

Research Article

On the Boundedness of the Fractional Bergman Operators

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We give a necessary and sufficient condition for the boundedness of the Bergman fractional operators.

1. Introduction and Statement of the Result

We are interested in this note in the boundedness of the Bergman fractional operator. The Bergman fractional operator has been shown recently to be quite useful in understanding off-diagonal questions for the Bergman operator (see [1, 2]). Our aim here is to provide a necessary and sufficient condition for the boundedness of this operator. In the next lines, we provide some notions and definitions needed in the sequel.

Let \mathcal{H} be the upper-half plane, that is, the set $\{z = x + iy \in \mathbb{C} : y > 0\}$. We denote by $L^p_\alpha(\mathcal{H})$ the Lebesgue space $L^p(\mathcal{H}, y^\alpha dx dy)$, that is, the space of all functions f such that

$$\|f\|_{p,\alpha}^p := \int_{\mathcal{H}} |f(x + iy)|^p y^\alpha dx dy < \infty. \quad (1)$$

For $\alpha > -1$ and $1 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathcal{H})$ is the subspace of $L^p_\alpha(\mathcal{H})$ consisting of analytic functions. It is well known that the Bergman space $A^2_\alpha(\mathcal{H})$ ($-1 < \alpha < \infty$) is a reproducing kernel Hilbert space with kernel $K_z^\alpha(w) = K^\alpha(z, w) = 1/(z - \bar{w})^{2+\alpha}$. That is, for any $f \in A^2_\alpha(\mathcal{H})$, the following representation holds:

$$\begin{aligned} f(w) &= P_\alpha f(w) = \langle f, K_w^\alpha \rangle_\alpha \\ &= \int_{\mathcal{H}} f(z) K^\alpha(w, z) dV_\alpha(z), \end{aligned} \quad (2)$$

where, for simplicity, we write $dV_\alpha(x + iy) = y^\alpha dx dy$. The positive Bergman operator P_α^+ is defined by

$$P_\alpha^+ f(w) = \int_{\mathcal{H}} f(z) |K^\alpha(w, z)| dV_\alpha(z). \quad (3)$$

Note that the boundedness of P_α^+ implies the boundedness of P_α . It is an elementary exercise to prove that the Bergman projection P_α is bounded on $L^p_\alpha(\mathcal{H})$ if and only if $1 < p < \infty$ (see, e.g., [3]).

The fractional Bergman operator $\mathcal{P}_{\alpha,\gamma}$ is defined by

$$\mathcal{P}_{\alpha,\gamma} f(w) = \int_{\mathcal{H}} \frac{f(z)}{(w - \bar{z})^{2+\alpha-\gamma}} dV_\alpha(z), \quad (4)$$

where $0 \leq \gamma < 2 + \alpha$. The corresponding positive operator will be denoted by $\mathcal{P}_{\alpha,\gamma}^+$ and can be seen as the upper-half space analogue of the Riesz potential also known as fractional operator (see [4]). Note also that, for $\gamma = 0$, $\mathcal{P}_{\alpha,\gamma}$ is just the Bergman projection.

We have the following necessary and sufficient condition for the boundedness of $\mathcal{P}_{\alpha,\gamma}$ and $\mathcal{P}_{\alpha,\gamma}^+$.

Theorem 1. *Let $\alpha > -1$, $0 \leq \gamma < 2 + \alpha$, and $1 < p \leq q < \infty$. Then the following conditions are equivalent:*

- The operator $\mathcal{P}_{\alpha,\gamma}$ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$.
- The operator $\mathcal{P}_{\alpha,\gamma}^+$ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$.
- The following relation holds:

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2 + \alpha}. \quad (5)$$

Unlike the case of the unit ball (see [5]), the above result can not be deduced from the boundedness of the families of Bergman-type operators considered in [2, 6].

We remark that the boundedness of the operator $\mathcal{P}_{\alpha,\gamma}^+$ from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$ implies the boundedness of P_α^+ from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\beta)$, where $\beta = q\gamma + \alpha$. It follows that we have the following.

Proposition 2. *Let $\alpha > -1, 0 \leq \gamma < 2 + \alpha$, and $1 < p \leq q < \infty$. Then the following conditions are equivalent:*

- (a) *The operator P_α is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_{q\gamma+\alpha})$.*
- (b) *The operator P_α^+ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_{q\gamma+\alpha})$.*
- (c) *The following relation holds:*

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2 + \alpha}. \tag{6}$$

For the proof of the sufficient part, we will use the off-diagonal Schur test due to Okikiolu [7].

2. Proof of Theorem 1 and Proposition 2

We start by recalling the following easy fact (see [3]).

Lemma 3. *Let α be real. Then the function $f(z) = ((z + it)/i)^{-\alpha}$, with $t > 0$, belongs to $L^p(\mathcal{H}, dV_\nu)$, if and only if $\nu > -1$ and $\alpha > (\nu + 2)/p$. In this case,*

$$\|f\|_{p,\nu}^p = C_{\alpha,p,q} t^{-p\alpha+\nu+2}. \tag{7}$$

The proof of the sufficient parts in our results is based on the following off-diagonal Schur-type test.

