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Research Article

Asymptotics for the Ostrovsky-Hunter Equation in the Critical Case

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We consider the Cauchy problem for the Ostrovsky-Hunter equation $\partial_x(\partial_t u-(b/3)\partial_x^3 u-\partial_x \mathscr{K}u^3) = au$, $(t,x)\in\mathbb{R}^2$, $u(0,x)=u_0(x)$, $x\in\mathbb{R}$, where $ab>0$. Define $\xi_0 = (27a/b)^{1/4}$. Suppose that $\mathscr{K}$ is a pseudodifferential operator with a symbol $\widehat{K}(\xi)$ such that $\widehat{K}(\pm\xi_0)=0$, $\operatorname{Im}\widehat{K}(\xi)=0$, and $|\widehat{K}(\xi)|\le C$. For example, we can take $\widehat{K}(\xi)=(\xi^2-\xi_0^2)/(\xi^2+1)$. We prove the global in time existence and the large time asymptotic behavior of solutions.

1. Introduction

We consider the Cauchy problem for the generalized Ostrovsky-Hunter equation

$$\partial_x\left(\partial_t u-\frac{b}{3}\partial_x^3 u-\partial_x f(u)\right)=au,\quad (t,x)\in\mathbb{R}^2,$$
$$u(0,x)=u_0(x),\quad x\in\mathbb{R},\tag{1}$$

where $ab>0$, $f(u)=\mathscr{K}u^3$. We assume that $\mathscr{K}$ is a pseudodifferential operator with a symbol $\widehat{K}(\xi)$ such that $\widehat{K}(\pm\xi_0)=0$ with $\xi_0=(27a/b)^{1/4}$. Also we suppose that $\operatorname{Im}\widehat{K}(\xi)=0$ and $|\widehat{K}(\xi)|\le C$. For example, we can choose $\widehat{K}(\xi)=(\xi^2-\xi_0^2)/(\xi^2+1)$. Denote by $\Lambda(\xi)=a/\xi+(b/3)\xi^3$ the symbol of the linear part of (1). The constant $\xi_0=(27a/b)^{1/4}$ is a positive root of $\Omega(\xi)=\Lambda(\xi)-3\Lambda(\xi/3)=(8b/27)\xi^{-1}(\xi^4-27a/b)=0$. Our strategy of the proof of the main result is similar to the one used in [1]. We translate (1) into the ordinary differential equation by using the evolution operator related to the linear problem; then we divide the nonlinear term into resonance and nonresonance parts. Nonresonance part has an oscillating term $e^{it\Omega(\xi)}$ which yields better time decay through the integration by parts; however the factor $1/\Omega(\xi)$ gives us a singularity at ξ_0; see (37) for details. This is the reason why we assume the additional condition $\widehat{K}(\pm\xi_0)=0$ on the symbol $\widehat{K}(\xi)$.

We define the evolution group $\mathscr{U}(t)=\mathscr{F}^{-1}E\mathscr{F}$, where the multiplication factor $E=e^{-it\Lambda(\xi)}$, $\Lambda(\xi)=a/\xi+(b/3)\xi^3$. It is well known that the operator $\mathscr{J}=\mathscr{U}(t)x\mathscr{U}(-t)$ is a useful tool for obtaining the $\mathbf{L}^\infty$-time decay estimates of solutions and has been used widely for studying the asymptotic behavior of solutions to various nonlinear dispersive equations. We have

$$\begin{aligned}\mathscr{J}&=\mathscr{U}(t)x\mathscr{U}(-t)=\mathscr{F}^{-1}e^{-it\Lambda(\xi)}i\partial_\xi e^{it\Lambda(\xi)}\mathscr{F}\\&=\mathscr{F}^{-1}\left(i\partial_\xi-t\Lambda'(\xi)\right)\mathscr{F}=x-t\Lambda'(-i\partial_x)\\&=x-ta\partial_x^{-2}+tb\partial_x^2,\end{aligned}\tag{2}$$

where $\Lambda'(-i\partial_x)=a\partial_x^{-2}-b\partial_x^2$, and the antiderivative ∂_x^{-1} is defined by the Fourier transform such that

$$\widehat{\partial_x^{-1}\phi}(\xi)=(i\xi)^{-1}\widehat{\phi}(\xi).\tag{3}$$

Note that the commutators are true $[\mathscr{J},\mathscr{L}]=0$, $[\partial_x,\mathscr{L}]=0$, $[\mathscr{J},\partial_x]=-1$, $[\partial_x^{-1},x\partial_x]=-\partial_x^{-1}$, where $\mathscr{L}=\partial_t+\Lambda(-i\partial_x)=\partial_t-a\partial_x^{-1}-(b/3)\partial_x^3$. However, it seems that $\mathscr{J}$ does not work well on the nonlinear terms. In order to avoid the derivative loss, when estimating the norm $\|\partial_x\mathscr{J}u\|_{\mathbf{L}^2}$ instead of the operators $\mathcal{J}$ we apply the modified dilation operator defined by

$$\mathcal{P} = t\partial_t + \frac{1}{3}x\partial_x - \frac{4}{3}a\partial_a. \tag{4}$$

Note that $\mathcal{P}$ acts well on the nonlinear terms as the first-order differential operator and it almost commutes with $\mathcal{L}$: $[\mathcal{P}, \mathcal{L}] = -\mathcal{L}$. Also $\mathcal{J}$ and $\mathcal{P}$ are related via the identity

$$\mathcal{P} = t\mathcal{L} + \frac{1}{3}\mathcal{J}\partial_x - \frac{4}{3}a\mathcal{I}, \tag{5}$$

where

$$\mathcal{I} = \partial_a - t\partial_x^{-1}. \tag{6}$$

Note that $[\mathcal{I}, \mathcal{L}] = 0$. In order to get the estimate of $\partial_x \mathcal{J} u$, we will show the a priori estimates of , $t\mathcal{L}u$, and $\mathcal{I}u$. Different point compared to the previous works is to consider the estimate of $\mathcal{I}u$ since $\mathcal{I}u$ contains the term $t\partial_x^{-1}$ with an additional explicit time growth.

When $f(u) = u^2$, then (1) was introduced in [2] for modelling the small-amplitude long waves in a rotating fluid of finite depth. Therefore (1) with $f(u) = u^2$ is called the Ostrovsky equation. It was studied by many authors (see, e.g., [3–5] and references cited therein). When $b = 0$, (1) is called the reduced Ostrovsky equation. Equation (1) has some conservation quantities, when $f(u) = \lambda |u|^{p-1} u$, $\lambda \in \mathbb{R}$. One of them is the zero mass conservation law which is obtained by integrating in space

$$a \int u(t,x)\, dx = 0 \tag{7}$$

under the restriction $\int u_0(x) dx = 0$. Rewrite (1) as

$$\partial_t u - \frac{b}{3}\partial_x^3 u - \partial_x f(u) = aD_x^{-1} u. \tag{8}$$

Multiplying both sides of (8) by u, integrating in space, using (7), we obtain

$$\frac{d}{dt}\int |u(t,x)|^2\, dx + \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\, dx = 0 \tag{9}$$

which is the conservation of the momentum. The same approach as in deriving (9) will be used for the high frequency part in order to avoid the derivative loss, when proving the existence of solutions of (1).

Local well-posedness for the Ostrovsky equation was shown in [5] in the case of the initial data

$$u_0 \in \mathbf{H}^s \cap \dot{\mathbf{H}}^{-1}, \quad s > \frac{3}{2} \tag{10}$$

by using the parabolic regularization technique and limiting arguments. Their method works also for the case of the generalized nonlinearity $f(u) = |u|^{p-1} u$ and also generalized reduced Ostrovsky equation (1), since the dispersive effects were not used in the proof. Thanks to the high frequency part u_{xxx}, the solutions to the linear equation $(u_t - \beta u_{xxx})_x = \gamma u$ obtain a smoothing property. By using this property, in [3], the local well-posedness for the Ostrovsky equation was shown under the condition

$$u_0 \in \mathbf{H}^s \cap \dot{\mathbf{H}}^{-1}, \quad s > \frac{3}{4}. \tag{11}$$

The method of [3] depends on the linear part of the equation and also works for the nonlinearities of a general order. In [4, 6–8] the local well-posedness for the Ostrovsky equation was treated by the Fourier restriction norm method of [9] and in [4] the $\mathbf{H}^{-3/4+}$ local well-posedness was shown. We note here that the Sobolev space $\mathbf{H}^{-3/4}$ is considered as critical regularity concerning the Korteweg-de Vries equation.

Global well-posedness in the energy class was obtained for the Ostrovsky equation in [3] through the energy conservation law, when the initial data

$$u_0 \in \mathbf{H}^1 \cap \dot{\mathbf{H}}^{-1}, \tag{12}$$

and $ab > 0$. After their work, the global well-posedness in

$$\mathbf{L}^2 \cap \dot{\mathbf{H}}^{-s}, \quad 0 \le s \le 1, \tag{13}$$

was proved in [4, 6] due to the $\mathbf{L}^2$-conservation law. The global well-posedness, in the negative order Sobolev space $\mathbf{H}^{-3/10+}$, was shown in [8] by using the I method of [10].

We now turn to the case of the reduced Ostrovsky equation. The local well-posedness was shown in the space $\mathbf{H}^2$ in paper [11] and after that in $\mathbf{H}^{3/2+}$ in [12]. Their methods work also in the case of the general nonlinear dispersive equations with different nonlinearities. We also refer to [13, 14] for the local well-posedness in the class

$$u_0 \in \mathbf{H}^m \cap \dot{\mathbf{H}}^{-1} \quad m \ge 2. \tag{14}$$

However there are few works on the global well-posedness for the reduced Ostrovsky equation due to the lack of the smoothing property. The global well-posedness for reduced Ostrovsky equation (1) with $b = 0$ and cubic nonlinearity $f(u) = u^3$ (which is called the short pulse equation) was obtained in [15], when the initial data

$$\left\|\partial_x u_0\right\|_{\mathbf{H}^1} < 1, \quad u_0 \in \mathbf{H}^2, \tag{15}$$

whereas for the quadratic nonlinearity $f(u) = u^2$ (which is called the reduced Ostrovsky equation or the Ostrovsky-Hunter equation; see [16, 17]), it was shown in [18] when the initial data

$$\left(1 - 3\partial_x^2\right) u_0(x) < 0, \quad u_0 \in \mathbf{H}^3, \tag{16}$$

for all $x \in \mathbf{R}$. The time decay properties of solutions to the corresponding linear problem can be studied if we assume that the initial data decay rapidly at infinity. So the global existence was shown in [12], for the nonlinearity $f(u) = u^p$ with an integer $p \ge 4$, when the initial data are small and sufficiently regular:

$$u_0 \in \mathbf{H}^5 \cap \mathbf{H}_1^3. \tag{17}$$

In [1, 19, 20], we considered the large time asymptotics for reduced Ostrovsky equation (1) with $b = 0$ and some conditions on the order of nonlinearity.

To state our results precisely we introduce *Notation and Function Spaces.* We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = (\int |\phi(x)|^p dx)^{1/p}$ for $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbf{R}}|\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{k,s} = \left\{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{k,s}} = \left\|\langle x\rangle^s \langle i\partial_x\rangle^k \phi\right\|_{\mathbf{L}^p} < \infty\right\}. \quad (18)$$

$k, s \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$, and $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{k,s} = \mathbf{H}_2^{k,s}$, $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly, if they do not cause any confusion. Let $\mathbf{C}(\mathbf{I}; \mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. We define the free evolution group $\mathscr{U}(t) = e^{-it\Lambda(-i\partial_x)} = \mathscr{F}^{-1}E\mathscr{F}$, where the multiplication factor $E(t,\xi) = e^{-it\Lambda(\xi)}$.

We are now in a position to state our main result.

Theorem 1. *Assume that the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^2\cap\mathbf{H}^{1,1}} \le \varepsilon$. Then there exists a unique global solution $u \in \mathbf{C}([0,\infty); \mathbf{H}^2)$ of Cauchy problem (1) satisfying the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-1/2}. \quad (19)$$

Moreover there exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ such that the asymptotics

$$u(t) = 2\text{Re}\, t^{-1/2} e^{-2it(a/\eta(x/t)-(b/3)\eta(x/t)^3)} \frac{W_+(\eta(x/t))}{\sqrt{\Lambda''(\eta(x/t))}} \cdot \exp\left(\frac{3i\eta(x/t)\widehat{K}(\eta(x/t))}{\langle\eta(x/t)\rangle\Lambda''(\eta(x/t))}\left|W_+\left(\eta\left(\frac{x}{t}\right)\right)\right|^2 \cdot \log t\right) + O\left(\varepsilon t^{-1/2-\delta}\right) \quad (20)$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\delta > 0$ is a small constant and

$$\eta(x) = \sqrt{\frac{1}{2b}\left(x + \sqrt{4ab + x^2}\right)}. \quad (21)$$

2. Factorization Technique

We now introduce the factorization formulas for (1). We have for the free evolution group $\mathscr{U}(t) = \mathscr{F}^{-1}E\mathscr{F}$, where the multiplication factor $E = e^{-it\Lambda(\xi)}$, $\Lambda(\xi) = a/\xi + (b/3)\xi^3$. Denote the Heaviside function $\theta(\xi) = 1$ for $\xi > 0$ and $\theta(\xi) = 0$ for $\xi \le 0$. Then for the real-valued function $u(t) = \mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi}$ we find

$$\begin{aligned}\mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi} &= 2\text{Re}\mathscr{F}^{-1}\theta E\widehat{\varphi} \\ &= 2\text{Re}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{it((x/t)\xi-\Lambda(\xi))}\widehat{\varphi}(\xi)\, d\xi \\ &= 2\text{Re}\mathscr{D}_t\frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{it(x\xi-\Lambda(\xi))}\widehat{\varphi}(\xi)\, d\xi,\end{aligned} \quad (22)$$

where the dilation operator $\mathscr{D}_t\phi = |t|^{-1/2}\phi(xt^{-1})$. Note that there is a unique stationary point in the integral $\int_0^\infty e^{it(x\xi-\Lambda(\xi))}\phi(\xi)d\xi$, which is defined by the root $\xi = \eta(x) = \sqrt{(1/2b)(x+\sqrt{4ab+x^2})} > 0$ of the equation $\Lambda'(\xi) = -a/\xi^2 + b\xi^2 = x$ for all $x \in \mathbb{R}$. Thus $\Lambda'(\eta(x)) = x$ and we introduce the so-called scaling operator

$$\left(\mathscr{B}^{-1}\phi\right)(x) = \frac{1}{\sqrt{\Lambda''(\eta(x))}}\phi(\eta(x)) \quad (23)$$

and the multiplication factor

$$M(t,\eta) = e^{it(\eta\Lambda'(\eta)-\Lambda(\eta))} = e^{-2it(a/\eta-(b/3)\eta^3)}. \quad (24)$$

Note that, in the case of $b = 0$, then $\eta(x)$ is defined by $\Lambda'(\xi) = -a/\xi^2 = x$; namely, $\eta(x) = \sqrt{a/|x|}$ for $x < 0$. Hence

$$\left(\mathscr{B}^{-1}\phi\right)(x) = \frac{1}{\sqrt{3a}}\left(\frac{a}{|x|}\right)^{3/4}\phi\left(\sqrt{\frac{a}{|x|}}\right), \quad (25)$$

for $b = 0$, $x < 0$; see [1]. Therefore $\mathscr{B}^{-1}$ is the scaling operator if the symbol $\Lambda(\xi)$ is homogeneous.

