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Research Article

Some New Volterra-Fredholm-Type Nonlinear Discrete Inequalities with Two Variables Involving Iterated Sums and Their Applications

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Some generalized discrete Volterra-Fredholm-type inequalities were developed, which can be used as effective tools in the qualitative analysis of the solution to difference equations.

1. Introduction

In recent years, various forms of inequalities played increasingly important roles in the study of quantitative properties of solutions of differential and integral equations [1–15]. Discrete inequalities, especially the discrete Volterra-Fredholm-type inequalities, have been applied to study the discrete equations widely. For example, see [1–3, 9–11] and the references therein. In this paper, some new Volterra-Fredholm-type discrete inequalities involving four iterated infinite sums were established. Furthermore, to illustrate the usefulness of the established results, some examples were provided for the studying of their solutions on the boundedness, uniqueness, and continuous dependence.

We design the needed symbols as follows:

(a) N_0 denotes the set of nonnegative integers and Z denotes the set of integers, while R denotes the set of real numbers $R_+ = [0, \infty)$.

(b) Let $\Omega := ([m_0, M] \times [n_0, N]) \cap Z^2$, where $m_0, n_0 \in Z$, and $M, N \in Z \cup \{\infty\}$ are two constants.

(c) $K_i > 0$ $(i = 1, 2, 3, 4)$ are all constants, and $l_1, l_2 \in Z$ are two constants.

(d) If U is a lattice, then we denote the set of all R−valued functions on U by $\wp(U)$ and denote the set of all R_+−valued functions on U by $\wp_+(U)$.

(e) For a function $g \in \wp_+(U)$, we have $\sum_{s=m_0}^{m_1} g(s) = 0$ provided $m_0 > m_1$.

We need the following lemmas in the discussions of our main results.

Lemma 1 (see [4]). *Let $u(m,n) \in \wp_+(\Omega)$, $b(s,t,m,n) \in \wp_+(\Omega^2)$ be nondecreasing in the third variable; $k \geq 0$ is a constant. For $(m,n) \in \Omega$, if*

$$u(m,n) \leq k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t,m,n)\, u(s,t), \tag{1}$$

then

$$u(m,n) \leq k \exp\left\{ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t,m,n) \right\}. \tag{2}$$

Lemma 2 (see [4]). *Let $u(m,n), a(m,n), c(m,n) \in \wp_+(\Omega)$. If $a(m,n)$ is nondecreasing in the first variable, then, for $(m,n) \in \Omega$,*

$$u(m,n) \leq a(m,n) + \sum_{s=m_0}^{m-1} c(s,n)\, u(s,n), \tag{3}$$

then

$$u(m,n) \leq a(m,n) \prod_{s=m_0}^{m-1} [1 + c(s,n)]. \tag{4}$$

Lemma 3 (see [5]). *Let $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$; then, for any $K > 0$,*

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}. \tag{5}$$

2. Main Results

Theorem 4. *Suppose that $u(m,n), a(m,n), b_1(m,n)$, $b_2(m,n) \in \wp_+(\Omega)$, $c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n), f_j(s,t,m,n)$, $g_j(s,t,m,n)$, $w_j(s,t,m,n) \in \wp_+(\Omega^2)$, and p, q_i, r_i, h_j, v_j are nonnegative constants with $p \geq q_i > 0$, $p \geq r_i > 0$ ($i = 1,2,\ldots,l_1$), $p \geq h_j > 0$, $p \geq v_j > 0$ ($j = 1,2,\ldots,l_2$), and $c_i, d_i, e_i, f_j, g_j, w_j$ being nondecreasing in the last two variables, $b_1(m,n)$ and $b_2(m,n)$ are also nondecreasing. If*

$$\begin{aligned} u^p(m,n) &\leq a(m,n) + b_1(m,n) \\ &\cdot \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c_i(s,t,m,n) u^{q_i}(s,t) + d_i(s,t,m,n)\right. \\ &\left.\cdot u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b_2(m,n) \\ &\cdot \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f_j(s,t,m,n) u^{h_j}(s,t)\right. \\ &\left.+ g_j(s,t,m,n) u^{v_j}(s,t) + w_j(s,t,m,n)\right], \end{aligned} \tag{6}$$

then, for $(m,n) \in \Omega$, we have

$$\begin{aligned} &u(m,n) \\ &\leq \left\{a(m,n) + b(m,n) \frac{J(M,N)}{1-\lambda(M,N)} C(m,n)\right\}^{1/p}, \end{aligned} \tag{7}$$

provided that $\lambda(M,N) < 1$, where

$$b(m,n) = \max\{b_1(m,n), b_2(m,n)\}, \tag{8}$$

$$C(m,n) = \exp\left\{\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n)\right\}, \tag{9}$$

$$B(s,t,m,n) = \sum_{i=1}^{l_1} \left[c_i(s,t,m,n) \frac{q_i}{p} K_1^{(q_i-p)/p} + d_i(s,t,m,n) \frac{r_i}{p} K_2^{(r_i-p)/p}\right] b(s,t), \tag{10}$$

$$\begin{aligned} J(m,n) = &\sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{c_i(s,t,m,n) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} a(s,t) + \frac{p-q_i}{p} K_1^{q_i/p}\right]\right. \\ &\left.+ d_i(s,t,m,n) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} a(s,t) + \frac{p-r_i}{p} K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} \\ &+ \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{f_j(s,t,m,n) \left[\frac{h_j}{p} K_3^{(h_j-p)/p} a(s,t) + \frac{p-h_j}{p} K_3^{h_j/p}\right]\right. \\ &\left.+ g_j(s,t,m,n) \left[\frac{v_j}{p} K_4^{(v_j-p)/p} a(s,t) + \frac{p-v_j}{p} K_4^{v_j/p}\right] + w_j(s,t,m,n)\right\}, \end{aligned} \tag{11}$$

$$\lambda(m,n) = \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f_j(s,t,m,n) \frac{h_j}{p} K_3^{(h_j-p)/p} + g_j(s,t,m,n) \frac{v_j}{p} K_4^{(v_j-p)/p}\right] b(s,t) C(s,t). \tag{12}$$

Proof. Given $b(m,n) = \max\{b_1(m,n), b_2(m,n)\}$, for $(m,n) \in \Omega$, we have

$$\begin{aligned} u^p(m,n) &\leq a(m,n) + b(m,n) \\ &\cdot \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c_i(s,t,m,n) u^{q_i}(s,t)\right. \\ &\left.+ d_i(s,t,m,n) u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b(m,n) \\ &\cdot \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f_j(s,t,m,n) u^{h_j}(s,t)\right. \\ &\left.+ g_j(s,t,m,n) u^{v_j}(s,t) + w_j(s,t,m,n)\right]. \end{aligned} \tag{13}$$