Lemma 4 (Okikiolu [7]). *Let p, r, q be positive numbers such that $1 < p \leq r$ and $1/p + 1/q = 1$. Let $K(x, y)$ be a complex-valued function measurable on $X \times Y$ and suppose that there exist $0 < t \leq 1$, measurable functions $\varphi_1 : X \rightarrow (0, \infty)$ and $\varphi_2 : Y \rightarrow (0, \infty)$, and nonnegative constants M_1 and M_2 such that*

$$\int_X |K(x, y)|^{tq} \varphi_1^q(y) d\mu(y) \leq M_1^q \varphi_2^q(x) \quad \text{a.e on } Y, \tag{8}$$

$$\int_Y |K(x, y)|^{(1-t)r} \varphi_2^r(x) d\nu(x) \leq M_2^r \varphi_1^r(y) \quad \text{a.e on } X.$$

If T is given by

$$Tf(x) = \int_X f(y) K(x, y) d\mu(y), \tag{9}$$

where $f \in L^p(X, d\mu)$, then $T : L^p(X, d\mu) \rightarrow L^r(Y, d\nu)$ is bounded and for each $f \in L^p(X, d\mu)$,

$$\|Tf\|_{L^r(Y, d\nu)} \leq M_1 M_2 \|f\|_{L^p(X, d\mu)}. \tag{10}$$

We prove the following.

Lemma 5. *Let $\alpha > -1, 0 \leq \gamma < 2 + \alpha$, and $1 \leq p \leq q < \infty$. If the operator $\mathcal{P}_{\alpha,\gamma}$ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$, then*

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2 + \alpha}. \tag{11}$$

Proof. We assume that the operator $\mathcal{P}_{\alpha,\gamma}$ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$. Let $R > 0$ and associate to any function f , the function f_R defined by $f_R(z) = f(Rz)$. Then it is easy to see that

$$\|f_R\|_{p,\alpha} = R^{-(2+\alpha)/p} \|f\|_{p,\alpha}. \tag{12}$$

It follows also from an easy change of variables that

$$\mathcal{P}_{\alpha,\gamma} f_R(w) = R^{-\gamma} \mathcal{P}_{\alpha,\gamma} f(Rw). \tag{13}$$

Hence

$$\|\mathcal{P}_{\alpha,\gamma} f_R\|_{q,\alpha} = R^{-\gamma-(2+\alpha)/q} \|\mathcal{P}_{\alpha,\gamma} f\|_{q,\alpha}. \tag{14}$$

It follows from the above considerations and the boundedness of $\mathcal{P}_{\alpha,\gamma}$ that there exists a constant $C > 0$ such that, for any $f \in L^p(\mathcal{H}, dV_\alpha)$,

$$\begin{aligned} R^{-\gamma-(2+\alpha)/q} \|\mathcal{P}_{\alpha,\gamma} f\|_{q,\alpha} &= \|\mathcal{P}_{\alpha,\gamma} f_R\|_{q,\alpha} \leq C \|f_R\|_{p,\alpha} \\ &= R^{-(2+\alpha)/p} \|f\|_{p,\alpha}. \end{aligned} \tag{15}$$

That is,

$$R^{-\gamma-(2+\alpha)/q+(2+\alpha)/p} \|\mathcal{P}_{\alpha,\gamma} f\|_{q,\alpha} \leq C \|f\|_{p,\alpha}. \tag{16}$$

As the latter holds for any $f \in L^p(\mathcal{H}, dV_\alpha)$ and any $R > 0$, we must have

$$-\gamma - \frac{2 + \alpha}{q} + \frac{2 + \alpha}{p} = 0. \tag{17}$$

That is, $1/p - 1/q = \gamma/(2 + \alpha)$. □

The following is obtained as above.

Lemma 6. *Let $\alpha > -1, 0 \leq \gamma < 2 + \alpha$, and $1 \leq p \leq q < \infty$. If the operator P_α is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_{q\gamma+\alpha})$, then*

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2 + \alpha}. \tag{18}$$

We next prove that condition (5) is sufficient for the boundedness of the fractional operator in the case $p > 1$.

Lemma 7. *Let $\alpha > -1, 0 \leq \gamma < 2 + \alpha$, and $1 < p \leq q < \infty$. Assume that*

$$\frac{1}{p} - \frac{1}{q} = \frac{\gamma}{2 + \alpha}. \tag{19}$$

Then the operator $\mathcal{P}_{\alpha,\gamma}^+$ is bounded from $L^p(\mathcal{H}, dV_\alpha)$ to $L^q(\mathcal{H}, dV_\alpha)$.

Proof. We are assuming that $1/p - 1/q = \gamma/(2 + \alpha)$.