By the definition of $\mathscr{B}^{-1}$, its inverse operator is defined by

$$(\mathscr{B}\phi)(\eta) = \sqrt{\Lambda''(\eta)}\phi\left(\Lambda'(\eta)\right). \quad (26)$$

Then we have

$$\begin{aligned}\mathscr{U}(t)\mathscr{F}^{-1}\phi &= 2\text{Re}\mathscr{D}_t\frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\, d\xi \\ &= 2\text{Re}\mathscr{D}_t\mathscr{B}^{-1}\mathscr{B}\frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{it(\Lambda'(\eta(x))\xi-\Lambda(\xi))}\phi(\xi)\, d\xi \\ &= 2\text{Re}\mathscr{D}_t\mathscr{B}^{-1}M\sqrt{\frac{|t|\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\phi(\xi)\, d\xi \\ &= 2\text{Re}\mathscr{D}_t\mathscr{B}^{-1}M\mathscr{V}\phi,\end{aligned} \quad (27)$$

where the phase function $S(\eta,\xi) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta)$ and the operator

$$\begin{aligned}\mathscr{V}\phi &= \overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\mathscr{F}^{-1}E\theta\phi \\ &= \sqrt{\frac{|t|\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\phi(\xi)\, d\xi.\end{aligned} \quad (28)$$

We have $\|\mathscr{D}_t^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$, $\|\mathscr{F}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$, and $\|\mathscr{B}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$. Hence

$$\|\mathscr{V}\phi\|_{\mathbf{L}^2} = \left\|\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\mathscr{F}^{-1}E\theta\phi\right\|_{\mathbf{L}^2} = \|\theta\phi\|_{\mathbf{L}^2} \le \|\phi\|_{\mathbf{L}^2}. \quad (29)$$

Also we decompose the inverse operator

$$\begin{aligned}\mathscr{F}\mathscr{U}(-t)\phi &= \overline{E}\mathscr{F}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-it((x/t)\xi-\Lambda(\xi))}\phi(x)\,dx \\ &= \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-it(\xi x-\Lambda(\xi))}\mathscr{D}_t^{-1}\phi(x)\,dx. \\ &= \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{-it(\xi\Lambda'(\eta)-\Lambda(\xi))}\sqrt{\Lambda''(\eta)}\left(\mathscr{B}\mathscr{D}_t^{-1}\phi\right)d\eta \\ &= \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}\overline{M}\left(\mathscr{B}\mathscr{D}_t^{-1}\phi\right)\sqrt{\Lambda''(\eta)}d\eta \\ &= \mathscr{V}^*\overline{M}\mathscr{B}\mathscr{D}_t^{-1}\phi.\end{aligned} \quad (30)$$

Since $x = \Lambda'(\eta)$, then

$$\begin{aligned}&\int_{\mathbb{R}} e^{-it(\xi x-\Lambda(\xi))}\mathscr{D}_t^{-1}\phi(x)\,dx \\ &= \int_{\mathbb{R}} e^{-it(\xi\Lambda'(\eta)-\Lambda(\xi))}\mathscr{D}_t^{-1}\phi\left(\Lambda'(\eta)\right)\Lambda''(\eta)\,d\eta \\ &= \int_0^\infty e^{-it(\xi\Lambda'(\eta)-\Lambda(\xi))}\sqrt{\Lambda''(\eta)}\left(\mathscr{B}\mathscr{D}_t^{-1}\phi\right)d\eta \\ &= \int_0^\infty e^{itS(\eta,\xi)}\overline{M}\left(\mathscr{B}\mathscr{D}_t^{-1}\phi\right)\sqrt{\Lambda''(\eta)}d\eta \\ &= \frac{\sqrt{2\pi}}{|t|^{1/2}}\mathscr{V}^*\overline{M}\mathscr{B}\mathscr{D}_t^{-1}\phi,\end{aligned} \quad (31)$$

where $\Lambda''(\xi) = 2\xi^{-3}(a+b\xi^4) > 0$ and the operator

$$\begin{aligned}\mathscr{V}^*\phi &= \mathscr{F}\mathscr{U}(-t)\mathscr{D}_t\mathscr{B}^{-1}M\phi \\ &= \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}\phi(\eta)\sqrt{\Lambda''(\eta)}d\eta.\end{aligned} \quad (32)$$

Define the new dependent variable $\widehat{\varphi} = \mathscr{F}\mathscr{U}(-t)u(t)$. Since $\mathscr{F}\mathscr{U}(-t)\mathscr{L} = \partial_t\mathscr{F}\mathscr{U}(-t)$, where $\mathscr{L} = \partial_t - \partial_x^{-1} - (1/3)\partial_x^3$, applying the operator $\mathscr{F}\mathscr{U}(-t)$ to (1) we get

$$\begin{aligned}\partial_t\widehat{\varphi} &= \partial_t\mathscr{F}\mathscr{U}(-t)u = \mathscr{F}\mathscr{U}(-t)\mathscr{L}u \\ &= \mathscr{F}\mathscr{U}(-t)\partial_x\mathscr{K}u^3 = i\xi\widehat{K}(\xi)\mathscr{F}\mathscr{U}(-t)u^3 \\ &= i\xi\widehat{K}(\xi)\mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\left(u^3\right).\end{aligned} \quad (33)$$

Then since

$$\begin{aligned}&\mathscr{U}(t)\mathscr{F}^{-1}\phi = 2\mathrm{Re}\mathscr{D}_t\mathscr{B}^{-1}M\mathscr{V}\phi, \\ &u = \mathscr{U}(t)\mathscr{F}^{-1}\widehat{\varphi} = 2\mathrm{Re}\mathscr{D}_t\mathscr{B}^{-1}M\mathscr{V}\widehat{\varphi} \\ &= \mathscr{D}_t\mathscr{B}^{-1}\left(M\mathscr{V}\widehat{\varphi} + \overline{M\mathscr{V}\widehat{\varphi}}\right),\end{aligned} \quad (34)$$

we find the following representation:

$$\begin{aligned}\partial_t\widehat{\varphi} &= \mathscr{V}^*\overline{M}\mathscr{B}\mathscr{D}_t^{-1}\left(\partial_x\mathscr{K}u^3\right) = i\xi\widehat{K}(\xi) \\ &\cdot\mathscr{V}^*\overline{M}\mathscr{B}\mathscr{D}_t^{-1}\left(\left(\mathscr{D}_t\mathscr{B}^{-1}\left(M\mathscr{V}\widehat{\varphi} + \overline{M\mathscr{V}\widehat{\varphi}}\right)\right)^3\right) \\ &= i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*\overline{M}\mathscr{B}\left(\left(\mathscr{B}^{-1}\left(M\mathscr{V}\widehat{\varphi} + \overline{M\mathscr{V}\widehat{\varphi}}\right)\right)^3\right) \\ &= i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*\frac{1}{\Lambda''}\overline{M}\left(M\mathscr{V}\widehat{\varphi} + \overline{M\mathscr{V}\widehat{\varphi}}\right)^3 \\ &= i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*M^2\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right)^3 + 3i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^* \\ &\cdot\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right)^2\left(\overline{\mathscr{V}\widehat{\varphi}}\right) + 3i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*\overline{M}^2\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right) \\ &\cdot\left(\overline{\mathscr{V}\widehat{\varphi}}\right)^2 + i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*\overline{M}^4\frac{1}{\Lambda''}\left(\overline{\mathscr{V}\widehat{\varphi}}\right)^3.\end{aligned} \quad (35)$$

Note that for $\alpha \neq -1$

$$\begin{aligned}\mathscr{V}^*(t)M^\alpha\phi &= \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}M^\alpha\phi(\eta)\Lambda''(\eta)\,d\eta \\ &= \frac{|t|^{1/2}}{\sqrt{2\pi}} \\ &\cdot e^{it(\Lambda(\xi)-(1+\alpha)\Lambda(\xi/(1+\alpha)))}\int_0^\infty e^{it(1+\alpha)S(\eta,\xi/(1+\alpha))}\phi(\eta) \\ &\cdot\Lambda''(\eta)\,d\eta = e^{it(\Lambda(\xi)-(1+\alpha)\Lambda(\xi/(1+\alpha)))}|1+\alpha|^{1/2} \\ &\cdot\mathscr{D}_{1+\alpha}\mathscr{V}^*((1+\alpha)t)\phi.\end{aligned} \quad (36)$$

Thus we obtain the following equation for the new dependent variable $\widehat{\varphi}(t,\xi) = \mathscr{F}\mathscr{U}(-t)u(t)$:

$$\begin{aligned}&\partial_t\widehat{\varphi}(t,\xi) \\ &= \sqrt{3}i\xi\widehat{K}(\xi)t^{-1}e^{it\Omega(\xi)}\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right)^3 \\ &+ 3i\xi\widehat{K}(\xi)t^{-1}\mathscr{V}^*(t)\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right)^2\left(\overline{\mathscr{V}\widehat{\varphi}}\right) \\ &+ 3i\xi\widehat{K}(\xi)t^{-1}\mathscr{D}_{-1}\mathscr{V}^*(-t)\frac{1}{\Lambda''}\left(\mathscr{V}\widehat{\varphi}\right)\left(\overline{\mathscr{V}\widehat{\varphi}}\right)^2 \\ &+ \sqrt{3}i\xi\widehat{K}(\xi)t^{-1}e^{it\Omega(\xi)}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\left(\overline{\mathscr{V}\widehat{\varphi}}\right)^3,\end{aligned} \quad (37)$$

where $\Omega(\xi) = \Lambda(\xi) - 3\Lambda(\xi/3)$.

Now we explain how to use (37) for estimating $|\widehat{\varphi}(t,\xi)|$ uniformly with respect to ξ. For the real-valued solution u, we have $\overline{\widehat{\varphi}(t,\xi)} = \widehat{\varphi}(t,-\xi)$; hence it is sufficient to consider the case $\xi > 0$ only. From Lemmas 2 and 3, we find that the last two terms of the right-hand side of (37) are the remainders. We need to consider the first and the second terms of the right-hand side of (37). Due to the oscillating factor $\widehat{K}(\xi)e^{it\Omega(\xi)}$, integrating by parts with respect to time, we will show that the first term of (37) is also a remainder, since $\widehat{K}(\xi)/\Omega(\xi)$ is bounded in view of the conditions for the symbol $\widehat{K}(\xi)$.

We organize the rest of our paper as follows. In Section 3, we state main estimates for the decomposition operators $\mathcal{V}(t)$ and $\mathcal{V}^*(t)$ related to the evolution group $\mathcal{U}(t)$. We prove a priori estimates of solutions in Section 4. Section 5 is devoted to the proof of Theorem 1.

3. Preliminaries

3.1. Two Kernels. Define the kernel

$$A_j(t,\eta)=\theta(\eta)\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\xi^j\chi(\xi\eta^{-1})\,d\xi, \tag{38}$$

where $j=-1,0,1,2$, the phase function $S(\eta,\xi)=\Lambda(\xi)-\Lambda(\eta)-\Lambda'(\eta)(\xi-\eta)=(1/3)\xi^{-1}\eta^{-2}(3a+2b\eta^3\xi+b\eta^2\xi^2)(\xi-\eta)^2$, $\Lambda''(\eta)=2\eta^{-3}(a+b\eta^4)$, the cut-off function $\chi(z)\in\mathbf{C}^2(\mathbb{R})$ is such that $\chi(z)=0$ for $z\le 1/3$ or $z\ge 3$ and $\chi(z)=1$ for $2/3\le z\le 3/2$, and the Heaviside function $\theta(\eta)=1$ for $\eta>0$ and $\theta(\eta)=0$ for $\eta\le 0$. We change $\xi=\eta y$; then we get

$$\begin{aligned}&A_j(t,\eta)=\eta^j\theta(\eta)\\&\cdot\sqrt{\frac{t\eta^2\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-it(ay^{-1}\eta^{-1}+(2/3)b\eta^3+(b/3)\eta^3y)(y-1)^2}y^j\chi(y)\,dy.\end{aligned} \tag{39}$$

To compute the asymptotics of the kernel $A_j(t,\eta)$ for large t we apply the stationary phase method (see [21, 22], p. 163):

$$\begin{aligned}&\int e^{itg(y)}f(y)\,dy\\&=e^{itg(y_0)}f(y_0)\sqrt{\frac{2\pi}{t|g''(y_0)|}}e^{i(\pi/4)\operatorname{sgn}g''(y_0)}\\&\quad+O(t^{-3/2})\end{aligned} \tag{40}$$

for $t\to+\infty$, where the stationary point y_0 is defined by $g'(y_0)=0$. By virtue of formula (40) with $g(y)=-(ay^{-1}\eta^{-1}+(2/3)b\eta^3+(b/3)\eta^3y)(y-1)^2$, $f(y)=y^j\chi(y)$, $y_0=1$, we get

$$A_j(t,\eta)=\eta^j\theta(\eta)\left(e^{-i(\pi/4)}+O(t^{-1})\right). \tag{41}$$

In particular we have the estimate $\|\eta^{-j}A_j(t)\|_{\mathbf{L}^\infty}\le C$. Also we define the kernel

$$\begin{aligned}&A^*(t,\xi)\\&=\theta(\xi)\frac{t^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}\chi(\eta\xi^{-1})\sqrt{\Lambda''(\eta)}\,d\eta.\end{aligned} \tag{42}$$

We change $\eta=\xi y$; then we get

$$\begin{aligned}&A^*(t,\xi)=\theta(\xi)\frac{t^{1/2}}{\sqrt{\pi\xi}}\\&\cdot\int_0^\infty e^{it(a\xi^{-1}y^{-2}+(2b/3)\xi^3y+(b/3)\xi^3)(y-1)^2}\chi(y)\\&\cdot\sqrt{y^{-3}(a+b\xi^4y^4)}\,dy.\end{aligned} \tag{43}$$

By virtue of formula (40) with $g(y)=(a\xi^{-1}y^{-2}+(2b/3)\xi^3y+(b/3)\xi^3)(y-1)^2$, $f(y)=\chi(y)\sqrt{y^{-3}(a+b\xi^4y^4)}$, $y_0=1$, we obtain

$$A^*(t,\xi)=\theta(\xi)e^{i(\pi/4)}+O(t^{-1}). \tag{44}$$

In particular we have the estimate $\|A^*(t)\|_{\mathbf{L}^\infty}\le C$.

3.2. Estimates in the Uniform Norm. In the next lemma we estimate the operator $\mathcal{V}$ in the uniform norm. Denote $\mu_{-1}=5/4$, $\mu_0=1/4$, $\mu_1=0$, $\nu_{-1}=-5/4$, $\nu_0=1$, and $\nu_1=1/4$.

Lemma 2. *Let $j=-1,0,1$. Then the estimates*

$$\begin{aligned}&\left\||\eta|^{\mu_j}\langle\eta\rangle^{\nu_j}\left(\mathcal{V}\xi^j\phi-A_j(t)\phi\right)\right\|_{\mathbf{L}^\infty(0,\infty)}\\&\le Ct^{-1/2}\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}+Ct^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2},\\&\left\||\eta|^{\nu_j}\langle\eta\rangle^{-2\nu_j+1}\mathcal{V}\xi^j\phi\right\|_{\mathbf{L}^\infty(-\infty,0)}\\&\le Ct^{-1/2}\left(\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}+\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right)\end{aligned} \tag{45}$$

are valid for all $t\ge 1$.

Proof. We write

$$\begin{aligned}&\mathcal{V}\xi^j\phi-A_j(t,\eta)\phi(\eta)\\&=\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}(\phi(\xi)-\phi(\eta))\\&\cdot\xi^j\chi(\xi\eta^{-1})\,d\xi+\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\phi(\xi)\\&\cdot\left(1-\chi(\xi\eta^{-1})\right)\xi^jd\xi=I_1+I_2\end{aligned} \tag{46}$$

for $\eta>0$. For the first summand I_1 we integrate by parts via the identity

$$e^{-itS(\eta,\xi)}=H_1\partial_\xi\left((\xi-\eta)e^{-itS(\eta,\xi)}\right) \tag{47}$$

with $H_1=(1-it(\xi-\eta)\partial_\xi S(\eta,\xi))^{-1}$, $\partial_\xi S(\eta,\xi)=\xi^{-2}\eta^{-2}(a+b\eta^2\xi^2)(\xi^2-\eta^2)$, to get

$$\begin{aligned}&I_1=C\sqrt{t\Lambda''(\eta)}\int_0^\infty e^{-itS(\eta,\xi)}(\phi(\xi)-\phi(\eta))(\xi-\eta)\\&\cdot\partial_\xi\left(\xi^jH_1\chi(\xi\eta^{-1})\right)+e^{-itS(\eta,\xi)}(\xi-\eta)\\&\cdot\xi^jH_1\chi(\xi\eta^{-1})\phi_\xi(\xi)\,d\xi.\end{aligned} \tag{48}$$

Using the estimates

$$\begin{aligned} &|\phi(\xi)-\phi(\eta)| \le C\eta^{-1}\int_{\xi}^{\eta}\xi\phi_{\xi}(\xi)\,d\xi \\ &\quad \le C\eta^{-1}|\xi-\eta|^{1/2}\|\xi\phi_{\xi}\|_{\mathbf{L}^2}, \\ &\left|(\xi-\eta)\xi^{j}H_1\chi(\xi\eta^{-1})\right| \le \frac{C|\xi-\eta|\eta^{j}}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2}, \\ &\left|(\xi-\eta)\partial_{\xi}\left(\xi^{j}H_1\chi(\xi\eta^{-1})\right)\right| \le \frac{C\eta^{j}}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2} \end{aligned} \tag{49}$$

in the domain $0<(1/3)\eta\le\xi\le 3\eta$, we find

$$\begin{aligned} |I_1| &\le Ct^{1/2}\eta^{j-5/2}\langle\eta\rangle^2\|\xi\phi_{\xi}\|_{\mathbf{L}^2} \\ &\cdot\int_{(1/3)\eta}^{3\eta}\frac{|\xi-\eta|^{1/2}\,d\xi}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2}+Ct^{1/2}\eta^{j-5/2}\langle\eta\rangle^2 \\ &\cdot\int_{(1/3)\eta}^{3\eta}\frac{|\xi-\eta|\,|\xi\phi_{\xi}(\xi)|\,d\xi}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2} \le Ct^{1/2}\eta^{j-5/2}\langle\eta\rangle^2 \\ &\cdot\|\xi\phi_{\xi}\|_{\mathbf{L}^2}\left(\int_{(1/3)\eta}^{3\eta}\frac{|\xi-\eta|^{1/2}\,d\xi}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2}\right. \\ &\left.+\left(\int_{(1/3)\eta}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2\right)^2}\right)^{1/2}\right). \end{aligned} \tag{50}$$

Changing $\xi=\eta y$ we have

$$\begin{aligned} &\int_{(1/3)\eta}^{3\eta}\frac{|\xi-\eta|^{1/2}\,d\xi}{1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2} \\ &\quad\le C\eta^{3/2}\int_{1/3}^{3}\frac{|y-1|^{1/2}\,dy}{1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2} \\ &\quad\le C\eta^{3/2}\left\langle t\eta^{-1}\langle\eta\rangle^4\right\rangle^{-3/4}\le Ct^{-3/4}\eta^{9/4}\langle\eta\rangle^{-3}, \\ &\int_{(1/3)\eta}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t\eta^{-3}\langle\eta\rangle^4(\xi-\eta)^2\right)^2} \\ &\quad\le C\eta^{3}\int_{1/3}^{3}\frac{(y-1)^2\,dy}{\left(1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2\right)^2} \\ &\quad\le C\eta^{3}\left\langle t\eta^{-1}\langle\eta\rangle^4\right\rangle^{-3/2}\le Ct^{-3/2}\eta^{9/2}\langle\eta\rangle^{-6}. \end{aligned} \tag{51}$$

Thus we obtain

$$|I_1|\le Ct^{-1/4}\eta^{j-1/4}\langle\eta\rangle^{-1}\|\xi\phi_{\xi}\|_{\mathbf{L}^2} \tag{52}$$

for all $t\ge 1$, $\eta>0$, and $j=-1,0,1$.