Define a function $z(m,n)$ by

$$z(m,n) = \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c_i(s,t,m,n) u^{q_i}(s,t) + d_i(s,t,m,n) u^{r_i}(s,t) + e_i(s,t,m,n) \right] + \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f_j(s,t,m,n) u^{h_j}(s,t) + g_j(s,t,m,n) u^{v_j}(s,t) + w_j(s,t,m,n) \right]. \tag{14}$$

Then

$$u^p(m,n) \le a(m,n) + b(m,n) z(m,n), \tag{15}$$

or

$$u(m,n) \le \left(a(m,n) + b(m,n) z(m,n)\right)^{1/p}. \tag{16}$$

By using Lemma 3, for any $K_i > 0$ $(i = 1, 2, 3, 4)$, we have

$$\begin{aligned} z(m,n) \le & \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left\{ c_i(s,t,m,n) \left[\frac{q_i}{p} K_1^{(q_i-p)/p} (a(s,t) + b(s,t) z(s,t)) + \frac{p-q_i}{p} K_1^{q_i/p} \right] \right. \\ & \left. + d_i(s,t,m,n) \left[\frac{r_i}{p} K_2^{(r_i-p)/p} (a(s,t) + b(s,t) z(s,t)) + \frac{p-r_i}{p} K_2^{r_i/p} \right] + e_i(s,t,m,n) \right\} \\ & + \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left\{ f_j(s,t,m,n) \left[\frac{h_j}{p} K_3^{(h_j-p)/p} (a(s,t) + b(s,t) z(s,t)) + \frac{p-h_j}{p} K_3^{h_j/p} \right] \right. \\ & \left. + g_j(s,t,m,n) \left[\frac{v_j}{p} K_4^{(v_j-p)/p} (a(s,t) + b(s,t) z(s,t)) + \frac{p-v_j}{p} K_4^{v_j/p} \right] + w_j(s,t,m,n) \right\} = R(m,n) \\ & + \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c_i(s,t,m,n) \frac{q_i}{p} K_1^{(q_i-p)/p} + d_i(s,t,m,n) \frac{r_i}{p} K_2^{(r_i-p)/p} \right] b(s,t) z(s,t), \end{aligned} \tag{17}$$

where

$$R(m,n) = J(m,n) + \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[f_j(s,t,m,n) \frac{h_j}{p} K_3^{(h_j-p)/p} + g_j(s,t,m,n) \frac{v_j}{p} K_4^{(v_j-p)/p} \right] b(s,t) z(s,t), \tag{18}$$

and $J(m,n)$ is defined in (11). Then, using that $R(m,n)$ is nondecreasing in every variable, we get

$$\begin{aligned} z(m,n) &\le R(M,N) + \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[c_i(s,t,m,n) \frac{q_i}{p} K_1^{(q_i-p)/p} + d_i(s,t,m,n) \frac{r_i}{p} K_2^{(r_i-p)/p} \right] b(s,t) z(s,t) \\ &= R(M,N) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n) z(s,t), \end{aligned} \tag{19}$$

where $B(s,t,m,n)$ is defined in (10).

Since $b(m,n)$ is nondecreasing and $c_i(s,t,m,n), d_i(s,t,m,n)$ are nondecreasing in the last two variables, then $B(s,t,m,n)$ is also nondecreasing in the last two variables, and, by Lemma 1 and (19), we get

$$z(m,n) \leq R(M,N) \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\right\} = R(M,N) C(m,n), \tag{20}$$

where $C(m,n)$ is defined in (9). Considering the definition of $R(m,n)$ and (20), we have

$$\begin{aligned} R(M,N) &= J(M,N) + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,M,N)\frac{h_j}{p}K_3^{(h_j-p)/p} + g_j(s,t,M,N)\frac{v_j}{p}K_4^{(v_j-p)/p}\right] b(s,t) z(s,t) \leq J(M,N) + R(M,N) \\ &\cdot \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,M,N)\frac{h_j}{p}K_3^{(h_j-p)/p} + g_j(s,t,M,N)\frac{v_j}{p}K_4^{(v_j-p)/p}\right] b(s,t) C(s,t) \\ &= J(M,N) + R(M,N)\lambda(M,N), \end{aligned} \tag{21}$$

where $\lambda(m,n)$ is defined in (12). Then,

$$R(M,N) \leq \frac{J(M,N)}{1-\lambda(M,N)}. \tag{22}$$

Combining (20) and (22), we deduce

$$z(m,n) \leq \frac{J(M,N)}{1-\lambda(M,N)} C(m,n), \tag{23}$$

where $C(m,n), \lambda(m,n)$ are defined in (9) and (12).

Then, combining (16) and (23), we obtain the desired result. □

Corollary 5. *Let* $r_{1i}(m,n), d_{1i}(m,n), c_{1i}(m,n), e_{1i}(m,n) \in \wp_+(\Omega)$, $(i = 1,2,\ldots,l_1)$, $f_{1j}(m,n), g_{1j}(m,n), w_{1j}(m,n), r_{2j}(m,n) \in \wp_+(\Omega)$, $(j = 1,2,\ldots,l_2)$, $r_{1i}(m,n), r_{2j}(m,n)$, $b_1(m,n)$ *and* $b_2(m,n)$ *be nondecreasing in every variable.* $u(m,n), a(m,n), b_1(m,n), b_2(m,n), p, q_i, r_i, h_j, v_j$ *are defined as in Theorem 4. If*

$$\begin{aligned} u^p(m,n) &\leq a(m,n) + b_1(m,n)\sum_{i=1}^{l_1} r_{1i}(m,n) \\ &\cdot \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_{1i}(s,t)u^{q_i}(s,t) + d_{1i}(s,t)u^{r_i}(s,t) + e_{1i}(s,t)\right] + b_2(m,n)\sum_{j=1}^{l_2} r_{2j}(m,n) \\ &\cdot \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_{1j}(s,t)u^{h_j}(s,t) + g_{1j}(s,t)u^{v_j}(s,t) + w_{1j}(s,t)\right], \end{aligned} \tag{24}$$

then, for $(m,n) \in \Omega$, *we have*

$$u(m,n) \leq \left\{a(m,n) + b(m,n)\frac{J(M,N)}{1-\lambda(M,N)} C(m,n)\right\}^{1/p}, \tag{25}$$

