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Research Article

Multiplicity of Positive Solutions for Fractional Differential Equation with p-Laplacian Boundary Value Problems

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We investigate the existence of multiple positive solutions of fractional differential equations with p-Laplacian operator $D_{a^+}^{\beta}(\phi_p(D_{a^+}^{\alpha}u(t))) = f(t,u(t))$, $a < t < b$, $u^{(j)}(a) = 0$, $j = 0,1,2,\dots,n-2$, $u^{(\alpha_1)}(b) = \xi u^{(\alpha_1)}(\eta)$, $\phi_p(D_{a^+}^{\alpha}u(a)) = 0 = D_{a^+}^{\beta_1}(\phi_p(D_{a^+}^{\alpha}u(b)))$, where $\beta \in (1,2]$, $\alpha \in (n-1,n]$, $n \geq 3$, $\xi \in (0,\infty)$, $\eta \in (a,b)$, $\beta_1 \in (0,1]$, $\alpha_1 \in \{1,2,\dots,\alpha-2\}$ is a fixed integer, and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $(1/p) + (1/q) = 1$, by applying Leggett–Williams fixed point theorems and fixed point index theory.

1. Introduction

The goal of differential equations is to understand the phenomena of nature by developing mathematical models. Fractional calculus is the field of mathematical analysis, which deals with investigation and applications of derivatives and integrals of an arbitrary order. Among all, a class of differential equations governed by nonlinear differential operators appears frequently and generated a great deal of interest in studying such problems. In this theory, the most applicable operator is the classical p-Laplacian, given by $\phi_p(u) = |u|^{p-2}$, $p > 1$.

The positive solutions of boundary value problems associated with ordinary differential equations were studied by many authors [1–4] and extended to p-Laplacian boundary value problems [5–8]. Later, these results are further extended to fractional order boundary value problems [9–12] by applying various fixed point theorems on cones. Recently, researchers are concentrating on the theory of fractional order boundary value problems associated with p-Laplacian operator [13–19]. The above few papers motivated this work.

In this paper, we are concerned with the existence of multiple positive solutions for the fractional differential equation with p-Laplacian operator

$$D_{a^+}^{\beta}\left(\phi_p\left(D_{a^+}^{\alpha}u(t)\right)\right) = f(t,u(t)), \quad a < t < b, \tag{1}$$

with the boundary conditions

$$\begin{gathered} u^{(j)}(a) = 0, \quad j = 0,1,2,\dots,n-2, \\ u^{(\alpha_1)}(b) = \xi u^{(\alpha_1)}(\eta), \\ \phi_p\left(D_{a^+}^{\alpha}u(a)\right) = D_{a^+}^{\beta_1}\left(\phi_p\left(D_{a^+}^{\alpha}u(b)\right)\right) = 0, \end{gathered} \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $(1/p) + (1/q) = 1$, $\beta \in (1,2]$, $\alpha \in (n-1,n]$, $n \geq 3$, $\beta_1 \in (0,1]$, $\alpha_1 \in [1,\alpha-2]$ is a fixed integer, and $\xi \in (0,\infty)$, $\eta \in (a,b)$ are constants. The function $f : [a,b] \times R^+ \to R^+$ is continuous and $D_{a^+}^{\alpha}$, $D_{a^+}^{\beta}$, $D_{a^+}^{\beta_1}$ are the standard Riemann-Liouville fractional order derivatives.

The rest of this paper is organized as follows. In Section 2, the Green functions for the homogeneous BVPs corresponding to (1)-(2) are constructed and the bounds for the Green functions are estimated. In Section 3, sufficient conditions for the existence of at least two or at least three positive solutions are established, by using fixed point index theory and Leggett-Williams fixed point theorems. In Section 4, as an application, an example is presented to illustrate our main result.

2. Green's Function and Bounds

In this section, we construct Green's function for the homogeneous boundary value problem and estimate bounds for Green's function that will be used to prove our main theorems.

Let $G(t,s)$ be Green's function for the homogeneous BVP

$$\begin{aligned} -D_{a^+}^{\alpha} u(t) &= 0, \quad a < t < b, \\ u^{(j)}(a) &= 0, \quad 0 \le j \le n-2, \\ u^{(\alpha_1)}(b) &= \xi u^{(\alpha_1)}(\eta). \end{aligned} \tag{3}$$

Lemma 1. *Let* $d = (b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1} > 0$. *If* $y \in C[a,b]$, *then the fractional order BVP*

$$D_{a^+}^{\alpha} u(t) + y(t) = 0, \quad a < t < b, \tag{4}$$

$$\begin{aligned} u^{(j)}(a) &= 0, \quad 0 \le j \le n-2, \\ u^{(\alpha_1)}(b) &= \xi u^{(\alpha_1)}(\eta) \end{aligned} \tag{5}$$

has a unique solution, $u(t) = \int_a^b G(t,s) y(s) ds$, *where*

$$G(t,s) = G_1(t,s) + \frac{\xi (t-a)^{\alpha-1}}{d} G_2(\eta,s); \tag{6}$$

here

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}, & a \le t \le s \le b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \end{cases} \\ G_2(\eta,s) &= \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} \dfrac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}, & \eta \le s \le b, \\ \dfrac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (\eta-s)^{\alpha-\alpha_1-1}, & a \le s \le \eta. \end{cases} \end{aligned} \tag{7}$$