To estimate the second integral I_2 we integrate by parts via the identity

$$e^{-itS(x,\xi)}=H_2\partial_{\xi}\left(\xi e^{-itS(x,\xi)}\right) \tag{53}$$

with $H_2=(1-it\xi\partial_{\xi}S(\eta,\xi))^{-1}$, $\partial_{\xi}S(\eta,\xi)=\xi^{-2}\eta^{-2}(a+b\eta^2\xi^2)(\xi^2-\eta^2)$, to get

$$\begin{aligned} I_2 &= C\sqrt{t\Lambda''(\eta)}\int_0^{\infty}e^{-itS(\eta,\xi)}\phi(\xi) \\ &\cdot\xi\partial_{\xi}\left(\left(1-\chi(\xi\eta^{-1})\right)H_2\xi^{j}\right) \\ &+e^{-itS(x,\xi)}H_2\left(1-\chi(\xi\eta^{-1})\right)\xi^{j+1}\phi_{\xi}(\xi)\,d\xi. \end{aligned} \tag{54}$$

Using the estimate

$$\begin{aligned} &\left|H_2\left(1-\chi(\xi\eta^{-1})\right)\xi^{j}\right|+\left|\xi\partial_{\xi}\left(\left(1-\chi(\xi\eta^{-1})\right)H_2\xi^{j}\right)\right| \\ &\quad\le\frac{C\xi^{j}}{1+t\xi^{-1}\eta^{-2}(1+\xi^2\eta^2)(\xi^2+\eta^2)} \end{aligned} \tag{55}$$

in the domain $0<\xi\le(2/3)\eta$, or $\xi\ge(3/2)\eta>0$, we obtain

$$\begin{aligned} |I_2| &\le Ct^{1/2}\eta^{-3/2}\langle\eta\rangle^2 \\ &\cdot\int_0^{\infty}\frac{\xi^{j}\left(|\phi(\xi)|+|\xi\phi_{\xi}(\xi)|\right)d\xi}{1+t\xi^{-1}\eta^{-2}(1+\xi^2\eta^2)(\xi^2+\eta^2)} \\ &\le Ct^{1/2}\eta^{-3/2}\langle\eta\rangle^2\|\xi^{1/2}\phi\|_{\mathbf{L}^{\infty}} \\ &\cdot\int_0^{\infty}\frac{\xi^{j-1/2}d\xi}{1+t\xi^{-1}\eta^{-2}(1+\xi^2\eta^2)(\xi^2+\eta^2)} \\ &+Ct^{1/2}\eta^{-3/2}\langle\eta\rangle^2\|\xi\phi_{\xi}\|_{\mathbf{L}^2} \\ &\cdot\left(\int_0^{\infty}\frac{\xi^{2j}d\xi}{\left(1+t\xi^{-1}\eta^{-2}(1+\xi^2\eta^2)(\xi^2+\eta^2)\right)^2}\right)^{1/2}. \end{aligned} \tag{56}$$

Changing $\xi=\eta y$ we obtain

$$\begin{aligned} &\int_0^{\infty}\frac{\xi^{j-1/2}d\xi}{1+t\xi^{-1}\eta^{-2}(1+\xi^2\eta^2)(\xi^2+\eta^2)} \\ &\quad\le\int_0^{\infty}\frac{\eta^{j+1/2}y^{j-1/2}dy}{1+t\eta^{-1}y^{-1}(1+\eta^4y^2)\langle y\rangle^2}. \end{aligned} \tag{57}$$

For $\eta\le 1$

$$\begin{aligned} &\int_0^{1}\frac{\eta^{j+1/2}y^{j-1/2}dy}{1+t\eta^{-1}y^{-1}(1+\eta^4y^2)}\le Ct^{-1}\eta^{j+3/2}\int_0^1 y^{j+1/2}dy \\ &\quad\le Ct^{-1}\eta^{j+3/2}, \\ &\int_1^{\infty}\frac{\eta^{j+1/2}y^{j-1/2}dy}{1+t\eta^{-1}y(1+\eta^4y^2)}\le Ct^{-1}\eta^{j+3/2}\int_1^{\infty}\frac{y^{j-3/2}dy}{1+\eta^4y^2} \\ &\quad\le Ct^{-1}\eta^{3/2-\nu_j} \end{aligned} \tag{58}$$

$\nu_{-1} = 1$, and $\nu_0 = \nu_1 = 0$. For $\eta > 1$

$$\begin{aligned}&\int_0^1 \frac{\eta^{j+1/2} y^{j-1/2} dy}{1 + t\eta^{-1} y^{-1} (1 + \eta^4 y^2)} \\ &\quad \leq Ct^{-1} \eta^{j+3/2} \int_0^{1/\eta^2} y^{j+1/2} dy \\ &\qquad + Ct^{-1} \eta^{j-5/2} \int_{1/\eta^2}^1 y^{j-3/2} dy \\ &\quad \leq Ct^{-1} \langle \eta \rangle^{-j-3/2} + Ct^{-1} \langle \eta \rangle^{j-5/2} \\ &\int_1^\infty \frac{\eta^{j+1/2} y^{j-1/2} dy}{1 + t\eta^{-1} y (1 + \eta^4 y^2)} \leq Ct^{-1} \eta^{j-5/2} \int_1^\infty y^{j-7/2} dy \\ &\quad \leq Ct^{-1} \langle \eta \rangle^{j-5/2}.\end{aligned} \tag{59}$$

Hence

$$\begin{aligned}&\eta^{-3/2} \langle \eta \rangle^2 \int_0^\infty \frac{\xi^{j-1/2} d\xi}{1 + t\xi^{-1}\eta^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)} \\ &\quad \leq Ct^{-1} \left(\{\eta\}^j + \{\eta\}^{-\nu_j} \right) \left(\langle \eta \rangle^{-j-1} + \langle \eta \rangle^{j-2} \right) \\ &\quad \leq Ct^{-1} \{\eta\}^{-\nu_j} \langle \eta \rangle^{\nu_j - 1} \leq Ct^{-1} \eta^{-\nu_j} \langle \eta \rangle^{2\nu_j - 1}.\end{aligned} \tag{60}$$

In the same manner changing $\xi = \eta y$ we get

$$\begin{aligned}&\int_0^\infty \frac{\xi^{2j} d\xi}{\left(1 + t\xi^{-1}\eta^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)\right)^2} \\ &\quad \leq C \int_0^\infty \frac{\eta^{2j+1} y^{2j} dy}{\left(1 + t\eta^{-1} y^{-1} (1 + \eta^4 y^2) \langle y \rangle^2\right)^2}.\end{aligned} \tag{61}$$

For $\eta \leq 1$

$$\begin{aligned}&\int_0^1 \frac{\eta^{2j+1} y^{2j} dy}{\left(1 + t\eta^{-1} y^{-1} (1 + \eta^4 y^2)\right)^2} \\ &\quad \leq Ct^{-2} \eta^{2j+3} \int_0^1 y^{2j+2} dy \leq Ct^{-2} \eta^{2j+3} \leq Ct^{-2} \eta^{3-2\nu_j}, \\ &\int_1^\infty \frac{\eta^{2j+1} y^{2j} dy}{\left(1 + t\eta^{-1} y (1 + \eta^4 y^2)\right)^2} \\ &\quad \leq Ct^{-2} \eta^{2j+3} \int_1^\infty \frac{y^{2j-2} dy}{\left(1 + \eta^4 y^2\right)^2} \leq Ct^{-2} \eta^{2j+3-2(2j-1)} \\ &\quad \leq Ct^{-2} \eta^{3-2\nu_j}\end{aligned} \tag{62}$$

$\nu_{-1} = 1$, and $\nu_0 = \nu_1 = 0$. For $\eta > 1$

$$\begin{aligned}&\int_0^1 \frac{\eta^{2j+1} y^{2j} dy}{\left(1 + t\eta^{-1} y^{-1} (1 + \eta^4 y^2)\right)^2} \\ &\quad \leq Ct^{-2} \eta^{2j+3} \int_0^{1/\eta^2} y^{2j+2} dy \\ &\qquad + Ct^{-2} \eta^{2j-5} \int_{1/\eta^2}^1 y^{2j-2} dy \\ &\quad \leq Ct^{-2} \langle \eta \rangle^{-2j-3} + Ct^{-2} \langle \eta \rangle^{2j-5} \\ &\int_1^\infty \frac{\eta^{2j+1} y^{2j} dy}{\left(1 + t\eta^{-1} y (1 + \eta^4 y^2)\right)^2} \leq Ct^{-2} \eta^{2j-5} \int_1^\infty y^{2j-6} dy \\ &\quad \leq Ct^{-2} \langle \eta \rangle^{2j-5}.\end{aligned} \tag{63}$$

Hence

$$\begin{aligned}&\eta^{-3} \langle \eta \rangle^4 \int_0^\infty \frac{\xi^{2j} d\xi}{\left(1 + t\xi^{-1}\eta^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)\right)^2} \\ &\quad \leq Ct^{-2} \{\eta\}^{-2\nu_j} \left(\langle \eta \rangle^{-2j-2} + \langle \eta \rangle^{2j-4} \right) \\ &\quad \leq Ct^{-2} \{\eta\}^{-2\nu_j} \langle \eta \rangle^{2\nu_j - 2} \leq Ct^{-2} \eta^{-2\nu_j} \langle \eta \rangle^{4\nu_j - 2}\end{aligned} \tag{64}$$

for $j = -1, 0, 1$. Thus we have

$$\left| I_2 \right| \leq Ct^{-1/2} \eta^{-\nu_j} \langle \eta \rangle^{2\nu_j - 1} \left(\left\| \xi \phi_\xi \right\|_{\mathbf{L}^2} + \left\| \xi^{1/2} \phi \right\|_{\mathbf{L}^\infty} \right) \tag{65}$$

for all $t \geq 1$, $\eta > 0$, and $j = -1, 0, 1$.

For the case of $\eta < 0$ we integrate by parts using identity (53):

$$\begin{aligned}\mathcal{V} \xi^j \phi &= C \sqrt{t \Lambda''(\eta)} \int_0^\infty e^{-itS(\eta,\xi)} \phi(\xi) \xi \partial_\xi \left(H_2 \xi^j \right) \\ &\quad + e^{-itS(x,\xi)} H_2 \xi^j \xi \phi_\xi(\xi) \, d\xi.\end{aligned} \tag{66}$$

Using the estimate

$$\begin{aligned}&\left| H_2 \xi^j \right| + \left| \xi \partial_\xi \left(H_2 \xi^j \right) \right| \\ &\quad \leq \frac{C\xi^j}{1 + t\xi^{-1} |\eta|^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)}\end{aligned} \tag{67}$$

in the domain $\xi > 0$, $\eta < 0$, we obtain

$$\begin{aligned}&\left| \mathcal{V} \xi^j \phi \right| \leq Ct^{1/2} |\eta|^{-3/2} \langle \eta \rangle^2 \\ &\quad \cdot \int_0^\infty \frac{\xi^j \left(|\phi(\xi)| + |\xi \phi_\xi(\xi)| \right) d\xi}{1 + t\xi^{-1} |\eta|^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)} \\ &\quad \leq Ct^{1/2} |\eta|^{-3/2} \langle \eta \rangle^2 \left\| \xi^{1/2} \phi \right\|_{\mathbf{L}^\infty}\end{aligned}$$

$$\cdot \int_0^\infty \frac{\xi^{j-1/2} d\xi}{1 + t\xi^{-1} |\eta|^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)} + Ct^{1/2} |\eta|^{-3/2} \langle \eta \rangle^2 \|\xi\phi_\xi\|_{\mathbf{L}^2} \cdot \left(\int_0^\infty \frac{\xi^{2j} d\xi}{\left(1 + t\xi^{-1} |\eta|^{-2} (1 + \xi^2\eta^2)(\xi^2 + \eta^2)\right)^2} \right)^{1/2}. \tag{68}$$

Then as above we get

$$\left|\mathscr{V}\xi^j\phi\right| \le Ct^{-1/2} |\eta|^{-\nu_j} \langle \eta \rangle^{2\nu_j - 1} \left(\|\xi\phi_\xi\|_{\mathbf{L}^2} + \left\||\xi|^{1/2} \phi\right\|_{\mathbf{L}^\infty} \right) \tag{69}$$

for all $t \ge 1$, $\eta < 0$, and $j = -1, 0, 1$. Lemma 2 is proved. □

By Lemma 2, we have the estimate

$$\begin{aligned} \left|\mathscr{V}\xi^j\phi\right| &\le C|\eta|^j |\phi(\eta)| + Ct^{-1/4} |\eta|^{-\mu_j} \langle \eta \rangle^{-\nu_j} \left(\left\||\xi|^{1/2} \phi\right\|_{\mathbf{L}^\infty} + \|\xi\phi_\xi\|_{\mathbf{L}^2} \right) \\ &\le C|\eta|^{j-1/2} \left\||\xi|^{1/2} \phi\right\|_{\mathbf{L}^\infty} + Ct^{-1/4} |\eta|^{-\mu_j} \langle \eta \rangle^{-\nu_j} \left(\left\||\xi|^{1/2} \phi\right\|_{\mathbf{L}^\infty} + \|\xi\phi_\xi\|_{\mathbf{L}^2} \right). \end{aligned} \tag{70}$$

We next consider the operator $\mathscr{V}^*$.

Lemma 3. *The estimates*

$$\begin{aligned} &\|\langle \xi \rangle (\mathscr{V}^*\phi - A^*(t,\xi)\phi(\xi))\|_{\mathbf{L}^\infty(0,\infty)} \le Ct^{-1/4} \left(\|\phi\|_{\mathbf{L}^\infty} + \left\||\eta|^{-9/4} \langle \eta \rangle^4 t\mathscr{A}_0\phi\right\|_{\mathbf{L}^2} \right), \\ &\|\langle \xi \rangle \mathscr{V}^*\phi\|_{\mathbf{L}^\infty(-\infty,0)} \le Ct^{-1/2} \left(\|\phi\|_{\mathbf{L}^\infty} + \left\||\eta|^{-9/4} \langle \eta \rangle^4 t\mathscr{A}_0\phi\right\|_{\mathbf{L}^2} \right) \end{aligned} \tag{71}$$

are valid for all $t \ge 1$, *where* $\mathscr{A}_0 = (1/t\sqrt{\Lambda''})\partial_\eta(1/\sqrt{\Lambda''})$.

Proof. We find

$$\begin{aligned} \mathscr{V}^*\phi - A^*(t,\xi)\phi(\xi) &= \frac{t^{1/2}}{\sqrt{2\pi}} \cdot \int_0^\infty e^{itS(\eta,\xi)} (\phi(\eta) - \phi(\xi)) \chi(\eta\xi^{-1}) \sqrt{\Lambda''(\eta)} d\eta \\ &+ \frac{t^{1/2}}{\sqrt{2\pi}} \int_0^\infty e^{itS(\eta,\xi)} \phi(\eta) \left(1 - \chi(\eta\xi^{-1})\right) \sqrt{\Lambda''(\eta)} d\eta \\ &= I_3 + I_4 \end{aligned} \tag{72}$$

for $\xi > 0$. In the first integral I_3 using the identity

$$e^{itS(x,\xi)} = H_3 \partial_\eta \left((\eta - \xi) e^{itS(\eta,\xi)} \right) \tag{73}$$

with $H_3 = (1 + it(\eta-\xi)S_\eta(\eta,\xi))^{-1}$, $\partial_\eta S(\eta,\xi) = 2\eta^{-3}(a + b\eta^4)(\eta-\xi)$, we integrate by parts

$$\begin{aligned} I_3 = Ct^{1/2} \int_0^\infty & e^{itS(\eta,\xi)} (\phi(\eta) - \phi(\xi))(\eta - \xi) \cdot \partial_\eta \left(H_3\chi(\eta\xi^{-1}) \sqrt{\Lambda''(\eta)} \right) + e^{itS(\eta,\xi)} \phi_\eta(\eta)(\eta - \xi) \\ &\cdot H_3\chi(\eta\xi^{-1}) \sqrt{\Lambda''(\eta)} d\eta. \end{aligned} \tag{74}$$

Then using the identity

$$\phi_\eta(\eta) = \Lambda''(\eta) t\mathscr{A}_0\phi(\eta) + \frac{\Lambda'''(\eta)}{2\Lambda''(\eta)} \phi(\eta) \tag{75}$$

we get

$$\begin{aligned} I_3 = Ct^{1/2} \int_0^\infty & e^{itS(\eta,\xi)} (\phi(\eta) - \phi(\xi))(\eta - \xi) \cdot \partial_\eta \left(H_3\chi(\eta\xi^{-1}) \sqrt{\Lambda''(\eta)} \right) + e^{itS(\eta,\xi)} (\eta - \xi) \\ &\cdot H_3\chi(\eta\xi^{-1}) (\Lambda''(\eta))^{3/2} t\mathscr{A}_0\phi(\eta) + e^{itS(\eta,\xi)} \\ &\cdot \frac{\Lambda'''(\eta)}{\sqrt{\Lambda''(\eta)}} \phi(\eta)(\eta - \xi) H_3\chi(\eta\xi^{-1}) d\eta. \end{aligned} \tag{76}$$