provided that $\lambda(M,N) < 1$, *where*

$$\begin{aligned} b(m,n) &= \max\{b_1(m,n), b_2(m,n)\}, \\ C(m,n) &= \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} B(s,t,m,n)\right\}, \\ B(s,t,m,n) &= \sum_{i=1}^{l_1} r_{1i}(m,n)\left[c_{1i}(s,t)\frac{q_i}{p}K_1^{(q_i-p)/p} + d_{1i}(s,t)\frac{r_i}{p}K_2^{(r_i-p)/p}\right] b(s,t), \\ J(m,n) &= \sum_{i=1}^{l_1} r_{1i}(m,n)\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_{1i}(s,t) \cdot \left[\frac{q_i}{p}K_1^{(q_i-p)/p}a(s,t) + \frac{p-q_i}{p}K_1^{q_i/p}\right] + d_{1i}(s,t) \cdot \left[\frac{r_i}{p}K_2^{(r_i-p)/p}a(s,t) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_{1i}(s,t)\right\} \\ &+ \sum_{j=1}^{l_2} r_{2j}(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_{1j}(s,t) \cdot \left[\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right] + g_{1j}(s,t) \cdot \left[\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right] + w_{1j}(s,t)\right\}, \\ \lambda(m,n) &= \sum_{j=1}^{l_2} r_{2j}(m,n)\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_{1j}(s,t)\frac{h_j}{p}K_3^{(h_j-p)/p} + g_{1j}(s,t)\frac{v_j}{p}K_4^{(v_j-p)/p}\right] b(s,t) C(s,t). \end{aligned} \tag{26}$$

The proof of Corollary 5 can be completed by setting $c_i(s,t,m,n) = r_{1i}(m,n)c_{1i}(s,t)$, $d_i(s,t,m,n) = r_{1i}(m,n)d_{1i}(s,t)$, $e_i(s,t,m,n) = r_{1i}(m,n)e_{1i}(s,t)$, $f_j(s,t,m,n) = r_{2j}(m,

$n) f_{1j}(s,t)$, $g_j(s,t,m,n) = r_{2j}(m,n) g_{1j}(s,t)$, $w_j(s,t,m,n) = r_{2j}(m,n) w_{1j}(s,t)$ in Theorem 4.

Letting $p = 1$, we get the following corollary.

Corollary 6. *Let $u(m,n), a(m,n), b_1(m,n), b_2(m,n), c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n), f_j(s,t,m,n), g_j(s,t,m,n), w_j(s,t,m,n)$ be defined as in Theorem 4. If*

$$u(m,n) \le a(m,n) + b_1(m,n) \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} [c_i(s,t,m,n) u(s,t) + d_i(s,t,m,n) u(s,t) + e_i(s,t,m,n)] + b_2(m,n) \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} [f_j(s,t,m,n) u(s,t) + g_j(s,t,m,n) u(s,t) + w_j(s,t,m,n)], \tag{27}$$

then, for $(m,n) \in \Omega$, we have

$$u(m,n) \le a(m,n) + b(m,n) \frac{J(M,N)}{1 - \lambda(M,N)} C(m,n), \tag{28}$$

provided that $\lambda(M,N) < 1$, where

$$\begin{aligned} b(m,n) &= \max\{b_1(m,n), b_2(m,n)\}, \\ C(m,n) &= \exp\left\{\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} B(s,t,m,n)\right\}, \\ B(s,t,m,n) &= \sum_{i=1}^{l_1} [c_i(s,t,m,n) + d_i(s,t,m,n)] b(s,t), \\ J(m,n) &= \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \{[c_i(s,t,m,n) + d_i(s,t,m,n)] a(s,t) + e_i(s,t,m,n)\} \\ &\quad + \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \{[f_j(s,t,m,n) + g_j(s,t,m,n)] a(s,t) + w_j(s,t,m,n)\}, \\ \lambda(m,n) &= \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} [f_j(s,t,m,n) + g_j(s,t,m,n)] b(s,t) C(s,t). \end{aligned} \tag{29}$$

Theorem 7. *Let $\varphi(m,n) \in \wp_+(\Omega)$, $u(m,n), a(m,n), b_1(m,n), b_2(m,n), c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n), f_j(s,t,m,n), g_j(s,t,m,n), w_j(s,t,m,n), p, q_i, r_i, h_j, v_j$ be defined as in Theorem 4. Assume that $a(m,n)$ is nondecreasing in the first variable. If*

$$\begin{aligned} u^p(m,n) &\le a(m,n) + \sum_{s=m_0}^{m-1} \varphi(s,n) u^p(s,n) + b_1(m,n) \\ &\quad \cdot \sum_{i=1}^{l_1} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} [c_i(s,t,m,n) u^{q_i}(s,t) \\ &\quad + d_i(s,t,m,n) u^{r_i}(s,t) + e_i(s,t,m,n)] + b_2(m,n) \\ &\quad \cdot \sum_{j=1}^{l_2} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} [f_j(s,t,m,n) \\ &\quad \cdot u^{h_j}(s,t) + g_j(s,t,m,n) u^{v_j}(s,t) + w_j(s,t,m,n)], \end{aligned} \tag{30}$$

then, for $(m,n) \in \Omega$, we have

$$\begin{aligned} u(m,n) &\le \left\{\left[a(m,n) + b(m,n) \frac{\widetilde{J}(M,N)}{1 - \widetilde{\lambda}(M,N)} \widetilde{C}(m,n)\right] \right. \\ &\quad \left. \cdot \widetilde{\varphi}(m,n)\right\}^{1/p}, \end{aligned} \tag{31}$$

provided that $\widetilde{\lambda}(M,N) < 1$, where

$$\widetilde{\varphi}(m,n) = \prod_{s=m_0}^{m-1} \left[1 + \varphi(s,n)\right], \tag{32}$$

$$b(m,n) = \max\left\{b_1(m,n), b_2(m,n)\right\}, \tag{33}$$

$$\begin{aligned}
\widetilde{c}_i(s,t,m,n) &= c_i(s,t,m,n)\left(\widetilde{\varphi}(s,t)\right)^{q_i/p}, \\
\widetilde{d}_i(s,t,m,n) &= d_i(s,t,m,n)\left(\widetilde{\varphi}(s,t)\right)^{r_i/p}, \quad i = 1,2,\ldots,l_1, \\
\widetilde{f}_j(s,t,m,n) &= f_j(s,t,m,n)\left(\widetilde{\varphi}(s,t)\right)^{h_j/p}, \\
\widetilde{g}_j(s,t,m,n) &= g_j(s,t,m,n)\left(\widetilde{\varphi}(s,t)\right)^{v_j/p}, \quad j = 1,2,\ldots,l_2,
\end{aligned} \tag{34}$$

$$\widetilde{C}(m,n) = \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \widetilde{B}(s,t,m,n)\right\}, \tag{35}$$

