlinear integral equations," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 3, pp. 705–726, 1995.

[2] D. O'Regan, "Singular integral equations arising in draining and coating flows," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 438–441, 2008.

[3] F. S. De Blasi, "On a property of the unit sphere in a Banach space," *Bulletin mathematiques de la Societe des sciences mathematiques de Roumanie*, vol. 21(69), no. 3-4, pp. 259–262, 1977.

[4] G. Emmanuele, "Measure of weak noncompactness and fixed point theorems," *Bulletin mathematiques de la Societe des sciences mathematiques de Roumanie*, vol. 25, no. 4, pp. 353–358, 1981.

[5] O. Arino, S. Gautier, and J.-P. Penot, "A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations," *Funkcialaj Ekvacioj. Serio Internacia*, vol. 27, no. 3, pp. 273–279, 1984.

[6] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.

[7] H. A. Salem, "On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 565–572, 2009.

[8] W. Y. Zhong and W. Lin, "Nonlocal and multiple-point boundary value problem for fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1345–1351, 2010.

[9] Z. Bai, "On solutions of some fractional m-point boundary value problems at resonance," *Electronic Journal of Qualitative Theory of Differential Equations*, pp. 1–15, 2010.

[10] Q. Song, X. Dong, Z. Bai, and B. Chen, "Existence for fractional Dirichlet boundary value problem under barrier strip conditions," *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 10, no. 7, pp. 3592–3598, 2017.

[11] Z. Bai, S. Zhang, S. Sun, and C. Yin, "Monotone iterative method for fractional differential equations," *Electronic Journal of Differential Equations*, vol. 2016, article 6, 2016.

[12] W. Zhou and Y. Chu, "Existence of solutions for fractional differential equations with multi-point boundary conditions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1142–1148, 2012.

[13] B. Li and H. Gou, "Weak solutions nonlinear fractional integrodifferential equations in nonreflexive Banach spaces," *Boundary Value Problems*, Paper No. 209, 13 pages, 2016.

[14] N. Hussain and M. A. Taoudi, "Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations," *Fixed Point Theory and Applications*, vol. 2013, no. 1, article 196, 2013.

[15] M. Cicho'n, I. Kubiaczyk, and A. Sikorska, "The Henstock-Kurzweil-Pettis integrals and existence theorems for the Cauchy problem," *Czechoslovak Mathematical Journal*, vol. 54(129), no. 2, pp. 279–289, 2004.

[16] S. B. Kaliaj, A. D. Tato, and F. D. Gumeni, "Controlled convergence theorem for the Henstock-Kurwzeil-Pettis integrals on m-dimiensional compact intervals," *Czechoslovak Mathematical Journal*, vol. 62, no. 1, pp. 243–255, 2012.

[17] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, vol. 373 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.

[18] J. Banaś and K. Goebel, *Measures of Noncompactness*, vol. 60 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[19] A. R. Mitchell and C. Smith, "An existence theorem for weak solutions of differential equations in Banach spaces," in *Nonlinear equations in abstract spaces (Proc. Internat. Sympos., Univ. TEXas, Arlington, TEX., 1977)*, pp. 387–403, Academic Press, New York, 1978.

[20] R. P. Agarwal, D. O'Regan, and M.-A. Taoudi, "Fixed point theorems for convex-power condensing operators relative to the weak topology and applications to Volterra integral equations," *Journal of Integral Equations and Applications*, vol. 24, no. 2, pp. 167–181, 2012.

[21] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[22] A. R. Mitchell and C. Smith, "An existence theorem for weak solutions of differential equations in Banach spaces," in *Nonlinear Equations in Abstract Spaces*, pp. 387–403, Orlando, 1978.

[23] J. Garcia-Falset, "Existence of fixed points and measures of weak noncompactness," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 71, no. 7-8, pp. 2625–2633, 2009.