Applying the estimates

$$\begin{aligned} |\phi(\eta) - \phi(\xi)| &= \left| \int_\eta^\xi \partial_\eta\phi d\eta \right| \\ &= \left| \int_\eta^\xi \Lambda''(\eta) t\mathscr{A}_0\phi(\eta) d\eta + \int_\eta^\xi \frac{\Lambda'''(\eta)}{2\Lambda''(\eta)} \phi(\eta) d\eta \right| \\ &\le C|\eta - \xi|^{1/2} |\xi|^{9/4} \langle \xi \rangle^{-3} \Lambda''(\xi) \cdot \left\| |\eta|^{-9/4} \langle \eta \rangle^3 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} + C|\xi|^{-3/2} \langle \xi \rangle^2 |\eta - \xi| \\ &\cdot \left\| |\eta|^{1/2} \langle \eta \rangle^{-2} \phi \right\|_{\mathbf{L}^\infty}, \\ |H_3| &\le C \left(1 + t\xi^{-3} \langle \xi \rangle^4 (\xi - \eta)^2 \right)^{-1}, \\ &\left| (\eta - \xi) H_3\chi(\eta\xi^{-1}) (\Lambda''(\eta))^{3/2} \right| \le \frac{C\xi^{-9/2} \langle \xi \rangle^6 |\eta - \xi|}{1 + t\xi^{-3} \langle \xi \rangle^4 (\xi - \eta)^2} \end{aligned} \tag{77}$$

and $\Lambda''(\xi) = O(\xi^{-3}\langle \xi \rangle^4)$, $\Lambda'''(\xi) = O(\xi^{-4}\langle \xi \rangle^4)$, and

$$\left| (\eta - \xi) \partial_\eta \left(H_3\chi(\eta\xi^{-1}) \sqrt{\Lambda''(\eta)} \right) \right| \le \frac{C\xi^{-3/2} \langle \xi \rangle^2}{1 + t\xi^{-3} \langle \xi \rangle^4 (\xi - \eta)^2} \tag{78}$$

for $(1/3)\xi \le \eta \le 3\xi$, we find

$$\begin{aligned} |I_3| &\le Ct^{1/2} |\xi|^{-9/4} \langle\xi\rangle^3 \left\| |\eta|^{-9/4} \langle\eta\rangle^3 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} \\ &\cdot \int_{(1/3)\xi}^{3\xi} \frac{|\eta-\xi|^{1/2} d\eta}{1+t\xi^{-3}\langle\xi\rangle^4(\xi-\eta)^2} + Ct^{1/2}\xi^{-3}\langle\xi\rangle^4 \\ &\cdot \left\| |\eta|^{1/2} \langle\eta\rangle^{-2} \phi \right\|_{\mathbf{L}^\infty} \\ &\cdot \int_{(1/3)\xi}^{3\xi} \frac{|\eta-\xi| d\eta}{1+t\xi^{-3}\langle\xi\rangle^4(\xi-\eta)^2} + Ct^{1/2} |\xi|^{-9/4} \\ &\cdot \langle\xi\rangle^3 \left\| |\eta|^{-9/4} \langle\eta\rangle^3 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} \\ &\cdot \left(\int_{(1/3)\xi}^{3\xi} \frac{(\eta-\xi)^2 d\eta}{\left(1+t\xi^{-3}\langle\xi\rangle^4(\xi-\eta)^2\right)^2} \right)^{1/2}. \end{aligned} \tag{79}$$

Hence

$$\begin{aligned} |I_3| &\le Ct^{1/2} |\xi|^{-9/4} \langle\xi\rangle^2 \left\langle t\xi^{-3}\langle\xi\rangle^4 \right\rangle^{-3/4} \\ &\cdot \left\| |\eta|^{-9/4} \langle\eta\rangle^4 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} + Ct^{1/2}\xi^{-5/2}\langle\xi\rangle^3 \\ &\cdot \left\langle t\xi^{-3}\langle\xi\rangle^4 \right\rangle^{\gamma-1} \|\phi\|_{\mathbf{L}^\infty} \le Ct^{-1/4}\langle\xi\rangle^{-1} \\ &\cdot \left\| |\eta|^{-9/4} \langle\eta\rangle^3 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} + Ct^{\gamma-1/2}\langle\xi\rangle^{-1}\|\phi\|_{\mathbf{L}^\infty}. \end{aligned} \tag{80}$$

In the second integral I_4, using the identity $e^{itS(\eta,\xi)} = H_4\partial_\eta(\eta e^{itS(\eta,\xi)})$ with $H_4 = (1+it\eta S_\eta(\eta,\xi))^{-1}$ we integrate by parts

$$\begin{aligned} I_4 &= Ct^{1/2} \int_0^\infty e^{itS(\eta,\xi)}\phi(\eta) \\ &\cdot \eta\partial_\eta \left(H_4 \left(1-\chi\left(\eta\xi^{-1}\right)\right) \sqrt{\Lambda''(\eta)} \right) \\ &+ e^{itS(\eta,\xi)}\phi_\eta(\eta)\,\eta H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right)\sqrt{\Lambda''(\eta)}d\eta. \end{aligned} \tag{81}$$

Then using

$$\phi_\eta(\eta) = \Lambda''(\eta)\, t\mathscr{A}_0\phi(\eta) + \frac{\Lambda'''(\eta)}{2\Lambda''(\eta)}\phi(\eta) \tag{82}$$

we get

$$\begin{aligned} |I_4| &\le Ct^{1/2} \int_0^\infty |\phi(\eta)| \\ &\cdot \left| \eta\partial_\eta \left(H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right)\sqrt{\Lambda''(\eta)} \right) \right| \\ &+ \left| \frac{\Lambda'''(\eta)}{\sqrt{\Lambda''(\eta)}} \eta H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right) \right| |\phi(\eta)| \\ &+ \left| \left(\Lambda''(\eta)\right)^{3/2} \eta H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right) \right| \\ &\cdot |t\mathscr{A}_0\phi(\eta)|\, d\eta. \end{aligned} \tag{83}$$

Then using the estimates

$$\begin{aligned} &\left| \eta\partial_\eta \left(H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right)\sqrt{\Lambda''(\eta)} \right) \right| \\ &\quad + \left| \frac{\Lambda'''(\eta)}{\sqrt{\Lambda''(\eta)}} \eta H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right) \right| \\ &\quad \le \frac{C\eta^{-3/2}\langle\eta\rangle^2}{1+t\eta^{-2}\langle\eta\rangle^4(\xi+\eta)}, \\ &\left| \left(\Lambda''(\eta)\right)^{3/2} \eta H_4\left(1-\chi\left(\eta\xi^{-1}\right)\right) \right| \\ &\quad \le \frac{C\eta^{-7/2}\langle\eta\rangle^6}{1+t\eta^{-2}\langle\eta\rangle^4(\xi+\eta)} \end{aligned} \tag{84}$$

in the domain $\eta \ge 3\xi > 0$ or $0 < \eta < (1/3)\xi$, we get

$$\begin{aligned} |I_4| &\le Ct^{1/2}\|\phi\|_{\mathbf{L}^\infty} \int_0^\infty \frac{\eta^{-3/2}\langle\eta\rangle^2 d\eta}{1+t\eta^{-2}\langle\eta\rangle^4(\xi+\eta)} \\ &+ Ct^{1/2} \left\| |\eta|^{-9/4}\langle\eta\rangle^4 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} \\ &\cdot \left(\int_0^\infty \frac{\eta^{-5/2}\langle\eta\rangle^5 d\eta}{\left(1+t\eta^{-2}\langle\eta\rangle^4(\xi+\eta)\right)^2} \right)^{1/2}. \end{aligned} \tag{85}$$

Therefore

$$\begin{aligned} |I_4| &\le Ct^{-1/2}\|\phi\|_{\mathbf{L}^\infty} \int_0^\infty \frac{\eta^{1/2}\langle\eta\rangle^{-2} d\eta}{\xi+\eta} \\ &+ Ct^{-1/2} \left\| |\eta|^{-9/4}\langle\eta\rangle^4 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2} \\ &\cdot \left(\int_0^\infty \frac{\eta^{3/2}\langle\eta\rangle^{-3} d\eta}{(\xi+\eta)^2} \right)^{1/2} \le Ct^{-1/2}\langle\xi\rangle^{-1}\|\phi\|_{\mathbf{L}^\infty} \\ &+ Ct^{-1/2}\langle\xi\rangle^{-1} \left\| |\eta|^{-9/4}\langle\eta\rangle^4 t\mathscr{A}_0\phi \right\|_{\mathbf{L}^2}. \end{aligned} \tag{86}$$

Next we consider $\xi < 0$. Using the identity $e^{itS(\eta,\xi)} = H_4\partial_\eta(\eta e^{itS(\eta,\xi)})$ with $H_4 = (1+it\eta S_\eta(\eta,\xi))^{-1}$ we integrate by parts

$$\begin{aligned} \mathscr{V}^*\phi &= Ct^{1/2} \int_0^\infty e^{itS(\eta,\xi)}\phi(\eta)\,\eta\partial_\eta\left(H_4\sqrt{\Lambda''(\eta)}\right) \\ &+ e^{itS(\eta,\xi)}\phi_\eta(\eta)\,\eta H_4\sqrt{\Lambda''(\eta)}d\eta. \end{aligned} \tag{87}$$

Then using formula (82), we get

$$\begin{aligned} |\mathscr{V}^*\phi| &\le Ct^{1/2} \int_0^\infty |\phi(\eta)| \left| \eta\partial_\eta\left(H_4\sqrt{\Lambda''(\eta)}\right) \right| \\ &+ \left| \frac{\Lambda'''(\eta)}{\sqrt{\Lambda''(\eta)}} \eta H_4 \right| |\phi(\eta)| \\ &+ \left| \left(\Lambda''(\eta)\right)^{3/2} \eta H_4 \right| |t\mathscr{A}_0\phi(\eta)|\, d\eta. \end{aligned} \tag{88}$$

Then using the estimates

$$\left|\eta\partial_\eta\left(H_4\sqrt{\Lambda''(\eta)}\right)\right|+\left|\frac{\Lambda'''(\eta)}{\sqrt{\Lambda''(\eta)}}\eta H_4\right| \le \frac{C\eta^{-3/2}\langle\eta\rangle^2}{1+t\eta^{-2}\langle\eta\rangle^4(|\xi|+\eta)}, \quad (89)$$
$$\left|\left(\Lambda''(\eta)\right)^{3/2}\eta H_4\right|\le\frac{C\eta^{-7/2}\langle\eta\rangle^6}{1+t\eta^{-2}\langle\eta\rangle^4(|\xi|+\eta)}$$

in the domain $\xi<0$ and $\eta>0$, we get

$$|\mathcal{V}^*\phi|\le Ct^{1/2}\|\phi\|_{\mathbf{L}^\infty}\int_0^\infty\frac{\eta^{-3/2}\langle\eta\rangle^2\,d\eta}{1+t\eta^{-2}\langle\eta\rangle^4(|\xi|+\eta)} + Ct^{1/2}\left\||\eta|^{-9/4}\langle\eta\rangle^4 t\mathcal{A}_0\phi\right\|_{\mathbf{L}^2} \cdot\left(\int_0^\infty\frac{\eta^{-5/2}\langle\eta\rangle^6\,d\eta}{\left(1+t\eta^{-2}\langle\eta\rangle^4(|\xi|+\eta)\right)^2}\right)^{1/2}. \quad (90)$$

Therefore

$$|\mathcal{V}^*\phi|\le Ct^{-1/2}\|\phi\|_{\mathbf{L}^\infty}\int_0^\infty\frac{\eta^{1/2}\langle\eta\rangle^{-2}\,d\eta}{|\xi|+\eta} + Ct^{-1/2}\left\||\eta|^{-9/4}\langle\eta\rangle^4 t\mathcal{A}_0\phi\right\|_{\mathbf{L}^2} \cdot\left(\int_0^\infty\frac{\eta^{3/2}\langle\eta\rangle^{-3}\,d\eta}{(|\xi|+\eta)^2}\right)^{1/2}\le Ct^{-1/2}\langle\xi\rangle^{-1}\|\phi\|_{\mathbf{L}^\infty} + Ct^{-1/2}\langle\xi\rangle^{-1}\left\||\eta|^{-9/4}\langle\eta\rangle^4 t\mathcal{A}_0\phi\right\|_{\mathbf{L}^2}. \quad (91)$$

Lemma 3 is proved. □

3.3. Estimates for Derivatives. Denote $\mathcal{A}(t)=\overline{M}(1/t\sqrt{\Lambda''})\partial_\eta(1/\sqrt{\Lambda''})M=\mathcal{A}_0(t)+i\eta$, $\mathcal{A}_0(t)=(1/t\sqrt{\Lambda''})\partial_\eta(1/\sqrt{\Lambda''})$ such that

$$i\xi\mathcal{V}^*(t)\phi=\frac{|t|^{1/2}}{\sqrt{2\pi}}e^{it\Lambda(\xi)}\int_0^\infty i\xi e^{-it\Lambda'(\eta)\xi}M(t,\eta)\phi(\eta) \cdot\sqrt{\Lambda''(\eta)}d\eta=\frac{|t|^{1/2}}{\sqrt{2\pi}} \cdot e^{it\Lambda(\xi)}\int_0^\infty e^{-it\Lambda'(\eta)\xi}\partial_\eta\left(\frac{1}{t\sqrt{\Lambda''(\eta)}}M(t,\eta) \cdot\phi(\eta)\right)d\eta=\mathcal{V}^*(t)\mathcal{A}(t)\phi. \quad (92)$$

Since $\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^\infty}\le C|t|^{1/2}\|\sqrt{\Lambda''}\phi\|_{\mathbf{L}^1(0,\infty)}$ and

$$\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^2}=\|\mathcal{F}\mathcal{U}(-t)\mathcal{D}_t\mathcal{B}M\phi\|_{\mathbf{L}^2} =\|\mathcal{D}_t\mathcal{B}M\phi\|_{\mathbf{L}^2}=\|\mathcal{B}M\phi\|_{\mathbf{L}^2}=\|\phi\|_{\mathbf{L}^2}, \quad (93)$$

then by the Riesz interpolation theorem (see [23], p. 52) we have

$$\|\mathcal{V}^*(t)\phi\|_{\mathbf{L}^p}\le C|t|^{1/2-1/p}\left\|\left(\Lambda''\right)^{1/2-1/p}\phi\right\|_{\mathbf{L}^{p/(p-1)}} \quad (94)$$

for $2\le p\le\infty$. We now estimate the derivative $\partial_\eta\mathcal{V}\phi$.

Lemma 4. *The estimate*

$$\left\|\eta^{\gamma+j}\langle\eta\rangle^{-2\gamma}t\mathcal{A}_0\mathcal{V}\xi^j\phi\right\|_{\mathbf{L}^2}\le C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}+\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty} \quad (95)$$

is true for all $t\ge1$, $j=-1,0$, where $\gamma>0$.

Proof. Since $\mathcal{A}_0=\mathcal{A}-i\eta$ and $\mathcal{A}\mathcal{V}=\mathcal{V}i\xi$, we have

$$t\mathcal{A}_0\mathcal{V}\xi^j\phi=it\left(\mathcal{V}\xi^{j+1}\phi-\eta\mathcal{V}\xi^j\phi\right) =\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}it(\xi-\eta)\xi^j\phi(\xi)\,d\xi \quad (96)$$

for $\eta>0$. So we need to estimate

$$\eta^{\gamma+j}\langle\eta\rangle^{-2\gamma}t\mathcal{A}_0\mathcal{V}\xi^j\phi=\eta^\gamma\langle\eta\rangle^{-2\gamma}\mathcal{V}\psi_1(\xi,\xi_1)\xi\phi_\xi +\eta^\gamma\langle\eta\rangle^{-2\gamma} \cdot\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\left(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1)\right) \cdot\xi\phi_\xi(\xi)\,d\xi+\eta^\gamma\langle\eta\rangle^{-2\gamma} \cdot\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\phi(\xi)\psi_2(\xi,\eta)\,d\xi=I_1 +I_2+I_3, \quad (97)$$

where $\xi_1=(a/b)^{1/4}$, and

$$\psi_1(\xi,\eta)=\frac{\eta^{2+j}\xi^{j+1}}{(a+b\eta^2\xi^2)(\xi+\eta)}, \quad \psi_2(\xi,\eta)=\partial_\xi\frac{\eta^{2+j}\xi^{j+2}}{(a+b\eta^2\xi^2)(\xi+\eta)}. \quad (98)$$

For the first summand using $\|\mathcal{V}\phi\|_{\mathbf{L}^2}\le\|\phi\|_{\mathbf{L}^2}$ we have for $j=-1,0$

$$\|I_1\|_{\mathbf{L}^2}=\left\|\mathcal{V}\psi_1(\xi,\xi_1)\xi\phi_\xi\right\|_{\mathbf{L}^2}\le\left\|\langle\xi\rangle^{-3}\xi^{j+2}\phi_\xi\right\|_{\mathbf{L}^2} \le\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}. \quad (99)$$

Consider the second summand

$$\begin{aligned}\|I_2\|^2_{\mathbf{L}^2(0,\infty)} &= Ct\int_0^\infty d\eta\,\Lambda''(\eta)\,\eta^{2\gamma}\langle\eta\rangle^{-4\gamma}\\ &\cdot\int_0^\infty e^{-itS(\eta,\xi)}\left(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1)\right)\xi\phi_\xi(\xi)\,d\xi\\ &\cdot\int_0^\infty e^{itS(\eta,\zeta)}\left(\psi_1(\zeta,\eta)-\psi_1(\zeta,\xi_1)\right)\zeta\overline{\phi_\zeta(\zeta)}d\zeta\\ &= C\int_0^\infty d\xi\,e^{-it\Lambda(\xi)}\xi\phi_\xi(\xi)\int_0^\infty d\zeta\\ &\cdot e^{it\Lambda(\zeta)}\overline{\zeta\phi_\zeta(\zeta)}K(t,\xi,\zeta),\end{aligned}\tag{100}$$

where

$$\begin{aligned}K(t,\xi,\zeta) &= t\int_0^\infty d\eta\,\Lambda''(\eta)\,\eta^{2\gamma}\langle\eta\rangle^{-4\gamma}\\ &\cdot e^{it\Lambda'(\eta)(\xi-\zeta)}\left(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1)\right)\\ &\cdot\left(\psi_1(\zeta,\eta)-\psi_1(\zeta,\xi_1)\right).\end{aligned}\tag{101}$$

Changing $x=\Lambda'(\eta)$ we get

$$\begin{aligned}K(t,\xi,\zeta) &= t\int_{-\infty}^\infty dx e^{itx(\xi-\zeta)}\eta^{2\gamma}\langle\eta\rangle^{-4\gamma}\\ &\cdot\left(\psi_1(\xi,\eta(x))-\psi_1(\xi,\xi_1)\right)\\ &\cdot\left(\psi_1(\zeta,\eta(x))-\psi_1(\zeta,\xi_1)\right).\end{aligned}\tag{102}$$