$$\widetilde{B}(s,t,m,n) = \sum_{i=1}^{l_1}\left[\widetilde{c}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} + \widetilde{d}_i(s,t,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]b(s,t), \tag{36}$$

$$\begin{aligned}
\widetilde{J}(m,n) &= \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{\widetilde{c}_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}a(s,t) + \frac{p-q_i}{p}K_1^{q_i/p}\right] + \widetilde{d}_i(s,t,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}a(s,t) + \frac{p-r_i}{p}\right.\right. \\
&\quad \left.\left. \cdot K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{\widetilde{f}_j(s,t,m,n)\left[\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right]\right. \\
&\quad \left. + \widetilde{g}_j(s,t,m,n)\left[\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right] + w_j(s,t,m,n)\right\},
\end{aligned} \tag{37}$$

$$\widetilde{\lambda}(m,n) = \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\widetilde{f}_j(s,t,m,n)\frac{h_j}{p}K_3^{(h_j-p)/p} + \widetilde{g}_j(s,t,m,n)\frac{v_j}{p}K_4^{(v_j-p)/p}\right]b(s,t)\widetilde{C}(s,t). \tag{38}$$

Proof. Given $b(m,n) = \max\{b_1(m,n), b_2(m,n)\}$, for $(m,n) \in \Omega$, we have

$$\begin{aligned}
u^p(m,n) &\le a(m,n) + \sum_{s=m_0}^{m-1}\varphi(s,n)u^p(s,n) + b(m,n) \\
&\quad \cdot \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t)\right. \\
&\quad \left. + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b(m,n) \\
&\quad \cdot \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)\right. \\
&\quad \left. \cdot u^{h_j}(s,t) + g_j(s,t,m,n)u^{v_j}(s,t) + w_j(s,t,m,n)\right].
\end{aligned} \tag{39}$$

Define function $\widetilde{z}(m,n)$ by

$$\begin{aligned}
\widetilde{z}(m,n) &= a(m,n) + b(m,n) \\
&\quad \cdot \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t)\right. \\
&\quad \left. + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b(m,n) \\
&\quad \cdot \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)u^{h_j}(s,t)\right. \\
&\quad \left. + g_j(s,t,m,n)u^{v_j}(s,t) + w_j(s,t,m,n)\right].
\end{aligned} \tag{40}$$

Then,

$$u^p(m,n) \le \widetilde{z}(m,n) + \sum_{s=m_0}^{m-1}\varphi(s,n)u^p(s,n). \tag{41}$$

Clearly $z(m,n)$ is nondecreasing in the first variable. Then, by Lemma 2, we get

$$\begin{aligned}
u^p(m,n) &\le \widetilde{z}(m,n)\prod_{s=m_0}^{m-1}\left[1 + \varphi(s,n)\right] \\
&= \widetilde{z}(m,n)\widetilde{\varphi}(m,n),
\end{aligned} \tag{42}$$

where $\widetilde{\varphi}(m,n)$ is defined in (32). Define function

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t) + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)u^{h_j}(s,t) + g_j(s,t,m,n)u^{v_j}(s,t) + w_j(s,t,m,n)\right]. \quad (43)$$

From (40), we get

$$\widetilde{z}(m,n) = a(m,n) + b(m,n)v(m,n). \quad (44)$$

Then (42) becomes

$$u(m,n) \le \left\{\left[a(m,n) + b(m,n)v(m,n)\right]\widetilde{\varphi}(m,n)\right\}^{1/p}. \quad (45)$$

By (45) and Lemma 3, from (43), we have

$$\begin{aligned} v(m,n) \le & \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_i(s,t,m,n)(\widetilde{\varphi}(s,t))^{q_i/p}\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+b(s,t)v(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right]\right. \\ & \left.+ d_i(s,t,m,n)(\widetilde{\varphi}(s,t))^{r_i/p}\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(s,t)+b(s,t)v(s,t)) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} \\ & + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_j(s,t,m,n)(\widetilde{\varphi}(s,t))^{h_j/p}\left[\frac{h_j}{p}K_3^{(h_j-p)/p}(a(s,t)+b(s,t)v(s,t)) + \frac{p-h_j}{p}K_3^{h_j/p}\right]\right. \\ & \left.+ g_j(s,t,m,n)(\widetilde{\varphi}(s,t))^{v_j/p}\left[\frac{v_j}{p}K_4^{(v_j-p)/p}(a(s,t)+b(s,t)v(s,t)) + \frac{p-v_j}{p}K_4^{v_j/p}\right] + w_j(s,t,m,n)\right\} = \widetilde{R}(m,n) \\ & + \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\widetilde{c}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} + \widetilde{d}_i(s,t,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]b(s,t)v(s,t), \end{aligned} \quad (46)$$

where

$$\widetilde{R}(m,n) = \widetilde{J}(m,n) + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\widetilde{f}_j(s,t,m,n)\frac{h_j}{p}K_3^{(h_j-p)/p} + \widetilde{g}_j(s,t,m,n)\frac{v_j}{p}K_4^{(v_j-p)/p}\right]b(s,t)v(s,t), \quad (47)$$

$\widetilde{c}_i, \widetilde{d}_i, \widetilde{f}_j, \widetilde{g}_j$ and $\widetilde{J}(m,n)$ are defined in (34) and (37), respectively.

Similar to the process of (17)–(23), we deduce that

$$v(m,n) \le \frac{\widetilde{J}(M,N)}{1-\widetilde{\lambda}(M,N)}\widetilde{C}(m,n), \quad (48)$$

where $\widetilde{C}(m,n)$, $\widetilde{\lambda}(m,n)$ are defined in (35) and (38).