Let us put

$$\omega = -(2 + \alpha) \left(\frac{1}{p'} + \frac{1}{q} \right) = -(2 + \alpha) \left(1 - \frac{\gamma}{2 + \alpha} \right). \tag{20}$$

Clearly, $\omega < 0$. As $\alpha + 1 > 0$, we can find two numbers s and r such that

$$\begin{aligned} 0 < r < \frac{\alpha + 1}{q}, \\ 0 < s < \frac{\alpha + 1}{p'} \end{aligned} \tag{21}$$

with $r > s$.

Let

$$t := \frac{-(\alpha + 2) / p' + s - r}{\omega} \tag{22}$$

so that

$$1 - t = \frac{r - s - (\alpha + 2) / q}{\omega}. \tag{23}$$

We observe that the operator $\mathcal{P}_{\alpha, \gamma}^+$ can be represented as

$$\mathcal{P}_{\alpha, \gamma}^+ f(z) = \int_{\mathcal{H}} f(w) K(z, w) dV_{\alpha}(w), \tag{24}$$

where $K(z, w) = 1/|z - \bar{w}|^{2+\alpha-\gamma}$. Let us define

$$\begin{aligned} \varphi_1(w) &= (\mathfrak{I}w)^{-s}, \\ \varphi_2(w) &= (\mathfrak{I}w)^{-r}. \end{aligned} \tag{25}$$

Applying Okikiolu's test to $\mathcal{P}_{\alpha, \gamma}^+$ we first obtain

$$\begin{aligned} &\int_{\mathcal{H}} K(z, w)^{tp'} \varphi_1^{p'}(w) (\mathfrak{I}w)^{\alpha} dV(w) \\ &= \int_{\mathcal{H}} \frac{v^{-sp'+\alpha}}{|z - \bar{w}|^{t(2+\alpha-\gamma)p'}} dV(w). \end{aligned} \tag{26}$$

From our choice of s we have $-sp' + \alpha > -1$. Using the definitions of ω and t , we obtain

$$\begin{aligned} &t(2 + \alpha - \gamma) p' + sp' - \alpha - 2 \\ &= (2 + \alpha) \left(1 - \frac{\gamma}{2 + \alpha}\right) tp' + sp' - \alpha - 2 \\ &= -\omega tp' + sp' - \alpha - 2 \\ &= \left(\frac{\alpha + 2}{p'} + r - s\right) p' + sp' - \alpha - 2 = rp' > 0. \end{aligned} \tag{27}$$

Hence we obtain from the above observations and Lemma 3 that

$$\begin{aligned} &\int_{\mathcal{H}} K(z, w)^{tp'} \varphi_1^{p'}(w) (\mathfrak{I}w)^{\alpha} dV(w) = Cy^{-rp'} \\ &= C\varphi_2(z)^{p'}. \end{aligned} \tag{28}$$

In the same way, we first have

$$\begin{aligned} &\int_{\mathcal{H}} K(z, w)^{(1-t)q} \varphi_2^q(z) (\mathfrak{I}z)^{\alpha} dV(z) \\ &= \int_{\mathcal{H}} \frac{(\mathfrak{I}z)^{-rq+\alpha}}{|z - \bar{w}|^{(1-t)(2+\alpha-\gamma)q}} dV(z). \end{aligned} \tag{29}$$

From our choice of s , we have $-rq + \alpha > -1$. From the definition of ω and $1 - t$, we obtain

$$\begin{aligned} &(1 - t)(2 + \alpha - \gamma)q + rq - \alpha - 2 \\ &= -\omega(1 - t)q + rq - \alpha - 2 \\ &= \left(\frac{2 + \alpha}{q} + s - r\right)q + rq - \alpha - 2 = sq > 0. \end{aligned} \tag{30}$$

Hence we obtain from the above observations and Lemma 3 that

$$\begin{aligned} &\int_{\mathcal{H}} K(z, w)^{(1-t)q} \varphi_2^q(z) (\mathfrak{I}z)^{\alpha} dV(w) = C(\mathfrak{I}w)^{-sq} \\ &= C\varphi_1(w)^q. \end{aligned} \tag{31}$$

The proof is complete. □

Proof of Theorem 1. It is obvious that (b) \Rightarrow (a). That (a) \Rightarrow (c) is Lemma 5. That (c) \Rightarrow (b) is Lemma 7. The proof is complete. □

Proof of Proposition 2. Clearly, (b) \Rightarrow (a). That (a) \Rightarrow (c) is Lemma 6. That (c) \Rightarrow (b) follows from Lemma 7 and the fact that the boundedness of $\mathcal{P}_{\alpha, \gamma}^+$ from $L^p(\mathcal{H}, dV_{\alpha})$ to $L^q(\mathcal{H}, dV_{\alpha})$ implies the boundedness of P_{α}^+ from $L^p(\mathcal{H}, dV_{\alpha})$ to $L^q(\mathcal{H}, dV_{q\gamma+\alpha})$. The proof is complete. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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