We can rotate the contour of integration $x=re^{i(\pi/8)\operatorname{sgn}(\xi-\zeta)}$, since we see that $\eta(x)=Cx^{1/2}$ for $x\to+\infty$, $\eta(x)=\eta(0)=(a/b)^{1/4}$ for $x\to 0$, and $\eta(x)=C|x|^{-1/2}$ for $x\to-\infty$, and hence

$$\begin{aligned}&\left|K(t,\xi,\zeta)\right|\\ &\le Ct\int_{-\infty}^\infty e^{-Ct|\xi-\zeta||r|}\left|\psi_1\left(\xi,\eta\left(re^{i(\pi/8)\operatorname{sgn}(\xi-\zeta)}\right)\right)\right.\\ &\left.-\psi_1(\xi,\eta(0))\right|\left|\psi_1\left(\zeta,\eta\left(re^{i(\pi/8)\operatorname{sgn}(\xi-\zeta)}\right)\right)\right.\\ &\left.-\psi_1(\zeta,\eta(0))\right|\eta^{2\gamma}\langle\eta\rangle^{-4\gamma}dr\\ &\le Ct\int_1^\infty e^{-Ct|\xi-\zeta||r|}\langle r\rangle^{-2\gamma}dr\\ &+Ct\int_{-1}^1 e^{-Ct|\xi-\zeta||r|}|r|\,dr\\ &+Ct\int_{-\infty}^{-1}e^{-Ct|\xi-\zeta||r|}|r|^{-\gamma}dr\le Ct\left(\left|\xi-\zeta\right|t\right)^{\gamma-1}\\ &\cdot\left\langle(\xi-\zeta)t\right\rangle^{-2\gamma}.\end{aligned}\tag{103}$$

Then by the Young inequality we obtain

$$\begin{aligned}\|I_2\|^2_{\mathbf{L}^2(0,\infty)} &\le Ct\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\\ &\cdot\left\|\int_{\mathbf{R}}\left(\left|\xi-\zeta\right|t\right)^{\gamma-1}\left\langle(\xi-\zeta)t\right\rangle^{-2\gamma}\left|\zeta\phi_\zeta(\zeta)\right|d\zeta\right\|_{\mathbf{L}^2}\\ &\le Ct\left\|\xi\phi_\xi\right\|^2_{\mathbf{L}^2}\left\|(\xi t)^{\gamma-1}\langle\xi t\rangle^{-2\gamma}\right\|_{\mathbf{L}^1}\le C\left\|\xi\phi_\xi\right\|^2_{\mathbf{L}^2}.\end{aligned}\tag{104}$$

To estimate I_3 we integrate by parts via identity (47)

$$\begin{aligned}I_3 &= Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\\ &\cdot\int_0^\infty e^{-itS(\eta,\xi)}(\xi-\eta)H_1\psi_2(\xi,\eta)\phi_\xi(\xi)\,d\xi\\ &+Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\\ &\cdot\int_0^\infty e^{-itS(\eta,\xi)}\phi(\xi)(\xi-\eta)\partial_\xi\left(H_1\psi_2(\xi,\eta)\right)d\xi.\end{aligned}\tag{105}$$

Using the estimates

$$\begin{aligned}&\left|(\xi-\eta)H_1\psi_2(\xi,\eta)\right|\\ &\le\frac{C\left|\xi-\eta\right|\eta^{2+j}\xi^{j+1}}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(1+\xi^2\eta^2)(\xi+\eta)},\\ &\left|(\xi-\eta)\partial_\xi\left(H_1\psi_2(\xi,\eta)\right)\right|\\ &\le\frac{C\eta^{2+j}\left(\left|\xi-\eta\right|\xi^j+\xi^{j+1}\right)}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(1+\xi^2\eta^2)(\xi+\eta)}\end{aligned}\tag{106}$$

we obtain

$$\begin{aligned}\left|I_3\right| &\le Ct^{1/2}\int_0^\infty\frac{\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\left|\xi-\eta\right|\eta^{2+j}\xi^j\left|\xi\phi_\xi(\xi)\right|d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(1+\xi^2\eta^2)(\xi+\eta)}\\ &+Ct^{1/2}\int_0^\infty\frac{\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\eta^{2+j}\left(\left|\xi-\eta\right|\xi^j+\xi^{j+1}\right)\left|\phi(\xi)\right|d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(1+\xi^2\eta^2)(\xi+\eta)}\end{aligned}$$

$$\leq Ct^{1/2}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\left(\int_0^\infty \frac{\eta^{2\gamma+2j+1}\langle\eta\rangle^{4-4\gamma}\left|\xi-\eta\right|^2\xi^{2j}d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}\left(1+\xi^2\eta^2\right)(\xi+\eta)\right)^2\left(1+\xi^2\eta^2\right)^2(\xi+\eta)^2}\right)^{1/2}$$
$$+Ct^{1/2}\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\int_0^\infty \frac{\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\eta^{2+j}\left(|\xi-\eta|\xi^j+\xi^{j+1}\right)\xi^{-1/2}d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}\left(1+\xi^2\eta^2\right)(\xi+\eta)\right)\left(1+\xi^2\eta^2\right)(\xi+\eta)}. \quad (107)$$

Since changing $\xi=\eta y$

$$\int_0^\infty \frac{t\eta^{2\gamma+2j+1}\langle\eta\rangle^{4-4\gamma}\left|\xi-\eta\right|^2\xi^{2j}d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}\left(1+\xi^2\eta^2\right)(\xi+\eta)\right)^2\left(1+\xi^2\eta^2\right)^2(\xi+\eta)^2}$$
$$=\int_0^\infty \frac{t\eta^{2\gamma+4j+2}\langle\eta\rangle^{4-4\gamma}\left|y-1\right|^2y^{2j}dy}{\left(1+t(y-1)^2y^{-2}\eta^{-1}\left(1+y^2\eta^4\right)(y+1)\right)^2\left(1+y^2\eta^4\right)^2(y+1)^2}\leq Ct^{-1}\eta^{2\gamma+4j+4}\langle\eta\rangle^{4-4\gamma}\int_0^{1/2}\frac{y^{2j+4}dy}{\left(1+y\eta^2\right)^8}$$
$$+Ct\eta^{2\gamma+4j+2}\langle\eta\rangle^{-4-4\gamma}\int_{1/2}^{3/2}\frac{\left|y-1\right|^2dy}{\left(1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2\right)^2}+Ct^{-1}\eta^{2\gamma+4j+4}\langle\eta\rangle^{4-4\gamma}\int_{3/2}^\infty\frac{y^{2j-2}dy}{\left(1+y\eta^2\right)^8}\leq C\eta^{2\gamma-1}\langle\eta\rangle^{-4\gamma}, \quad (108)$$

$$t^{1/2}\int_0^\infty \frac{\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\eta^{2+j}\left(|\xi-\eta|\xi^j+\xi^{j+1}\right)\xi^{-1/2}d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}\left(1+\xi^2\eta^2\right)(\xi+\eta)\right)\left(1+\xi^2\eta^2\right)(\xi+\eta)}$$
$$=t^{1/2}\int_0^\infty \frac{\eta^{2j+1+\gamma}\langle\eta\rangle^{2-2\gamma}\left(|y-1|+y\right)y^{j-1/2}dy}{\left(1+t(y-1)^2y^{-2}\eta^{-1}\left(1+y^2\eta^4\right)(y+1)\right)\left(1+y^2\eta^4\right)(y+1)}\leq Ct^{-1/2}\eta^{2j+2+\gamma}\langle\eta\rangle^{2-2\gamma}\int_0^{1/2}\frac{y^{j+3/2}dy}{\left(1+y\eta^2\right)^4}$$
$$+Ct^{1/2}\eta^{2j+1+\gamma}\langle\eta\rangle^{-2-2\gamma}\int_{1/2}^{3/2}\frac{dy}{1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2}+Ct^{-1/2}\eta^{2j+2+\gamma}\langle\eta\rangle^{2-2\gamma}\int_{3/2}^\infty\frac{y^{j-3/2}dy}{\left(1+y\eta^2\right)^4}\leq C\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma},$$

we get

$$\left\|I_3\right\|_{\mathbf{L}^2(0,\infty)}\leq C\left(\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}+\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\right)\left\|\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma}\right\|_{\mathbf{L}^2(0,\infty)} \quad (109)$$
$$\leq C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}+C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}.$$

In the case of $\eta<0$, the same estimate is obtained easier than the case of the positive line. Lemma 4 is proved. □

We need estimate of $\mathscr{V}_t$.

Lemma 5. *The estimate*

$$\left\|\eta^\gamma\langle\eta\rangle^{-2\gamma-2}t\partial_t\mathscr{V}\phi\right\|_{\mathbf{L}^2}\leq C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}+C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty} \quad (110)$$

is true for all $t\geq 1$, where $\gamma>0$.

Proof. Since

$$\frac{S(\eta,\xi)}{\partial_\xi S(\eta,\xi)}=\frac{\left(3a+2b\eta^3\xi+b\eta^2\xi^2\right)(\xi-\eta)\xi}{3\left(a+b\eta^2\xi^2\right)(\xi+\eta)}, \quad (111)$$

integrating by parts we get with $\xi_1=(a/b)^{1/4}$

$$\eta^\gamma\langle\eta\rangle^{-2\gamma-2}t\mathscr{V}_t\phi=\eta^\gamma\langle\eta\rangle^{-2\gamma-2}\frac{1}{2}\mathscr{V}\phi$$
$$-it\eta^\gamma\langle\eta\rangle^{-2\gamma-2}\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}S(\eta,\xi)$$
$$\cdot\phi(\xi)\,d\xi=\frac{1}{2}\eta^\gamma\langle\eta\rangle^{-2\gamma-2}\mathscr{V}\phi-\eta^\gamma\langle\eta\rangle^{-2\gamma-1}$$
$$\cdot\mathscr{V}\left(\psi_1(\xi,\xi_1)\xi\phi_\xi\right)-\eta^\gamma\langle\eta\rangle^{-2\gamma-2}$$
$$\cdot\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\left(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1)\right)$$
$$\cdot\xi\phi_\xi(\xi)\,d\xi-\eta^\gamma\langle\eta\rangle^{-2\gamma-2}$$
$$\cdot\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^\infty e^{-itS(\eta,\xi)}\psi_2(\xi,\eta)\phi(\xi)\,d\xi$$

$$-\eta^{\gamma}\langle\eta\rangle^{-2\gamma-2}\sqrt{\frac{t\Lambda''(\eta)}{2\pi}}\int_0^{\infty}e^{-itS(\eta,\xi)}\psi_3(\xi,\eta)\cdot\phi(\xi)\,d\xi = I_1+I_2+I_3+I_4+I_5, \quad (112)$$

where

$$\psi_1(\xi,\eta)=\frac{(3a+2b\eta^3\xi+b\eta^2\xi^2)(\xi-\eta)}{3(a+b\eta^2\xi^2)(\xi+\eta)},$$
$$\psi_2(\xi,\eta)=(\xi-\eta)\,\partial_{\xi}\frac{(3a+2b\eta^3\xi+b\eta^2\xi^2)\xi}{3(a+b\eta^2\xi^2)(\xi+\eta)} \quad (113)$$
$$\psi_3(\xi,\eta)=\frac{(3a+2b\eta^3\xi+b\eta^2\xi^2)\xi}{3(a+b\eta^2\xi^2)(\xi+\eta)}.$$

Using the estimate $\|\mathscr{V}\phi\|_{\mathbf{L}^2}\le\|\phi\|_{\mathbf{L}^2}$ we find for the first summand

$$\|I_1\|_{\mathbf{L}^2}\le\|\mathscr{V}\phi\|_{\mathbf{L}^2}\le\|\phi\|_{\mathbf{L}^2} \quad (114)$$

and for the second summand

$$\|I_2\|_{\mathbf{L}^2(0,\infty)}\le\left\|\eta^{\gamma}\langle\eta\rangle^{-2\gamma-2}\mathscr{V}(\psi_1(\xi,\xi_1)\xi\phi_{\xi})\right\|_{\mathbf{L}^2(0,\infty)} \le\left\|\psi_1(\xi,\xi_1)\xi\phi_{\xi}\right\|_{\mathbf{L}^2(0,\infty)}\le C\left\|\xi\phi_{\xi}\right\|_{\mathbf{L}^2}. \quad (115)$$

Consider the third summand

$$\left\|\eta^{\gamma}\langle\eta\rangle^{-2\gamma-2}I_3\right\|^2_{\mathbf{L}^2(0,\infty)}=Ct\int_0^{\infty}d\eta\Lambda''(\eta)\cdot\eta^{2\gamma}\langle\eta\rangle^{-4\gamma-4}\cdot\int_0^{\infty}e^{-itS(\eta,\xi)}(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1))\xi\phi_{\xi}(\xi)\,d\xi\cdot\int_0^{\infty}e^{itS(\eta,\zeta)}(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1))\zeta\overline{\phi_{\zeta}(\zeta)}d\zeta = C\int_0^{\infty}d\xi\,e^{-it\Lambda(\xi)}\xi\phi_{\xi}(\xi)\int_0^{\infty}d\zeta\cdot e^{it\Lambda(\zeta)}\overline{\zeta\phi_{\zeta}(\zeta)}K_1(t,\xi,\zeta), \quad (116)$$

where

$$K_1(t,\xi,\zeta)=t\int_0^{\infty}d\eta\,\Lambda''(\eta)\eta^{2\gamma}\langle\eta\rangle^{-4\gamma-4}\cdot e^{it\Lambda'(\eta)(\xi-\zeta)}(\psi_1(\xi,\eta)-\psi_1(\xi,\xi_1))\cdot(\psi_1(\zeta,\eta)-\psi_1(\zeta,\xi_1)). \quad (117)$$

Changing $x=\Lambda'(\eta)$, $\eta(x)=\sqrt{(1/2b)(x+\sqrt{4ab+x^2})}$, we get

$$K_1(t,\xi,\zeta)=t\int_{-\infty}^{\infty}dx\,\eta^{2\gamma}(x)\langle\eta(x)\rangle^{-4\gamma-4}\cdot e^{itx(\xi-\zeta)}(\psi_1(\xi,\eta(x))-\psi_1(\xi,\eta(0)))\cdot(\psi_1(\zeta,\eta(x))-\psi_1(\zeta,\eta(0))). \quad (118)$$

We can rotate the contour of integration $x=re^{i(\pi/8)\mathrm{sgn}(\xi-\zeta)}$, since we see that $\eta(x)=Cx^{1/2}$ for $x\to+\infty$, $\eta(x)=\eta(0)=\xi_1$ for $x\to 0$, and $\eta(x)=C|x|^{-1/2}$ for $x\to-\infty$, and hence

$$\left|K_1(t,\xi,\zeta)\right| \le Ct\int_{-\infty}^{\infty}e^{-Ct|\xi-\zeta||r|}\psi_1\left(\psi_1\left(\xi,\eta\left(re^{i(\pi/8)\mathrm{sgn}(\xi-\zeta)}\right)\right)-\psi_1(\xi,\eta(0))\right)\left(\psi_1\left(\zeta,\eta\left(re^{i(\pi/8)\mathrm{sgn}(\xi-\zeta)}\right)\right)-\psi_1(\zeta,\eta(0))\right)\eta^{2\gamma}\langle\eta\rangle^{-4\gamma-4}dr \le Ct\int_1^{\infty}e^{-Ct|\xi-\zeta||r|}\langle r\rangle^{-\gamma}dr + Ct\int_{-1}^{1}e^{-Ct|\xi-\zeta||r|}|r|\,dr + Ct\int_{-\infty}^{-1}e^{-Ct|\xi-\zeta||r|}|r|^{-2\gamma}dr\le Ct(|\xi-\zeta|t)^{\gamma-1}\cdot\langle(\xi-\zeta)t\rangle^{-2\gamma}. \quad (119)$$

Then by the Young inequality we obtain

$$\|I_3\|^2_{\mathbf{L}^2(0,\infty)}\le Ct\left\|\xi\phi_{\xi}\right\|_{\mathbf{L}^2}\cdot\left\|\int_{\mathbf{R}}(|\xi-\zeta|t)^{\gamma-1}\langle(\xi-\zeta)t\rangle^{-2\gamma}\left|\zeta\phi_{\zeta}(\zeta)\right|d\zeta\right\|_{\mathbf{L}^2} \le Ct\left\|\xi\phi_{\xi}\right\|^2_{\mathbf{L}^2}\left\|(\xi t)^{\gamma-1}\langle\xi t\rangle^{-2\gamma}\right\|_{\mathbf{L}^1}\le C\left\|\xi\phi_{\xi}\right\|^2_{\mathbf{L}^2}. \quad (120)$$

Next we estimate I_4. We integrate by parts via identity (53) to get

$$I_4=Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{-2\gamma}\cdot\int_0^{\infty}e^{-itS(\eta,\xi)}\phi(\xi)\,\xi\partial_{\xi}(H_2\psi_2(\xi,\eta)) + e^{-itS(x,\xi)}H_2\psi_2(\xi,\eta)\,\xi\phi_{\xi}(\xi)\,d\xi. \quad (121)$$

Using the estimates $|\psi_2(\xi,\eta)|\le C(|\xi-\eta|\langle\eta\rangle^2/(\xi+\eta))$,

$$\left|H_2\psi_2(\xi,\eta)\right| \le\frac{C|\xi-\eta|\langle\eta\rangle^2}{(1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)|\xi^2-\eta^2|)(\xi+\eta)},$$
$$\left|\xi\partial_{\xi}(H_2\psi_2)\right|\le\frac{C\langle\eta\rangle^2}{1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)|\xi^2-\eta^2|} \quad (122)$$

we obtain

$$\begin{aligned}\left|I_4\right| &\le Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma} \cdot \int_0^\infty \frac{|\phi(\xi)|\,d\xi}{1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|} \\ &+ Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma} \cdot \int_0^\infty \frac{|\xi-\eta|\left|\xi\phi_\xi(\xi)\right|d\xi}{\left(1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|\right)(\xi+\eta)} \\ &\le C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\xi^{-1/2}d\xi}{1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|} \\ &+ C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2} \cdot \left(\int_0^\infty \frac{t\eta^{2\gamma-3}\langle\eta\rangle^{4-4\gamma}(\xi-\eta)^2d\xi}{\left(1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|\right)^2(\xi+\eta)^2}\right)^{1/2}.\end{aligned} \tag{123}$$