Combining (45) and (48), we get the desired result. □

Theorem 8. *Let $u(m,n), a(m,n), b_1(m,n), b_2(m,n), c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n), f_j(s,t,m,n), g_j(s,t,m,n), w_j(s,t,m,n), p, q_i, r_i, h_j, v_j$ be defined as in Theorem 4. $H_j, L_j : \Omega \times R_+ \to R_+$ $(j = 1,2,\ldots,l_2)$ satisfies $0 \le H_j(m,n,u) - H_j(m,n,v) \le L_j(m,n,v)(u-v)$ for $u \ge v \ge 0$. If*

$$\begin{aligned} u^p(m,n) \le & \, a(m,n) + b_1(m,n) \\ & \cdot \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t)\right. \\ & \left.+ d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b_2(m,n) \\ & \cdot \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)H_j\left(s,t,u^{h_j}(s,t)\right)\right. \\ & \left.+ g_j(s,t,m,n)H_j\left(s,t,u^{v_j}(s,t)\right) + w_j(s,t,m,n)\right], \end{aligned} \quad (49)$$

then, for $(m,n) \in \Omega$, we have

$$u(m,n) \le \left\{a(m,n) + b(m,n)\frac{\overline{J}(M,N)}{1-\overline{\lambda}(M,N)}\overline{C}(m,n)\right\}^{1/p}, \quad (50)$$

provided that $\overline{\lambda}(M,N) < 1$, *where*

$$b(m,n) = \max\{b_1(m,n), b_2(m,n)\}, \tag{51}$$

$$\overline{C}(m,n) = \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{B}(s,t,m,n)\right\}, \tag{52}$$

$$\overline{B}(s,t,m,n) = \sum_{i=1}^{l_1}\left[c_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} + d_i(s,t,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]b(s,t), \tag{53}$$

$$\begin{aligned}\overline{J}(m,n) = {}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}a(s,t) + \frac{p-q_i}{p}K_1^{q_i/p}\right]\right.\\ &\left. + d_i(s,t,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}a(s,t) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\}\\ &+ \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_j(s,t,m,n)H_j\left[s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right]\right.\\ &\left. + g_j(s,t,m,n)H_j\left[s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right] + w_j(s,t,m,n)\right\},\end{aligned} \tag{54}$$

$$\overline{f}_j(s,t,m,n) = f_j(s,t,m,n)L_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right),\quad j=1,2,\ldots,l_2, \tag{55}$$

$$\overline{g}_j(s,t,m,n) = g_j(s,t,m,n)L_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right),\quad j=1,2,\ldots,l_2, \tag{56}$$

$$\overline{\lambda}(m,n) = \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\overline{f}_j(s,t,m,n)\frac{h_j}{p}K_3^{(h_j-p)/p} + \overline{g}_j(s,t,m,n)\frac{v_j}{p}K_4^{(v_j-p)/p}\right]b(s,t)\overline{C}(s,t). \tag{57}$$

Proof. Given $b(m,n) = \max\{b_1(m,n), b_2(m,n)\}$, for $(m,n) \in \Omega$, we have

$$\begin{aligned}u^p(m,n) \le {}& a(m,n) + b(m,n)\\ &\cdot\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t)\right.\\ &\left. + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] + b(m,n)\\ &\cdot\sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)H_j\left(s,t,u^{h_j}(s,t)\right)\right.\\ &\left. + g_j(s,t,m,n)H_j\left(s,t,u^{v_j}(s,t)\right) + w_j(s,t,m,n)\right].\end{aligned} \tag{58}$$

Define function $\overline{v}(m,n)$ by

$$\begin{aligned}\overline{v}(m,n) = {}& \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t) + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right]\\ &+ \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)H_j\left(s,t,u^{h_j}(s,t)\right) + g_j(s,t,m,n)H_j\left(s,t,u^{v_j}(s,t)\right) + w_j(s,t,m,n)\right].\end{aligned} \tag{59}$$

Then

$$u^p(m,n) \le a(m,n) + b(m,n)\overline{v}(m,n), \tag{60}$$

or

$$u(m,n) \le \left(a(m,n) + b(m,n)\overline{v}(m,n)\right)^{1/p}. \tag{61}$$

By Lemma 3, we have

$$
\begin{aligned}
\overline{v}(m,n) \leq & \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\{c_i(s,t,m,n)(a(s,t)+b(s,t)\overline{v}(s,t))^{q_i/p} \\
& + d_i(s,t,m,n)(a(s,t)+b(s,t)\overline{v}(s,t))^{r_i/p} + e_i(s,t,m,n)\} \\
& + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\{f_j(s,t,m,n)H_j\left(s,t,(a(s,t)+b(s,t)\overline{v}(s,t))^{h_j/p}\right) \\
& + g_j(s,t,m,n)H_j\left(s,t,(a(s,t)+b(s,t)\overline{v}(s,t))^{v_j/p}\right) + w_j(s,t,m,n)\} \\
& \leq \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right]\right. \\
& \left. + d_i(s,t,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} \\
& + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_j(s,t,m,n)\left[H_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-h_j}{p}K_3^{h_j/p}\right)\right.\right. \\
& - H_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right) + H_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right)\Bigg] + g_j(s,t,m,n) \\
& \cdot\left[H_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-v_j}{p}K_4^{v_j/p}\right) - H_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right)\right. \\
& \left.\left. + H_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right)\right] + w_j(s,t,m,n)\right\} \\
& \leq \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-q_i}{p}K_1^{q_i/p}\right]\right. \\
& \left. + d_i(s,t,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}(a(s,t)+b(s,t)\overline{v}(s,t)) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} \\
& + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_j(s,t,m,n)\left[L_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right)\right.\right. \\
& \cdot\frac{h_j}{p}K_3^{(h_j-p)/p}b(s,t)\overline{v}(s,t) + H_j\left(s,t,\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right)\Bigg] + g_j(s,t,m,n)\left[L_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t)\right.\right. \\
& \left.\left.\left. + \frac{p-v_j}{p}K_4^{v_j/p}\right)\frac{v_j}{p}K_4^{(v_j-p)/p}b(s,t)\overline{v}(s,t) + H_j\left(s,t,\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right)\right] + w_j(s,t,m,n)\right\} = \overline{R}(m,n) \\
& + \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} + d_i(s,t,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]b(s,t)\overline{v}(s,t),
\end{aligned}
\tag{62}
$$

where

$$\overline{R}(m,n) = \overline{J}(m,n) + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\overline{f}_j(s,t,m,n)\frac{h_j}{p}K_3^{(h_j-p)/p} + \overline{g}_j(s,t,m,n)\frac{v_j}{p}K_4^{(v_j-p)/p}\right]b(s,t)\,\overline{v}(s,t), \tag{63}$$

and $\overline{J}(m,n), \overline{f}_j(s,t,m,n), \overline{g}_j(s,t,m,n)$ are defined in (54)–(56).

Similar to the process of (17)–(23), we get

$$\overline{v}(m,n) \le \frac{\overline{J}(M,N)}{1-\overline{\lambda}(M,N)}\overline{C}(m,n), \tag{64}$$

where $\overline{C}(m,n), \overline{\lambda}(m,n)$ are defined in (52) and (57).