Changing $\xi=\eta y$ we find

$$\begin{aligned}&\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\xi^{-1/2}d\xi}{1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|} = \int_0^\infty \frac{t^{1/2}\eta^{\gamma-1}\langle\eta\rangle^{2-2\gamma}y^{-1/2}dy}{1+t\eta^{-1}y^{-1}(1+\eta^4y^2)(y+1)|y-1|} \\ &\le Ct^{-1/2}\eta^{\gamma}\langle\eta\rangle^{2-2\gamma}\int_0^{1/2}\frac{y^{1/2}dy}{(1+\eta^2y)^2} + Ct^{1/2}\eta^{\gamma-1}\langle\eta\rangle^{2-2\gamma}\int_{1/2}^{3/2}\frac{dy}{1+t\eta^{-1}\langle\eta\rangle^4|y-1|} \\ &+ Ct^{-1/2}\eta^{\gamma}\langle\eta\rangle^{2-2\gamma}\int_{3/2}^\infty\frac{y^{-3/2}dy}{1+\eta^4y^2} \le C\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma}, \\ &\int_0^\infty \frac{t\eta^{2\gamma-3}\langle\eta\rangle^{4-4\gamma}(\xi-\eta)^2d\xi}{\left(1+t\xi^{-1}\eta^{-2}(1+\eta^2\xi^2)\left|\xi^2-\eta^2\right|\right)^2(\xi+\eta)^2} = \int_0^\infty \frac{t\eta^{2\gamma-2}\langle\eta\rangle^{4-4\gamma}(y-1)^2dy}{\left(1+t\eta^{-1}y^{-1}(1+\eta^4y^2)(y+1)|y-1|\right)^2(y+1)^2} \\ &\le Ct^{-1}\eta^{2\gamma}\langle\eta\rangle^{4-4\gamma}\int_0^{1/2}\frac{y^2dy}{(1+\eta^2y)^4} + Ct^{-1}\eta^{2\gamma}\langle\eta\rangle^{-4-4\gamma}\int_{1/2}^{3/2}dy \\ &+ Ct^{-1}\eta^{2\gamma}\langle\eta\rangle^{4-4\gamma}\int_{3/2}^\infty\frac{dy}{y^2(1+\eta^2y)^4} \le C\eta^{2\gamma-1}\langle\eta\rangle^{-4\gamma}.\end{aligned} \tag{124}$$

Hence we get

$$\begin{aligned}&\left\|I_4\right\|_{\mathbf{L}^2(0,\infty)} \\ &\le C\left(\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}+\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\right)\left\|\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma}\right\|_{\mathbf{L}^2(0,\infty)} \\ &\le C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2(0,\infty)}+C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}.\end{aligned} \tag{125}$$

In the last integral I_5 we integrate by parts via identity (47)

$$\begin{aligned}I_5 &= Ct^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{-2\gamma} \cdot \int_0^\infty e^{-itS(\eta,\xi)}(\xi-\eta)H_1\psi_3(\xi,\eta)\phi_\xi(\xi) \\ &+ e^{-itS(\eta,\xi)}\phi(\xi)(\xi-\eta)\partial_\xi\left(H_1\psi_3(\xi,\eta)\right)d\xi.\end{aligned} \tag{126}$$

Using the estimates $|\psi_3(\xi,\eta)|\le C(\xi\langle\eta\rangle^2/(\xi+\eta))$,

$$\begin{aligned}&\left|(\xi-\eta)H_1\psi_3(\xi,\eta)\right| \le \frac{C\xi|\xi-\eta|\langle\eta\rangle^2}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(\xi+\eta)}, \\ &\left|(\xi-\eta)\partial_\xi\left(H_1\psi_3(\xi,\eta)\right)\right| \le \frac{C\langle\eta\rangle^2}{1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)}\end{aligned} \tag{127}$$

we obtain

$$\begin{aligned}\left|I_5\right| &\le C\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}|\xi-\eta|\left|\xi\phi_\xi(\xi)\right|d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)(\xi+\eta)} + C\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}|\phi(\xi)|\,d\xi}{1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)} \\ &\le C\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\left(\int_0^\infty \frac{t\eta^{2\gamma-3}\langle\eta\rangle^{4-4\gamma}(\xi-\eta)^2d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)^2(\xi+\eta)^2}\right)^{1/2} \\ &+ C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\xi^{-1/2}d\xi}{1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)}.\end{aligned} \tag{128}$$

Since changing $\xi = \eta y$

$$\int_0^\infty \frac{t\eta^{2\gamma-3}\langle\eta\rangle^{4-4\gamma}(\xi-\eta)^2\,d\xi}{\left(1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)\right)^2(\xi+\eta)^2}$$
$$= \int_0^\infty \frac{t\eta^{2\gamma-2}\langle\eta\rangle^{4-4\gamma}(y-1)^2\,dy}{\left(1+t(y-1)^2y^{-2}\eta^{-1}(1+\eta^4y^2)(y+1)\right)^2(y+1)^2}$$
$$\le Ct^{-1}\eta^{2\gamma}\langle\eta\rangle^{4-4\gamma}\int_0^{1/2}\frac{y^4dy}{(1+\eta^2y)^4}$$
$$+ Ct\eta^{2\gamma-2}\langle\eta\rangle^{4-4\gamma}\int_{1/2}^{3/2}\frac{(y-1)^2dy}{\left(1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2\right)^2}$$
$$+ Ct^{-1}\eta^{2\gamma}\langle\eta\rangle^{4-4\gamma}\int_{3/2}^\infty\frac{dy}{(1+\eta^2y)^4y^2} \le C\eta^{2\gamma-1}\langle\eta\rangle^{-4\gamma}, \tag{129}$$
$$\int_0^\infty \frac{t^{1/2}\eta^{\gamma-3/2}\langle\eta\rangle^{2-2\gamma}\xi^{-1/2}d\xi}{1+t(\xi-\eta)^2\xi^{-2}\eta^{-2}(1+\xi^2\eta^2)(\xi+\eta)}$$
$$= \int_0^\infty \frac{t^{1/2}\eta^{\gamma-1}\langle\eta\rangle^{2-2\gamma}y^{-1/2}dy}{1+t(y-1)^2y^{-2}\eta^{-1}(1+y^2\eta^4)(y+1)}$$
$$\le Ct^{-1/2}\eta^{\gamma}\langle\eta\rangle^{2-2\gamma}\int_0^{1/2}\frac{y^{3/2}dy}{(1+\eta^2y)^2}$$
$$+ Ct^{1/2}\eta^{\gamma-1}\langle\eta\rangle^{2-2\gamma}\int_{1/2}^{3/2}\frac{dy}{1+t\eta^{-1}\langle\eta\rangle^4(y-1)^2}$$
$$+ Ct^{-1/2}\eta^{\gamma}\langle\eta\rangle^{2-2\gamma}\int_{3/2}^\infty\frac{y^{-3/2}dy}{(1+\eta^2y)^2} \le C\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma},$$

we get

$$\|I_5\|_{\mathbf{L}^2(0,\infty)} \le C\left(\|\xi\phi_\xi\|_{\mathbf{L}^2} + \left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}\right)\left\|\eta^{\gamma-1/2}\langle\eta\rangle^{-2\gamma}\right\|_{\mathbf{L}^2(0,\infty)} \le C\|\xi\phi_\xi\|_{\mathbf{L}^2(0,\infty)} + C\left\||\xi|^{1/2}\phi\right\|_{\mathbf{L}^\infty}. \tag{130}$$

In the case of $\eta < 0$, the same estimate is obtained easier than the case of the positive line. Lemma 5 is proved. □

3.4. Asymptotics for the Nonlinearity. We obtain the asymptotic representation for the nonlinear term. Define the norm

$$\|\widehat{\varphi}\|_{\mathbf{Z}} = \left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + \|\xi\widehat{\varphi}_\xi\|_{\mathbf{L}^2}. \tag{131}$$

Lemma 6. *The asymptotics*

$$\mathcal{F}\mathcal{U}(-t)\partial_x\mathcal{K}u^3 = \frac{\sqrt{3}i\xi\widehat{K}}{it}e^{it\Omega(\xi)}\mathcal{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3 + \frac{3i\xi\widehat{K}}{t\Lambda''}|\widehat{\varphi}|^2\widehat{\varphi} + O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right) \tag{132}$$

is true for all $t \ge 1$, $\xi \ge 0$, where $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$.

Proof. In view of (37) we find for the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$

$$\mathcal{F}\mathcal{U}(-t)\partial_x\mathcal{K}u^3 = \sqrt{3}i\xi\widehat{K}t^{-1}e^{it\Omega(\xi)}\mathcal{D}_3\mathcal{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3 + 3i\xi\widehat{K}t^{-1}\mathcal{V}^*(t)\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0} + 3i\xi\widehat{K}t^{-1}\mathcal{D}_{-1}\mathcal{V}^*(-t)\frac{1}{\Lambda''}\psi_0\overline{\psi_0}^2 + \sqrt{3}i\xi\widehat{K}t^{-1}e^{it\Omega(\xi)}\mathcal{D}_{-3}\mathcal{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3, \tag{133}$$

where $\psi_j = \mathcal{V}(i\xi)^j\widehat{\varphi}$. By Lemma 3 we have

$$3i\xi\widehat{K}t^{-1}\mathcal{V}^*(t)\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0} = 3i\xi\widehat{K}t^{-1}A^*(t)\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0} + Ct^{-5/4}\left\||\eta|^{-9/4}\langle\eta\rangle^4t\mathcal{A}_0\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0}\right\|_{\mathbf{L}^2} + Ct^{-5/4}\left\|\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0}\right\|_{\mathbf{L}^\infty}. \tag{134}$$

By Lemma 2

$$\psi_0 = A_0(t)\widehat{\varphi} + O\left(t^{-1/4}|\eta|^{-1/4}\langle\eta\rangle^{-1}\left(\left\|\xi^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + \|\xi\widehat{\varphi}_\xi\|_{\mathbf{L}^2}\right)\right), \tag{135}$$
$$|\psi_0| \le C|\eta|^{-1/4}\langle\eta\rangle^{-1/4}\|\widehat{\varphi}\|_{\mathbf{Z}}.$$

Therefore

$$\left\|\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0}\right\|_{\mathbf{L}^\infty} \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^3\left\|\frac{1}{\Lambda''}|\eta|^{-3/4}\langle\eta\rangle^{-3/4}\right\|_{\mathbf{L}^\infty} \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^3. \tag{136}$$

Also by the Leibnitz rule

$$\mathcal{A}_0\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0} = \frac{1}{t\sqrt{\Lambda''}}\partial_\eta\left(\left(\frac{\psi_0}{\sqrt{\Lambda''}}\right)^2\overline{\frac{\psi_0}{\sqrt{\Lambda''}}}\right) = \frac{1}{\Lambda''}\psi_0^2\overline{\mathcal{A}_0\psi_0} + \frac{2}{\Lambda''}\psi_0\overline{\psi_0}\mathcal{A}_0\psi_0. \tag{137}$$

Then by Lemma 4 we get

$$\left\||\eta|^{-9/4}\langle\eta\rangle^4t\mathcal{A}_0\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0}\right\|_{\mathbf{L}^2} \le C\left\||\eta|^{3/4}|\psi_0|^2t\mathcal{A}_0\psi_0\right\|_{\mathbf{L}^2} \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^2\left\||\eta|^{1/4}\langle\eta\rangle^{-1/2}t\mathcal{A}_0\psi_0\right\|_{\mathbf{L}^2} \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^3. \tag{138}$$

Hence we obtain

$$\begin{aligned} &3i\xi\widehat{K}t^{-1}\mathcal{V}^*(t)\frac{1}{\Lambda''}\psi_0^2\overline{\psi_0} \\ &= 3i\xi\widehat{K}t^{-1}A^*(t)\frac{1}{\Lambda''}\left(A_0(t)\widehat{\varphi}\right)^2\overline{A_0(t)\widehat{\varphi}} \\ &+ O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right) \\ &= \frac{3i\xi\widehat{K}}{t\Lambda''}|\widehat{\varphi}|^2\widehat{\varphi} + O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right). \end{aligned} \tag{139}$$

In the same manner

$$\begin{aligned} &\sqrt{3}i\xi\widehat{K}t^{-1}e^{it\Omega(\xi)}\mathcal{D}_3\mathcal{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3 \\ &= \frac{\sqrt{3}i\xi\widehat{K}}{it}e^{it\Omega(\xi)}\mathcal{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3 + O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right), \\ &3i\xi\widehat{K}t^{-1}\mathcal{D}_{-1}\mathcal{V}^*(-t)\frac{1}{\Lambda''}\psi_0\overline{\psi_0}^2 = O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right), \\ &\sqrt{3}i\xi\widehat{K}t^{-1}e^{it\Omega(\xi)}\mathcal{D}_{-3}\mathcal{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3 \\ &= O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right). \end{aligned} \tag{140}$$

Hence the result of the lemma follows. □

4. A Priori Estimates

We define

$$\begin{aligned} &\mathbf{X}_T = \left\{\mathcal{U}(-t)u \in \mathbf{C}\left([0,T];\mathbf{H}^2\right); \|u\|_{\mathbf{X}_T} < \infty\right\}, \\ &\|u\|_{\mathbf{X}_T} = \sup_{t\in[0,T]}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\gamma}\|u(t)\|_{\mathbf{H}^2}\right. \\ &\left.+ \langle t\rangle^{-\gamma}\left\|\partial_x\mathcal{J}u(t)\right\|_{\mathbf{L}^2}\right), \end{aligned} \tag{141}$$

where $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t)$, $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, and $\gamma > 0$ is small. We have the local in time existence of solutions.

Theorem 7. *Let the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,1}$. Then there exists a time $T_0 > 0$ such that (1) has a unique solution u in $\mathbf{X}_{T_0}$.*

To get the desired results, we prove a priori estimates of solutions uniformly in time.

Lemma 8. *Assume $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,1}$ and the norm $\|u_0\|_{\mathbf{H}^{1,1}} + \|u_0\|_{\mathbf{H}^2} \le \varepsilon$. Then the estimate*

$$\|u\|_{\mathbf{X}_T} < C\varepsilon \tag{142}$$

is true for all $T \ge 1$.

Proof. By continuity of the norm $\|u\|_{\mathbf{X}_T}$ with respect to T, arguing by the contradiction we can find the first time $T \ge 1$ such that $\|u\|_{\mathbf{X}_T} = C\varepsilon$. Consider a priori estimates of $\|\partial_x\mathcal{J}u(t)\|_{\mathbf{L}^2} = \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}$. To avoid the derivative loss in (1) we apply the operator

$$\mathcal{P} = t\partial_t + \frac{1}{3}x\partial_x - \frac{4}{3}a\partial_a \tag{143}$$

and use the commutators $[\mathcal{P},\mathcal{L}] = -\mathcal{L}$, $[\mathcal{P},\partial_x\mathcal{K}] = \mathcal{F}^{-1}\widehat{K}_1(\xi)\mathcal{F} = \mathcal{K}_1$, where $\widehat{K}_1(\xi) = (i/3)\xi\partial_\xi(i\xi\widehat{K}(\xi)) - (4/3)i\xi a\partial_a\widehat{K}(\xi) = O(\xi)$. Also we represent $\widehat{K}(\xi) = \lambda + \widehat{K}_2(\xi)$ with $\widehat{K}_2(\xi) = O(\langle\xi\rangle^{-1})$; that is, $\mathcal{K} = \mathcal{K}_2 + \lambda$. Then we get

$$\begin{aligned} \mathcal{L}\mathcal{P}u &= (\mathcal{P}+1)\mathcal{L}u = \mathcal{P}\partial_x\mathcal{K}u^3 + \partial_x\mathcal{K}u^3 \\ &= 3\partial_x\mathcal{K}\left(u^2\mathcal{P}u\right) + \partial_x\mathcal{K}u^3 + \left[\mathcal{P},\partial_x\mathcal{K}\right]u^3 \\ &= 3\lambda\partial_x\left(u^2\mathcal{P}u\right) + 3\partial_x\mathcal{K}_2\left(u^2\mathcal{P}u\right) + \partial_x\mathcal{K}u^3 \\ &+ \mathcal{K}_1u^3. \end{aligned} \tag{144}$$

Define the high and short frequency projectors $\mathcal{Q}_j\phi = \mathcal{F}^{-1}\chi_j\widehat{\phi}$, where $\chi_1(\xi) = 1$ for $|\xi| \ge 1$ and $\chi_1(\xi) = 0$ for $|\xi| \le 1$, and also $\chi_2(\xi) = 1 - \chi_1(\xi)$. Then we get

$$\begin{aligned} \mathcal{L}\mathcal{Q}_1\mathcal{P}u &= 3\lambda\partial_x\mathcal{Q}_1\left(u^2\mathcal{Q}_1\mathcal{P}u\right) \\ &+ 3\lambda\partial_x\mathcal{Q}_1\left(u^2\mathcal{Q}_2\mathcal{P}u\right) \\ &+ 3\mathcal{Q}_1\partial_x\mathcal{K}_2\left(u^2\mathcal{P}u\right) + \mathcal{Q}_1\partial_x\mathcal{K}u^3 \\ &+ \mathcal{Q}_1\mathcal{K}_1u^3 \end{aligned} \tag{145}$$

and the integral equation

$$\begin{aligned} \mathcal{Q}_2\mathcal{P}u &= \mathcal{U}(t)\mathcal{Q}_2\mathcal{P}u_0 + \int_0^t d\tau\,\mathcal{U}(t-\tau) \\ &\cdot\left(3\lambda\partial_x\mathcal{Q}_2\left(u^2\mathcal{Q}_1\mathcal{P}u\right) + 3\lambda\partial_x\mathcal{Q}_2\left(u^2\mathcal{Q}_2\mathcal{P}u\right)\right. \\ &\left.+ 3\mathcal{Q}_2\partial_x\mathcal{K}_2\left(u^2\mathcal{P}u\right) + \mathcal{Q}_2\partial_x\mathcal{K}u^3 + \mathcal{Q}_2\mathcal{K}_1u^3\right). \end{aligned} \tag{146}$$

Hence applying the energy method to the first equation we find

$$\begin{aligned} \frac{d}{dt}\|\mathcal{Q}_1\mathcal{P}u\|_{\mathbf{L}^2}^2 &\le C\left(\|uu_x\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty}^2\right) \\ &\cdot\left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right)\|\mathcal{Q}_1\mathcal{P}u\|_{\mathbf{L}^2} \end{aligned} \tag{147}$$

and by the integral equation

$$\begin{aligned} &\|\mathcal{Q}_2\mathcal{P}u\|_{\mathbf{L}^2} \\ &\le \|\mathcal{Q}_2\mathcal{P}u_0\|_{\mathbf{L}^2} \\ &+ C\int_0^t\left(\|uu_x\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty}^2\right)\left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right)d\tau. \end{aligned} \tag{148}$$

Applying the estimate of Lemma 2 we have

$$\begin{aligned} &\left|\mathcal{V}\xi^j\phi\right| \\ &\le C\langle\eta\rangle^{1/2}|\eta|^{-1/4}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right). \end{aligned} \tag{149}$$

Hence

$$\begin{aligned}\left\|\partial_x^j u\right\|_{\mathbf{L}^\infty} &\le \left\|2\operatorname{Re}\mathscr{D}_t\mathscr{B}M\mathscr{V}(i\xi)^j\widehat{\varphi}\right\|_{\mathbf{L}^\infty}\\ &\le Ct^{-1/2}\left\|\eta^{3/2}\langle\eta\rangle^{-2}\mathscr{V}\xi^j\widehat{\varphi}\right\|_{\mathbf{L}^\infty}\\ &\le Ct^{-1/2}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty}+t^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right).\end{aligned} \tag{150}$$

Therefore

$$\begin{aligned}&\frac{d}{dt}\left\|\mathscr{Q}_1\mathscr{P}u\right\|_{\mathbf{L}^2}\le C\langle t\rangle^{-1}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty}\right.\\ &\qquad\left.+t^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right)\left(\|\mathscr{P}u\|_{\mathbf{L}^2}+\|u\|_{\mathbf{L}^2}\right),\\ &\left\|\mathscr{Q}_2\mathscr{P}u\right\|_{\mathbf{L}^2}\le\left\|\mathscr{Q}_2\mathscr{P}u_0\right\|_{\mathbf{L}^2}+C\int_0^t\langle t\rangle^{-1}\\ &\qquad\cdot\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty}+t^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right)\\ &\qquad\cdot\left(\|\mathscr{P}u\|_{\mathbf{L}^2}+\|u\|_{\mathbf{L}^2}\right)d\tau.\end{aligned} \tag{151}$$

And similarly

$$\begin{aligned}&\frac{d}{dt}\|u\|_{\mathbf{H}^2}\\ &\quad\le C\langle t\rangle^{-1}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty}+t^{-1/4}\left\|\xi\phi_\xi\right\|_{\mathbf{L}^2}\right)\|u\|_{\mathbf{H}^2},\end{aligned} \tag{152}$$

from which it follows that

$$\|\mathscr{P}u\|_{\mathbf{L}^2}+\|u\|_{\mathbf{H}^2}\le\varepsilon+C\varepsilon^3\langle t\rangle^\gamma. \tag{153}$$

By the identity $\mathscr{P}=t\mathscr{L}+(1/3)\mathscr{J}\partial_x-(4/3)a\mathscr{I}$, we obtain

$$\begin{aligned}\left\|\xi\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}&=\left\|\partial_x\mathscr{J}u\right\|_{\mathbf{L}^2}\\ &\le C\|\mathscr{P}u\|_{\mathbf{L}^2}+t\|\mathscr{L}u\|_{\mathbf{L}^2}+C\|\mathscr{I}u\|_{\mathbf{L}^2}\\ &\le C\|\mathscr{P}u\|_{\mathbf{L}^2}+Ct\left\|uu_x\right\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{L}^2}\\ &\quad+C\|\mathscr{I}u\|_{\mathbf{L}^2}\le C\varepsilon+C\varepsilon^3t^\gamma+C\|\mathscr{I}u\|_{\mathbf{L}^2}.\end{aligned} \tag{154}$$

Next we estimate the norm $\|\mathscr{I}u\|_{\mathbf{L}^2}$. Denote $\widehat{K}_3(\xi)=i\xi\partial_a\widehat{K}(\xi)=O(1)$. Applying the operator $\mathscr{I}=\partial_a-t\partial_x^{-1}$ to (1) via the commutator $[\mathscr{I},\mathscr{L}]=0$, we get

$$\begin{aligned}\mathscr{L}\mathscr{I}u&=\mathscr{I}\mathscr{L}u=\mathscr{I}\partial_x\mathscr{K}u^3\\ &=3\lambda\partial_x\left(u^2\mathscr{I}u\right)+\mathscr{K}_3u^3+3i\partial_x\mathscr{K}_2\left(u^2\mathscr{I}u\right)\\ &\quad+3t\mathscr{K}\partial_x\left(u^2\partial_x^{-1}u\right)-t\mathscr{K}u^3\\ &=3\lambda\partial_x\left(u^2\mathscr{I}u\right)+\mathscr{K}_3u^3+3\mathscr{K}_2\left(u^2\mathscr{I}u\right)+N,\end{aligned} \tag{155}$$

where $N=2t\mathscr{K}(3uu_x\partial_x^{-1}u+u^3)$. Using the factorization formulas as in the derivation of (37) we find

$$\begin{aligned}&\mathscr{F}\mathscr{U}(-t)N=6t\widehat{K}(\xi)\mathscr{F}\mathscr{U}(-t)\left(uu_x\partial_x^{-1}u\right)\\ &\quad+2t\widehat{K}_1(\xi)\mathscr{F}\mathscr{U}(-t)\left(u^3\right)=6t\widehat{K}(\xi)\\ &\quad\cdot\mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\left(\mathscr{D}_t\mathscr{B}\left(M\psi_0+\overline{M}\overline{\psi_0}\right)\right)\\ &\quad\cdot\left(\mathscr{D}_t\mathscr{B}\left(M\psi_1+\overline{M}\overline{\psi_1}\right)\right)\\ &\quad\cdot\left(\mathscr{D}_t\mathscr{B}\left(M\psi_{-1}+\overline{M}\overline{\psi_{-1}}\right)\right)+2t\widehat{K}(\xi)\\ &\quad\cdot\mathscr{V}^*\overline{M}\mathscr{B}^{-1}\mathscr{D}_t^{-1}\left(\mathscr{D}_t\mathscr{B}\left(M\psi_0+\overline{M}\overline{\psi_0}\right)^3\right)\\ &\quad=6\widehat{K}(\xi)\mathscr{V}^*\frac{1}{\Lambda''}\overline{M}\left(M\psi_0+\overline{M}\overline{\psi_0}\right)\left(M\psi_1+\overline{M}\overline{\psi_1}\right)\\ &\quad\cdot\left(M\psi_{-1}+\overline{M}\overline{\psi_{-1}}\right)+2\widehat{K}(\xi)\mathscr{V}^*\frac{1}{\Lambda''}\\ &\quad\cdot\overline{M}\left(M\psi_0+\overline{M}\overline{\psi_0}\right)^3,\end{aligned} \tag{156}$$

where we denote $\psi_j=\mathscr{V}(i\xi)^j\widehat{\varphi}$. Then we get

$$\begin{aligned}&\mathscr{F}\mathscr{U}(-t)N=2\widehat{K}(\xi)\mathscr{V}^*M^2\frac{1}{\Lambda''}\left(3\psi_0\psi_1\psi_{-1}+\psi_0^3\right)\\ &\quad+6\widehat{K}(\xi)\mathscr{V}^*\\ &\quad\cdot\frac{1}{\Lambda''}\left(\overline{\psi_0}\psi_1\psi_{-1}+\psi_0\overline{\psi_1}\psi_{-1}+\psi_0\psi_1\overline{\psi_{-1}}+\psi_0^2\overline{\psi_0}\right)\\ &\quad+6\widehat{K}(\xi)\mathscr{V}^*\overline{M}^2\\ &\quad\cdot\frac{1}{\Lambda''}\left(\overline{\psi_0\psi_1}\psi_{-1}+\overline{\psi_0}\psi_1\overline{\psi_{-1}}+\psi_0\overline{\psi_1\psi_{-1}}+\psi_0\overline{\psi_0}^2\right)\\ &\quad+2\widehat{K}(\xi)\mathscr{V}^*\overline{M}^4\frac{1}{\Lambda''}\left(3\overline{\psi_0\psi_1\psi_{-1}}+\overline{\psi_0}^3\right).\end{aligned} \tag{157}$$

Next using identity (36) we find

$$\begin{aligned}&\mathscr{F}\mathscr{U}(-t)N=2\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_3\mathscr{V}^*(3t)\\ &\quad\cdot\frac{1}{\Lambda''}\left(3\psi_0\psi_1\psi_{-1}+\psi_0^3\right)+6\widehat{K}(\xi)\mathscr{V}^*\\ &\quad\cdot\frac{1}{\Lambda''}\left(\overline{\psi_0}\psi_1\psi_{-1}+\psi_0\overline{\psi_1}\psi_{-1}+\psi_0\psi_1\overline{\psi_{-1}}+\psi_0^2\overline{\psi_0}\right)\\ &\quad+6\widehat{K}(\xi)\mathscr{D}_{-1}\mathscr{V}^*(-t)\\ &\quad\cdot\frac{1}{\Lambda''}\left(\overline{\psi_0\psi_1}\psi_{-1}+\overline{\psi_0}\psi_1\overline{\psi_{-1}}+\psi_0\overline{\psi_1\psi_{-1}}+\psi_0\overline{\psi_0}^2\right)\\ &\quad+2\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\\ &\quad\cdot\frac{1}{\Lambda''}\left(3\overline{\psi_0\psi_1\psi_{-1}}+\overline{\psi_0}^3\right)\end{aligned} \tag{158}$$

with $\Omega(\xi)=\Lambda(\xi)-3\Lambda(\xi/3)$. Next using the relations $\psi_j=\mathscr{V}(i\xi)^j\widehat{\varphi}=i\eta\psi_{j-1}+\mathscr{A}_0\psi_{j-1}$ and $i\eta\psi_j=\psi_{j+1}-\mathscr{A}_0\psi_j$, we get

$\psi_1\psi_{-1} = \psi_0^2 + R_1$, $\overline{\psi_1}\psi_{-1} = -\overline{\psi_0}\psi_0 + R_2$, and $\overline{\psi_1\psi_{-1}} = \overline{\psi_0}^2 + \overline{R_1}$, where $R_1 = -\psi_0\mathscr{A}_0\psi_{-1} + \psi_{-1}\mathscr{A}_0\psi_0$, and $R_2 = \overline{\psi_0}\mathscr{A}_0\psi_{-1} + \psi_{-1}\overline{\mathscr{A}_0\psi_0}$. Therefore we obtain

$$\begin{aligned} &\mathscr{F}\mathscr{U}(-t)N \\ &\quad = 8\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3 \\ &\qquad + 8\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3 + R_3, \end{aligned} \tag{159}$$

where

$$\begin{aligned} R_3 &= 6\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0R_1 + 6\widehat{K}(\xi) \\ &\quad\cdot\mathscr{V}^*\frac{1}{\Lambda''}\left(\overline{\psi_0}R_1 + \psi_0\left(R_2 + \overline{R_2}\right)\right) + 6\widehat{K}(\xi) \\ &\quad\cdot\mathscr{D}_{-1}\mathscr{V}^*(-t)\frac{1}{\Lambda''}\left(\psi_0\overline{R_1} + \overline{\psi_0}\left(R_2 + \overline{R_2}\right)\right) \\ &\quad + 6\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}\overline{R_1}. \end{aligned} \tag{160}$$

By Lemma 2 we have

$$\begin{aligned} &\left|\psi_j\right| \\ &\quad \le C|\eta|^{j-1/4}\langle\eta\rangle^{1/4-j}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-1/4}\left\|\xi\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2}\right) \end{aligned} \tag{161}$$

for $j = -1, 0$, and then by Lemma 4 we obtain

$$\begin{aligned} &\left\|\mathscr{U}(t)\mathscr{F}^{-1}R_3\right\|_{\mathbf{L}^2} \le \left\|R_3\right\|_{\mathbf{L}^2} \le C\left\|\frac{1}{\Lambda''}\left|\psi_0\right|^2\mathscr{A}_0\psi_{-1}\right\|_{\mathbf{L}^2} \\ &\quad + C\left\|\frac{1}{\Lambda''}\left|\psi_0\psi_{-1}\right|\mathscr{A}_0\psi_0\right\|_{\mathbf{L}^2} \\ &\quad \le C\left\|\eta^{4-\gamma}\langle\eta\rangle^{2\gamma-4}\left|\psi_0\right|^2\right\|_{\mathbf{L}^\infty}\left\|\eta^{\gamma-1}\langle\eta\rangle^{-2\gamma}\mathscr{A}_0\psi_{-1}\right\|_{\mathbf{L}^2} \\ &\quad + C\left\|\eta^{3-\gamma}\langle\eta\rangle^{2\gamma-4}\left|\psi_0\psi_{-1}\right|\right\|_{\mathbf{L}^\infty}\left\|\eta^{\gamma}\langle\eta\rangle^{-2\gamma}\mathscr{A}_0\psi_0\right\|_{\mathbf{L}^2} \\ &\quad \le Ct^{-1}\left(\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty} + t^{-1/4}\left\|\xi\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2}\right)^2 \\ &\quad \cdot\left(\left\|\xi\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2} + \left\|\xi^{1/2}\widehat{\varphi}\right\|_{\mathbf{L}^\infty}\right). \end{aligned} \tag{162}$$

Then we represent

$$\begin{aligned} &\mathscr{F}\mathscr{U}(-t)N \\ &\quad = 8\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3 \\ &\qquad + 8\sqrt{3}\widehat{K}(\xi)e^{it\Omega(\xi)}\mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3 + R_3 \\ &\quad = \partial_t\Psi + R_3 + R_4, \end{aligned} \tag{163}$$

where

$$\begin{aligned} \Psi &= 8\sqrt{3}\frac{\widehat{K}(\xi)}{i\Omega(\xi)}e^{it\Omega(\xi)}\left(\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3\right. \\ &\quad \left. + \mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3\right), \\ R_4 &= 8\sqrt{3}\frac{\widehat{K}(\xi)}{i\Omega(\xi)}e^{it\Omega(\xi)}\partial_t\left(\mathscr{D}_3\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0^3\right. \\ &\quad \left. + \mathscr{D}_{-3}\mathscr{V}^*(-3t)\frac{1}{\Lambda''}\overline{\psi_0}^3\right). \end{aligned} \tag{164}$$

We need to estimate the derivative $\mathscr{V}_t^*$. We have

$$\begin{aligned} &\mathscr{V}_t^*\phi \\ &\quad = \frac{1}{2t}\mathscr{V}^*\phi \\ &\quad + \frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}iS(\eta,\xi)\phi(\eta)\sqrt{\Lambda''(\eta)}d\eta. \end{aligned} \tag{165}$$

Since $S(\eta,\xi) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta) = (1/3)\xi^{-1}\eta^{-2}(3a + 2b\eta^3\xi + b\eta^2\xi^2)(\xi-\eta)^2$ and also $i\xi\mathscr{V}^*(t)\phi - \mathscr{V}^*(t)i\eta\phi = \mathscr{V}^*(t)\mathscr{A}_0(t)\phi$ we find for the second summand

$$\begin{aligned} &\frac{|t|^{1/2}}{\sqrt{2\pi}}\int_0^\infty e^{itS(\eta,\xi)}iS(\eta,\xi)\phi(\eta)\sqrt{\Lambda''(\eta)}d\eta = \frac{i}{3}\frac{|t|^{1/2}}{\sqrt{2\pi}} \\ &\quad\cdot\int_0^\infty e^{itS(\eta,\xi)}\left(b\eta\xi + 3\frac{a}{\eta^2} - 2b\eta^2 + b\xi^2 - 3\frac{a}{\eta\xi}\right) \\ &\quad\cdot(\xi-\eta)\phi(\eta)\sqrt{\Lambda''(\eta)}d\eta = \frac{ib}{3}\xi\mathscr{V}^*(t)\eta\mathscr{A}_0(t)\phi \\ &\quad + ia\mathscr{V}^*(t)\eta^{-2}\mathscr{A}_0(t)\phi - \frac{2i}{3}b\mathscr{V}^*(t)\eta^2\mathscr{A}_0(t)\phi \\ &\quad + \frac{i}{3}b\xi^2\mathscr{V}^*(t)\mathscr{A}_0(t)\phi - ia\xi^{-1}\mathscr{V}^*(t)\eta^{-1}\mathscr{A}_0(t)\phi. \end{aligned} \tag{166}$$