Combining (61) and (64), we get the desired result. □

Theorem 9. *Let $\varphi(m,n) \in \wp_+(\Omega), u(m,n), a(m,n), b_1(m,n), b_2(m,n), c_i(s,t,m,n), d_i(s,t,m,n), e_i(s,t,m,n), f_j(s,t,m,n), g_j(s,t,m,n), w_j(s,t,m,n), p, q_i, r_i, h_j, v_j$ be defined as in Theorem 4. Assume that $a(m,n)$ is nondecreasing in the first variable. H_j, L_j $(j = 1,2,\ldots,l_2)$ are defined as in Theorem 7. If*

$$\begin{aligned} u^p(m,n) \le{}& a(m,n) + \sum_{s=m_0}^{m-1}\varphi(s,n)u^p(s,n) \\ &+ b_1(m,n)\sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_i(s,t,m,n)u^{q_i}(s,t) + d_i(s,t,m,n)u^{r_i}(s,t) + e_i(s,t,m,n)\right] \\ &+ b_2(m,n)\sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_j(s,t,m,n)H_j\left(s,t,u^{h_j}(s,t)\right) + g_j(s,t,m,n)H_j\left(s,t,u^{v_j}(s,t)\right) + w_j(s,t,m,n)\right], \end{aligned} \tag{65}$$

then, for $(m,n) \in \Omega$, we have

$$u(m,n) \le \left\{a(m,n) + b(m,n)\frac{\widehat{J}(M,N)}{1-\widehat{\lambda}(M,N)}\widehat{C}(m,n)\right\}^{1/p}, \tag{66}$$

provided that $\widehat{\lambda}(M,N) < 1$, where

$$\begin{aligned} \widehat{\varphi}(m,n) &= \prod_{s=m_0}^{m-1}\left[1+\varphi(s,n)\right], \\ b(m,n) &= \max\left\{b_1(m,n), b_2(m,n)\right\}, \\ \widehat{C}(m,n) &= \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widehat{B}(s,t,m,n)\right\}, \\ \widehat{B}(s,t,m,n) &= \sum_{i=1}^{l_1}\left[\widehat{c}_i(s,t,m,n)\frac{q_i}{p}K_1^{(q_i-p)/p} + \widehat{d}_i(s,t,m,n)\frac{r_i}{p}K_2^{(r_i-p)/p}\right]b(s,t), \\ \widehat{J}(m,n) &= \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{\widehat{c}_i(s,t,m,n)\left[\frac{q_i}{p}K_1^{(q_i-p)/p}a(s,t) + \frac{p-q_i}{p}K_1^{q_i/p}\right]\right. \\ &\quad \left. + \widehat{d}_i(s,t,m,n)\left[\frac{r_i}{p}K_2^{(r_i-p)/p}a(s,t) + \frac{p-r_i}{p}K_2^{r_i/p}\right] + e_i(s,t,m,n)\right\} \\ &\quad + \sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_j(s,t,m,n)H_j\left[s,t,(\widehat{\varphi}(s,t))^{h_j/p}\left(\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t) + \frac{p-h_j}{p}K_3^{h_j/p}\right)\right]\right. \\ &\quad \left. + g_j(s,t,m,n)H_j\left[s,t,(\widehat{\varphi}(s,t))^{v_j/p}\left(\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t) + \frac{p-v_j}{p}K_4^{v_j/p}\right)\right] + w_j(s,t,m,n)\right\}, \end{aligned}$$

$$\widehat{\lambda}(m,n)=\sum_{j=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\widehat{f}_j(s,t,m,n)\frac{h_j}{p}K_3^{(h_j-p)/p}+\widehat{g}_j(s,t,m,n)\frac{v_j}{p}K_4^{(v_j-p)/p}\right]b(s,t)\widehat{C}(s,t),$$

$$\widetilde{c}_i(s,t,m,n)=c_i(s,t,m,n)(\widehat{\varphi}(s,t))^{q_i/p},$$

$$\widetilde{d}_i(s,t,m,n)=d_i(s,t,m,n)(\widehat{\varphi}(s,t))^{r_i/p},\quad i=1,2,\ldots,l_1,$$

$$\widehat{f}_j(s,t,m,n)=f_j(s,t,m,n)(\widehat{\varphi}(s,t))^{h_j/p}L_j\left[s,t,(\widehat{\varphi}(s,t))^{h_j/p}\left(\frac{h_j}{p}K_3^{(h_j-p)/p}a(s,t)+\frac{p-h_j}{p}K_3^{h_j/p}\right)\right],$$

$$\widehat{g}_j(s,t,m,n)=g_j(s,t,m,n)(\widehat{\varphi}(s,t))^{v_j/p}L_j\left[s,t,(\widehat{\varphi}(s,t))^{v_j/p}\left(\frac{v_j}{p}K_4^{(v_j-p)/p}a(s,t)+\frac{p-v_j}{p}K_4^{v_j/p}\right)\right],$$

$$j=1,2,\ldots,l_2. \tag{67}$$

The proof for Theorem 9 is similar to the combination of Theorems 7 and 8, and we omit the details here.

3. Applications

In this section, we will present some applications for the established results to study boundedness, uniqueness, and continuous dependence of solutions of certain difference equations.

Consider the following Volterra-Fredholm sum-difference equations:

$$\begin{aligned}u^p(m,n)&=a(m,n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[C(s,t,m,n,u(s,t))\\&\quad+D(s,t,m,n,u(s,t))+E(s,t,m,n)]\\&\quad+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[F(s,t,m,n,u(s,t))\\&\quad+G(s,t,m,n,u(s,t))+W(s,t,m,n)],\end{aligned}\tag{68}$$

where $u(m,n), a(m,n) \in \wp(\Omega)$, $p \geq 1$ is an odd number, $C, D, F, G : \Omega^2 \times R \to R$, $E, W \in \wp(\Omega^2)$.

Theorem 10. *Assume that functions C, D, E, F, G, W in equation (68) satisfy the following conditions:*

$$\begin{aligned}|C(s,t,m,n,u_1)|&\leq c_1(s,t,m,n)|u_1^q|,\\|D(s,t,m,n,u_1)|&\leq d_1(s,t,m,n)|u_1^r|,\\|E(s,t,m,n)|&\leq e_1(s,t,m,n),\\|F(s,t,m,n,u_1)|&\leq f_1(s,t,m,n)|u_1^h|,\\|G(s,t,m,n,u_1)|&\leq g_1(s,t,m,n)|u_1^v|,\\|W(s,t,m,n)|&\leq w_1(s,t,m,n)\end{aligned}\tag{69}$$

for $(m,n) \in \Omega$, $u_1 \in R$, where q, r, h, v are nonnegative constants satisfying $p \geq q > 0$, $p \geq r > 0$, $p \geq h > 0$, $p \geq v > 0$, $c_1, d_1, e_1, f_1, g_1, w_1 \in \wp_+(\Omega^2)$ which are nondecreasing in the last two variables; then one has