Since $\partial_t\psi_0 = \mathscr{V}_t\widehat{\varphi} + \mathscr{V}\widehat{\varphi_t}$, we obtain

$$\begin{aligned} &\left\|\mathscr{U}(t)\mathscr{F}^{-1}R_4\right\|_{\mathbf{L}^2} \le \left\|R_4\right\|_{\mathbf{L}^2} \\ &\quad \le \left\|\xi\langle\xi\rangle^{-4}\mathscr{V}_t^*(3t)\frac{1}{\Lambda''}\psi_0^3\right\|_{\mathbf{L}^2} \\ &\qquad + \left\|\mathscr{V}^*(3t)\frac{1}{\Lambda''}\psi_0^2\partial_t\psi_0\right\|_{\mathbf{L}^2} \\ &\quad \le Ct^{-1}\left\|\frac{1}{\Lambda''}\psi_0^3\right\|_{\mathbf{L}^2} \\ &\qquad + C\left\|\left(\eta^2 + \eta^{-2}\right)\mathscr{A}_0(3t)\frac{1}{\Lambda''}\psi_0^3\right\|_{\mathbf{L}^2} \\ &\qquad + C\left\|\frac{1}{\Lambda''}\left|\psi_0\right|^2\partial_t\psi_0\right\|_{\mathbf{L}^2}. \end{aligned} \tag{167}$$

By Lemmas 4 and 5

$$\begin{aligned}
&\left\|\frac{1}{\Lambda''}\psi_0^3\right\|_{\mathbf{L}^2} \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^3\left\|\frac{1}{\Lambda''}|\eta|^{-3/4}\langle\eta\rangle^{-3/4}\right\|_{\mathbf{L}^2} \\
&\quad \le C\|\widehat{\varphi}\|_{\mathbf{Z}}^3, \\
&\left\|\left(\eta^2+\eta^{-2}\right)\mathscr{A}_0(3t)\frac{1}{\Lambda''}\psi_0^3\right\|_{\mathbf{L}^2} \le C\left\|\eta|\psi_0|^2\mathscr{A}_0\psi_0\right\|_{\mathbf{L}^2} \\
&\quad \le Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Z}}^2\left\||\eta|^{1/4}\langle\eta\rangle^{-1/2}t\mathscr{A}_0\psi_0\right\|_{\mathbf{L}^2} \\
&\quad \le Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Z}}^3, \\
&\left\|\frac{1}{\Lambda''}|\psi_0|^2\partial_t\psi_0\right\|_{\mathbf{L}^2} \le Ct^{-1}\left\|\frac{1}{\Lambda''}|\psi_0|^2\eta^{-\gamma}\langle\eta\rangle^{2\gamma+2}\right\|_{\mathbf{L}^\infty} \\
&\quad \cdot\left\|\eta^\gamma\langle\eta\rangle^{-2\gamma-2}t\partial_t\psi_0\right\|_{\mathbf{L}^2} \le Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Z}}^3.
\end{aligned} \tag{168}$$

Therefore

$$\left\|\mathscr{U}(t)\mathscr{F}^{-1}R_4\right\|_{\mathbf{L}^2} \le Ct^{-1}\|\widehat{\varphi}\|_{\mathbf{Z}}^3. \tag{169}$$

Thus we get

$$\begin{aligned}
\mathscr{L}\mathscr{J}u &= 3\lambda\partial_x\left(u^2\mathscr{J}u\right)+\mathscr{K}_3u^3+3\mathscr{K}_2\left(u^2\mathscr{J}u\right) \\
&\quad +\mathscr{U}(t)\mathscr{F}^{-1}\partial_t\Psi+\mathscr{U}(t)\mathscr{F}^{-1}\left(R_3+R_4\right) \\
&= 3\lambda\partial_x\left(u^2\mathscr{J}u\right)+\mathscr{K}_3u^3+3\mathscr{K}_2\left(u^2\mathscr{J}u\right) \\
&\quad +\mathscr{L}\mathscr{U}(t)\mathscr{F}^{-1}\Psi+\mathscr{U}(t)\mathscr{F}^{-1}\left(R_3+R_4\right).
\end{aligned} \tag{170}$$

Hence

$$\begin{aligned}
&\mathscr{L}\left(\mathscr{J}u-\mathscr{U}(t)\mathscr{F}^{-1}\Psi\right) \\
&\quad = 3\lambda\partial_x\left(u^2\left(\mathscr{J}u-\mathscr{U}(t)\mathscr{F}^{-1}\Psi\right)\right) \\
&\qquad +3\lambda\partial_x\left(u^2\left(\mathscr{U}(t)\mathscr{F}^{-1}\Psi\right)\right)+\mathscr{K}_3u^3 \\
&\qquad +3\mathscr{K}_2\left(u^2\mathscr{J}u\right)+\mathscr{U}(t)\mathscr{F}^{-1}\left(R_3+R_4\right).
\end{aligned} \tag{171}$$

Then as the above using the projectors $\mathscr{Q}_1$ and $\mathscr{Q}_2$ we find

$$\left\|\mathscr{J}u-\mathscr{U}(t)\mathscr{F}^{-1}\Psi\right\|_{\mathbf{L}^2} \le C\varepsilon+C\varepsilon^3t^{\gamma-1}. \tag{172}$$

We have

$$\begin{aligned}
\left\|\mathscr{U}(t)\mathscr{F}^{-1}\Psi\right\|_{\mathbf{L}^2} &\le \|\Psi\|_{\mathbf{L}^2} \le C\left\|\eta^3\langle\eta\rangle^{-7}\psi_0^3\right\|_{\mathbf{L}^2} \\
&\le C\varepsilon^3t^{\gamma-1},
\end{aligned} \tag{173}$$

and then we get

$$\|\mathscr{J}u\|_{\mathbf{L}^2} \le C\varepsilon+C\varepsilon^3t^{\gamma-1}. \tag{174}$$

Therefore

$$\left\|\xi\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} \le 5\varepsilon+C\varepsilon^3t^{\gamma}. \tag{175}$$

Next we estimate $\|\langle\xi\rangle^{1/2}\widehat{\varphi}\|_{\mathbf{L}^\infty}$. In the domain $|\xi| \ge \langle t\rangle^\nu$ we get by the Sobolev imbedding theorem

$$\begin{aligned}
&\left\|\langle\xi\rangle^{1/2}\widehat{\varphi}(t,\xi)\right\|_{\mathbf{L}^\infty(|\xi|\ge\langle t\rangle^\nu)} \\
&\quad \le C\langle t\rangle^{-\nu}\left\|\langle\xi\rangle^{3/2}\widehat{\varphi}(t,\xi)\right\|_{\mathbf{L}^\infty(|\xi|\ge\langle t\rangle^\nu)} \\
&\quad \le C\langle t\rangle^{-\nu}\left(\left\|\xi\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}+\left\|\langle\xi\rangle^2\widehat{\varphi}\right\|_{\mathbf{L}^2}\right) \le C\varepsilon\langle t\rangle^{-\nu+\gamma},
\end{aligned} \tag{176}$$

if $\nu > \gamma$, so we need to estimate the function $\langle\xi\rangle^{(1/2)}\widehat{\varphi}(t,\xi)$ in the domain $|\xi| \le \langle t\rangle^\nu$. Next by (37) for $\widehat{\varphi} = \mathscr{F}\mathscr{U}(-t)u(t)$, using Lemma 6, we get

$$\begin{aligned}
\partial_t\widehat{\varphi}(t,\xi) &= \mathscr{F}\mathscr{U}(-t)\partial_x\mathscr{K}u^3 \\
&= \frac{\sqrt{3}i\xi\widehat{K}}{it}e^{it\Omega(\xi)}\mathscr{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3+\frac{3i\xi\widehat{K}}{t\Lambda''}|\widehat{\varphi}|^2\widehat{\varphi} \\
&\quad +O\left(t^{-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right)
\end{aligned} \tag{177}$$

for all $t \ge 1$, $|\xi| \le \langle t\rangle^\nu$. Multiplying this formula by $\langle\xi\rangle^{1/2}$ we get

$$\begin{aligned}
\partial_t\langle\xi\rangle^{1/2}\widehat{\varphi}(t,\xi) &= \frac{\sqrt{3}i\xi\widehat{K}}{it}\langle\xi\rangle^{1/2}e^{it\Omega(\xi)}\mathscr{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3 \\
&\quad +\frac{3i\xi\widehat{K}}{t\Lambda''}\langle\xi\rangle^{1/2}|\widehat{\varphi}|^2\widehat{\varphi} \\
&\quad +O\left(t^{\nu-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right)
\end{aligned} \tag{178}$$

in the domain $|\xi| \le \langle t\rangle^\nu$. Define the cut-off function $\chi \in \mathbf{C}^1(\mathbf{R})$, such that $\chi(x) = 1$ for $|x| < 1$ and $\chi(x) = 0$ for $|x| > 2$, and define $\widehat{\varphi}_1(t,\xi) = \chi(\xi\langle t\rangle^{-\nu})\langle\xi\rangle^{1/2}\widehat{\varphi}(t,\xi)$. Thus we get

$$\begin{aligned}
\partial_t\widehat{\varphi}_1 &= \frac{\sqrt{3}i\xi\widehat{K}}{it}\chi\left(\xi\langle t\rangle^{-\nu}\right)\langle\xi\rangle^{1/2}e^{it\Omega(\xi)}\mathscr{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3 \\
&\quad +\frac{3i\xi\widehat{K}}{t\Lambda''}|\widehat{\varphi}|^2\widehat{\varphi}_1 \\
&\quad +\langle\xi\rangle^{1/2}\xi\langle t\rangle^{-1-\nu}\chi'\left(\xi\langle t\rangle^{-\nu}\right)\widehat{\varphi}(t,\xi) \\
&\quad +O\left(t^{\nu-5/4}\|\widehat{\varphi}\|_{\mathbf{Z}}^3\right)
\end{aligned} \tag{179}$$

for all $t \ge 1$. The third term is estimated by $C\varepsilon\langle t\rangle^{-1-\nu+\gamma}$. To exclude the resonant term we make a change $\widehat{\varphi}_1(t,\xi) = y(t,\xi)\Theta(t,\xi)$, where

$$\Theta(t,\xi) = \exp\left(\frac{3i\xi\widehat{K}}{\Lambda''}\int_1^t\left|\widehat{\varphi}(\tau,\xi)\right|^2\frac{d\tau}{\tau}\right). \tag{180}$$

Then we get

$$\begin{aligned}
&y_t(t,\xi) \\
&\quad = \frac{\sqrt{3}i\xi\widehat{K}}{it}\chi\left(\xi\langle t\rangle^{-\nu}\right)\langle\xi\rangle^{1/2}e^{it\Omega(\xi)}\overline{\Theta(t,\xi)}\mathscr{D}_3\frac{1}{\Lambda''}\widehat{\varphi}^3 \\
&\qquad +O\left(\varepsilon^3t^{-1-\delta}\right)
\end{aligned} \tag{181}$$

with $\delta > 0$. Integrating by parts we obtain

$$y(t,\xi) - y(1,\xi) = \int_1^t \sqrt{3} i\xi \widehat{K} \chi(\xi \langle\tau\rangle^{-\gamma}) \langle\xi\rangle^{1/2} \cdot e^{i\tau\Omega(\xi)} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\Lambda''} \widehat{\varphi}^3 \frac{d\tau}{i\tau} + O(\varepsilon^3) = \sqrt{3} \cdot \frac{i\xi\widehat{K}}{i\Omega(\xi)} e^{i\tau\Omega(\xi)} \chi(\xi\langle\tau\rangle^{-\gamma}) \langle\xi\rangle^{1/2} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\tau\Lambda''} \cdot \widehat{\varphi}^3 \Big|_{\tau=1}^{\tau=t} + \int_1^t \sqrt{3} \frac{i\xi\widehat{K}}{i\Omega(\xi)} e^{i\tau\Omega(\xi)} \partial_\tau \Big(\chi(\xi\langle\tau\rangle^{-\gamma}) \cdot \langle\xi\rangle^{1/2} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\tau\Lambda''} \widehat{\varphi}^3 \Big) d\tau + O(\varepsilon^3) = O(\varepsilon^3). \tag{182}$$

Thus we get the estimate $|\widehat{\varphi}_1(\xi)| = |y(t,\xi)| \le |\widehat{\varphi}_1(1,\xi)| + O(\varepsilon^3)$ in the domain $|\xi| \le \langle t\rangle^{\gamma}$. Therefore we find the desired estimate $\|\langle\xi\rangle^{1/2}\widehat{\varphi}\|_{\mathbf{L}^\infty} \le C\varepsilon$. This is the desired contradiction. Lemma 8 is proved. □

5. Proof of Theorem 1

The global existence of solution $u \in \mathbf{C}([0,T]; \mathbf{H}^2)$ to Cauchy problem (1) satisfying a priori estimate

$$\left\| \langle\xi\rangle^{1/2} \widehat{\varphi} \right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\gamma} \|u(t)\|_{\mathbf{H}^2} + \langle t\rangle^{-\gamma} \left\|\partial_x \mathscr{J} u(t)\right\|_{\mathbf{L}^2} \le C\varepsilon \tag{183}$$

follows by a standard continuation argument from Lemma 8 and local existence Theorem 7. We need only to prove asymptotic formula (20).

We need to compute the asymptotics of the function $\widehat{\varphi}(t,\xi)$. As in the proof of Lemma 8 we get

$$y(t,\xi) - y(s,\xi) = \int_s^t \sqrt{3} i\xi \widehat{K} \chi(\xi \langle\tau\rangle^{-\gamma}) \langle\xi\rangle^{1/2} \cdot e^{i\tau\Omega(\xi)} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\Lambda''} \widehat{\varphi}^3 \frac{d\tau}{i\tau} + O(\varepsilon^3 s^{-\delta}) = \sqrt{3} \cdot \frac{i\xi\widehat{K}}{i\Omega(\xi)} e^{i\tau\Omega(\xi)} \chi(\xi\langle\tau\rangle^{-\gamma}) \langle\xi\rangle^{1/2} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\tau\Lambda''} \cdot \widehat{\varphi}^3 \Big|_{\tau=s}^{\tau=t} + \int_s^t \sqrt{3} \frac{i\xi\widehat{K}}{i\Omega(\xi)} e^{i\tau\Omega(\xi)} \partial_\tau \Big(\chi(\xi\langle\tau\rangle^{-\gamma}) \cdot \langle\xi\rangle^{1/2} \overline{\Theta(\tau,\xi)} \mathscr{D}_3 \frac{1}{\tau\Lambda''} \widehat{\varphi}^3 \Big) d\tau + O(\varepsilon^3 s^{-\delta}) = O(\varepsilon^3 s^{-\delta}) \tag{184}$$

for any $t > s > 0$. Therefore there exists a unique final state $y_+ \in \mathbf{L}^\infty$ such that

$$\|y(t) - y_+\|_{\mathbf{L}^\infty} \le C\varepsilon^3 t^{-\delta} \tag{185}$$

for all $t > 0$. We write

$$\frac{3\widehat{K}}{\Lambda''} \int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau} = \frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} \int_1^t |y(\tau,\xi)|^2 \frac{d\tau}{\tau} = \frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} |y_+|^2 \log t + \Phi_2(t). \tag{186}$$

We study the asymptotics in time of the remainder term $\Phi_2(t)$. We have

$$\Phi_2(t) - \Phi_2(s) = \frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} \int_s^t \left(|y(\tau)|^2 - |y(t)|^2\right) \frac{d\tau}{\tau} + \frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} \left(|y(t)|^2 - |y_+|^2\right) \log\frac{t}{s}. \tag{187}$$

By (185) we obtain $\|\Phi_2(t) - \Phi_2(s)\|_{\mathbf{L}^\infty} \le C\varepsilon^3 s^{-\delta}$ for any $t > s > 0$. Hence there exists a unique real-valued function $\Phi_+ \in \mathbf{L}^\infty$ such that

$$\|\Phi_2(t) - \Phi_+\|_{\mathbf{L}^\infty} \le C\varepsilon^3 t^{-\delta} \tag{188}$$

for all $t > 0$. Representation (186) and estimate (188) yield

$$\left\| \Theta(\tau,\xi) - \exp\left(\frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} |y_+|^2 \log t + \Phi_+ \right) \right\|_{\mathbf{L}^\infty} \le Ct^{-\delta} \tag{189}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\left\| \chi(\xi\langle t\rangle^{-\gamma}) \langle\xi\rangle^{1/2} \widehat{\varphi}(t,\xi) - y_+ \Theta(\tau,\xi) \right\|_{\mathbf{L}^\infty} = \|y(t) - y_+\|_{\mathbf{L}^\infty} \le Ct^{-\delta}, \tag{190}$$

$$\left\| y_+ \Theta(\tau,\xi) - y_+ \exp\left(\frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} |y_+|^2 \log t + \Phi_+ \right) \right\|_{\mathbf{L}^\infty} \le Ct^{-\delta}. \tag{191}$$

Therefore we obtain the estimate

$$\left\| \chi(\xi\langle t\rangle^{-\gamma}) \widehat{\varphi}(t,\xi) - W_+ \exp\left(\frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''} |W_+|^2 \log t \right) \right\|_{\mathbf{L}^\infty} \le Ct^{-\delta} \tag{192}$$

with $W_+ = y_+ e^{\Phi_+}$. Using the factorization of $\mathscr{U}(t)$ we have

$$\begin{aligned}&\Big\| u(t) - 2\mathrm{Re}\,\mathscr{D}_t\mathscr{B}^{-1} M W_+ \exp\left(\frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''}\left|W_+\right|^2\log t\right)\Big\|_{\mathbf{L}^\infty} \\ &\le C\Big\|\mathscr{D}_t\mathscr{B}^{-1}M\Big(\chi\left(\xi\langle t\rangle^{-\nu}\right)\widehat{\varphi}(t,\xi) - W_+\exp\left(\frac{3i\xi\widehat{K}}{\langle\xi\rangle\Lambda''}\left|W_+\right|^2\log t\right)\Big)\Big\|_{\mathbf{L}^\infty} + Ct^{-1/2-\delta} \\ &\le Ct^{-1/2-\delta}.\end{aligned} \tag{193}$$

This completes the proof of asymptotics (20). Theorem 1 is proved.

Competing Interests

The authors declare that they have no competing interests.

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