$$|u(m,n)|\leq\left\{|a(m,n)|+\frac{J_1(M,N)}{1-\lambda_1(M,N)}C_1(m,n)\right\}^{1/p},\tag{70}$$

provided that $\lambda_1(M,N) < 1$, where

$$C_1(m,n)=\exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_1(s,t,m,n)\right\},$$

$$B_1(s,t,m,n)=c_1(s,t,m,n)\frac{q}{p}K_1^{(q-p)/p}+d_1(s,t,m,n)\cdot\frac{r}{p}K_2^{(r-p)/p},$$

$$\begin{aligned}J_1(m,n)&=\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left\{c_1(s,t,m,n)\cdot\left[\frac{q}{p}K_1^{(q-p)/p}|a(s,t)|+\frac{p-q}{p}K_1^{q/p}\right]\right.\\&\quad+d_1(s,t,m,n)\left[\frac{r}{p}K_2^{(r-p)/p}|a(s,t)|+\frac{p-r}{p}K_2^{r/p}\right]\\&\quad\left.+e_1(s,t,m,n)\right\}+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left\{f_1(s,t,m,n)\cdot\left[\frac{h}{p}K_3^{(h-p)/p}|a(s,t)|+\frac{p-h}{p}K_3^{h/p}\right]\right.\\&\quad+g_1(s,t,m,n)\left[\frac{v}{p}K_4^{(v-p)/p}|a(s,t)|+\frac{p-v}{p}K_4^{v/p}\right]\\&\quad\left.+w_1(s,t,m,n)\right\},\end{aligned}$$

$$\lambda_1(m,n) = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_1(s,t,m,n)\frac{h}{p}K_3^{(h-p)/p} + g_1(s,t,m,n)\frac{\nu}{p}K_4^{(\nu-p)/p}\right]C_1(s,t). \quad (71)$$

Proof. Using conditions (69) to (68), we have

$$\begin{aligned} \left|u^p(m,n)\right| &\le |a(m,n)| + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[|C(s,t,m,n,u(s,t))| + |D(s,t,m,n,u(s,t))| + |E(s,t,m,n)|] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[|F(s,t,m,n,u(s,t))| + |G(s,t,m,n,u(s,t))| + |W(s,t,m,n)|] \le |a(m,n)| \\ &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[c_1(s,t,m,n)\left|u^q(s,t)\right| + d_1(s,t,m,n)\left|u^r(s,t)\right| + e_1(s,t,m,n)] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_1(s,t,m,n)\left|u^h(s,t)\right| + g_1(s,t,m,n)\left|u^\nu(s,t)\right| + w_1(s,t,m,n)\right]. \end{aligned} \quad (72)$$

Then a suitable application of Theorem 4 (with $l_1 = l_2 = 1$) to (72) yields the desired result.

The following theorem deals with the uniqueness of the solutions of (68). □

Theorem 11. *Supposing that*

$$\begin{aligned} &\left|C(s,t,m,n,u_1) - C(s,t,m,n,u_2)\right| \le c_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|F(s,t,m,n,u_1) - F(s,t,m,n,u_2)\right| \le f_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|D(s,t,m,n,u_1) - D(s,t,m,n,u_2)\right| \le d_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|G(s,t,m,n,u_1) - G(s,t,m,n,u_2)\right| \le g_1(s,t,m,n)\left|u_1^p - u_2^p\right| \end{aligned} \quad (73)$$

hold for $u_1, u_2 \in R$, where $c_1, d_1, f_1, g_1 \in \wp_+(\Omega^2)$ are nondecreasing in the last two variables,

$$\lambda(M,N) = \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[f_1(s,t,M,N) + g_1(s,t,M,N)]C(s,t) < 1, \quad (74)$$

$$B(s,t,m,n) = c_1(s,t,m,n) + d_1(s,t,m,n),$$

$$C(s,t) = \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B(s,t,m,n)\right\},$$

then (68) has at most one solution.

Proof. Assume that $u(m,n), \overline{u}(m,n)$ are two solutions of (68). Then

$$\begin{aligned} &\left|u^p(m,n) - \overline{u}^p(m,n)\right| \\ &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[|C(s,t,m,n,u(s,t)) - C(s,t,m,n,\overline{u}(s,t))| + |D(s,t,m,n,u(s,t)) - D(s,t,m,n,\overline{u}(s,t))|] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[|F(s,t,m,n,u(s,t)) - F(s,t,m,n,\overline{u}(s,t))| + |G(s,t,m,n,u(s,t)) - G(s,t,m,n,\overline{u}(s,t))|] \\ &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}[c_1(s,t,m,n) + d_1(s,t,m,n)]\left|u^p(s,t) - \overline{u}^p(s,t)\right| + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}[f_1(s,t,m,n) + g_1(s,t,m,n)] \\ &\cdot \left|u^p(s,t) - \overline{u}^p(s,t)\right|. \end{aligned} \quad (75)$$

Treat $|u^p(m,n) - \overline{u}^p(m,n)|$ as one variable, and a suitable application of Corollary 6 yields $|u^p(m,n) - \overline{u}^p(m,n)| \le 0$, which implies that $u^p(m,n) \equiv \overline{u}^p(m,n)$. Since p is an odd number, then we have $u^p(m,n) = \overline{u}^p(m,n)$, and the proof is complete. □

Finally we study the continuous dependence of the solutions of (68) on functions a, C, D, E, F, G, W. For this, we consider the following variation of (68):

$$\begin{aligned} \widetilde{u}^p(m,n) &= \widetilde{a}(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\widetilde{C}(s,t,m,n,\widetilde{u}(s,t)) + \widetilde{D}(s,t,m,n,\widetilde{u}(s,t)) + \widetilde{E}(s,t,m,n)\right] \\ &+ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\widetilde{F}(s,t,m,n,\widetilde{u}(s,t)) + \widetilde{G}(s,t,m,n,\widetilde{u}(s,t)) + \widetilde{W}(s,t,m,n)\right], \end{aligned} \quad (76)$$

where $\widetilde{C}, \widetilde{D}, \widetilde{F}, \widetilde{G} : \Omega^2 \times R \to R$, $\widetilde{E}, \widetilde{W} \in \wp(\Omega^2)$ and $p \geq 1$ is an odd number.

Theorem 12. *Consider (68) and (76). If*

$$\begin{aligned} &\left|C(s,t,m,n,u_1(s,t)) - C(s,t,m,n,u_2(s,t))\right| \\ &\quad \leq c_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|D(s,t,m,n,u_1(s,t)) - D(s,t,m,n,u_2(s,t))\right| \\ &\quad \leq d_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|F(s,t,m,n,u_1(s,t)) - F(s,t,m,n,u_2(s,t))\right| \\ &\quad \leq f_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \\ &\left|G(s,t,m,n,u_1(s,t)) - G(s,t,m,n,u_2(s,t))\right| \\ &\quad \leq g_1(s,t,m,n)\left|u_1^p - u_2^p\right|, \end{aligned} \tag{77}$$

hold for $u_1, u_2 \in R$, where $c_1, d_1, f_1, g_1 \in \wp_+(\Omega^2)$, and are nondecreasing in the last two variables, furthermore, for all solution $\widetilde{u}$ of (76), the following conditions hold for $(m,n) \in \Omega$:

$$\begin{aligned} \left|a(m,n) - \widetilde{a}(m,n)\right| &\leq \frac{\varepsilon}{4}, \\ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|E(s,t,m,n) - \widetilde{E}(s,t,m,n)\right| &\leq \frac{\varepsilon}{8}, \\ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left|W(s,t,m,n) - \widetilde{W}(s,t,m,n)\right| &\leq \frac{\varepsilon}{8}, \\ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F(s,t,m,n,\widetilde{u}) - \widetilde{F}(s,t,m,n,\widetilde{u})\right| &\leq \frac{\varepsilon}{8}, \\ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left|C(s,t,m,n,\widetilde{u}) - \widetilde{C}(s,t,m,n,\widetilde{u})\right| &\leq \frac{\varepsilon}{8}, \\ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|D(s,t,m,n,\widetilde{u}) - \widetilde{D}(s,t,m,n,\widetilde{u})\right| &\leq \frac{\varepsilon}{8}, \\ \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left|G(s,t,m,n,\widetilde{u}) - \widetilde{G}(s,t,m,n,\widetilde{u})\right| &\leq \frac{\varepsilon}{8}, \end{aligned} \tag{78}$$

where $\varepsilon > 0$ is an arbitrary constant. Then

$$\begin{aligned} &\left|u^p(m,n) - \widetilde{u}^p(m,n)\right| \\ &\quad \leq \varepsilon\left[1 + \frac{J_2(M,N)}{1 - \lambda_2(M,N)}C_2(m,n)\right], \end{aligned} \tag{79}$$

where $\lambda_2(M,N) < 1$, and

$$\begin{aligned} C_2(m,n) &= \exp\left\{\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}B_2(s,t,m,n)\right\}, \\ B_2(s,t,m,n) &= \left[c_1(s,t,m,n) + d_1(s,t,m,n)\right], \\ \lambda_2(m,n) &= \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_1(s,t,m,n) + g_1(s,t,m,n)\right]C_2(s,t), \\ J_2(m,n) &= \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_1(s,t,m,n) + d_1(s,t,m,n)\right] \\ &\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_1(s,t,m,n) + g_1(s,t,m,n)\right] \end{aligned} \tag{80}$$

for $(m,n) \in \Omega$. That is, u^p depends continuously on the functions a, C, D, E, F, G, W.

Proof. Let $u(m,n)$ and $\widetilde{u}(m,n)$ be solutions of (68) and (76), respectively. Then $u(m,n)$ satisfies (68) and $\widetilde{u}(m,n)$ satisfies (76). Hence

$$\begin{aligned} &\left|u^p(m,n) - \widetilde{u}^p(m,n)\right| \leq \left|a(m,n) - \widetilde{a}(m,n)\right| \\ &\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\left|C(s,t,m,n,u(s,t)) - \widetilde{C}(s,t,m,n,\widetilde{u}(s,t))\right|\right. \\ &\quad + \left|D(s,t,m,n,u(s,t)) - \widetilde{D}(s,t,m,n,\widetilde{u}(s,t))\right| \\ &\quad + \left.\left|E(s,t,m,n) - \widetilde{E}(s,t,m,n)\right|\right] \\ &\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\left|F(s,t,m,n,u(s,t)) - \widetilde{F}(s,t,m,n,\widetilde{u}(s,t))\right|\right. \\ &\quad + \left|G(s,t,m,n,u(s,t)) - \widetilde{G}(s,t,m,n,\widetilde{u}(s,t))\right| \\ &\quad + \left.\left|W(s,t,m,n) - \widetilde{W}(s,t,m,n)\right|\right] \leq \left|a(m,n) - \widetilde{a}(m,n)\right| \\ &\quad + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[\left|C(s,t,m,n,u(s,t)) - C(s,t,m,n,\widetilde{u}(s,t))\right|\right. \\ &\quad + \left|C(s,t,m,n,\widetilde{u}(s,t)) - \widetilde{C}(s,t,m,n,\widetilde{u}(s,t))\right| \\ &\quad + \left|D(s,t,m,n,u(s,t)) - D(s,t,m,n,\widetilde{u}(s,t))\right| \\ &\quad + \left|D(s,t,m,n,\widetilde{u}(s,t)) - \widetilde{D}(s,t,m,n,\widetilde{u}(s,t))\right| \\ &\quad + \left.\left|E(s,t,m,n) - \widetilde{E}(s,t,m,n)\right|\right] \\ &\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[\left|F(s,t,m,n,u(s,t)) - F(s,t,m,n,\widetilde{u}(s,t))\right|\right. \\ &\quad + \left|F(s,t,m,n,\widetilde{u}(s,t)) - \widetilde{F}(s,t,m,n,\widetilde{u}(s,t))\right| \end{aligned}$$

$$\begin{aligned}&+\left|G(s,t,m,n,u(s,t))-G(s,t,m,n,\widetilde{u}(s,t))\right|\\&+\left|G(s,t,m,n,\widetilde{u}(s,t))-\widetilde{G}(s,t,m,n,\widetilde{u}(s,t))\right|\\&+\left|W(s,t,m,n)-\widetilde{W}(s,t,m,n)\right|\Big]\leq\varepsilon\\&+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left[c_1(s,t,m,n)+d_1(s,t,m,n)\right]\left[u^p-\widetilde{u}^p\right]\\&+\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[f_1(s,t,m,n)+g_1(s,t,m,n)\right]\left|u^p-\widetilde{u}^p\right|.\end{aligned} \tag{81}$$

Treat $|u^p(m,n)-\widetilde{u}^p(m,n)|$ as one variable, and a suitable application of Corollary 6 (with $l_1=l_2=1$) yields the desired result (79). Hence u^p depends continuously on a, C, D, E, F, G, W. □

4. Conclusions

The author carried out some new Volterra-Fredholm-type discrete inequalities involving four iterated infinite sums and their corresponding applications. The results are more effective to qualitative analysis of solutions for sum-difference equations, such as the boundedness, uniqueness, and continuous dependence on solutions.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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