Proof. Assume that $u \in C^{[\alpha]+1}[a,b]$ is a solution of fractional order BVP (4)-(5) and is uniquely expressed as $I_{a^+}^{\alpha} D_{a^+}^{\alpha} u(t) = -I_{a^+}^{\alpha} y(t)$, so that

$$\begin{aligned} u(t) = {} & \frac{-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds + c_1 (t-a)^{\alpha-1} \\ & + c_2 (t-a)^{\alpha-2} + \cdots + c_n (t-a)^{\alpha-n}. \end{aligned} \tag{8}$$

From $u^{(j)}(a) = 0$, $0 \le j \le n-2$, we have $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Then

$$\begin{aligned} u(t) &= \frac{-1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds + c_1 (t-a)^{\alpha-1}, \\ u^{(\alpha_1)}(t) &= c_1 \prod_{i=1}^{\alpha_1} (\alpha-i)(t-a)^{\alpha-\alpha_1-1} \\ & \quad - \prod_{i=1}^{\alpha_1} (\alpha-i) \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-\alpha_1-1} y(s) ds. \end{aligned} \tag{9}$$

From $u^{(\alpha_1)}(b) = \xi u^{(\alpha_1)}(\eta)$, we have

$$\begin{aligned} & c_1 \prod_{i=1}^{\alpha_1} (\alpha-i)(b-a)^{\alpha-\alpha_1-1} - \prod_{i=1}^{\alpha_1} (\alpha-i) \frac{1}{\Gamma(\alpha)} \\ & \quad \cdot \int_a^b (b-s)^{\alpha-\alpha_1-1} y(s) ds \\ & \quad = \xi \left[c_1 \prod_{i=1}^{\alpha_1} (\alpha-i)(\eta-a)^{\alpha-\alpha_1-1} \right. \\ & \quad \left. - \prod_{i=1}^{\alpha_1} (\alpha-i) \frac{1}{\Gamma(\alpha)} \int_a^\eta (\eta-s)^{\alpha-\alpha_1-1} y(s) ds \right]. \end{aligned} \tag{10}$$

Therefore

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha)\left[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ & \quad \times \left[\int_a^b (b-s)^{\alpha-\alpha_1-1} y(s) ds \right. \\ & \quad \left. - \xi \int_a^\eta (\eta-s)^{\alpha-\alpha_1-1} y(s) ds \right] = \frac{1}{\Gamma(\alpha)\left[(b-a)^{\alpha-\alpha_1-1}\right]} \\ & \quad \cdot \int_a^b (b-s)^{\alpha-\alpha_1-1} y(s) ds \\ & \quad + \frac{\xi(\eta-a)^{\alpha-\alpha_1-1}}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}\left[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ & \quad \cdot \int_a^b (b-s)^{\alpha-\alpha_1-1} y(s) ds \\ & \quad - \frac{\xi}{\Gamma(\alpha)\left[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ & \quad \cdot \int_a^\eta (\eta-s)^{\alpha-\alpha_1-1} y(s) ds = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}} \\ & \quad \cdot \int_a^b (b-s)^{\alpha-\alpha_1-1} y(s) ds \\ & \quad + \frac{\xi}{\left[(b-a)^{\alpha-\alpha_1-1} - \xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ & \quad \cdot \int_a^b G_2(\eta,s) y(s) ds. \end{aligned} \tag{11}$$

Thus, the unique solution of (4)-(5) is

$$\begin{aligned} u(t) &= \frac{-1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} y(s)\,ds \\ &+ \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-\alpha_1-1}}\int_a^b (b-s)^{\alpha-\alpha_1-1} y(s)\,ds \\ &+ \frac{\xi(t-a)^{\alpha-1}}{\left[(b-a)^{\alpha-\alpha_1-1}-\xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ &\cdot \int_a^b G_2(\eta,s) y(s)\,ds = \int_a^b G_1(t,s) y(s)\,ds \\ &+ \frac{\xi(t-a)^{\alpha-1}}{\left[(b-a)^{\alpha-\alpha_1-1}-\xi(\eta-a)^{\alpha-\alpha_1-1}\right]} \\ &\cdot \int_a^b G_2(\eta,s) y(s)\,ds = \int_a^b G(t,s) y(s)\,ds, \end{aligned} \tag{12}$$

where $G(t,s)$ is given in (6). □

Lemma 2. *If $h \in C[a,b]$, then the fractional order differential equation*

$$D_{a^+}^{\beta}\left(\phi_p\left(D_{a^+}^{\alpha}u(t)\right)\right) = h(t), \quad a < t < b, \tag{13}$$

satisfying (5) and

$$\begin{aligned} \phi_p\left(D_{a^+}^{\alpha}u(a)\right) &= 0, \\ D_{a^+}^{\beta_1}\left(\phi_p\left(D_{a^+}^{\alpha}u(b)\right)\right) &= 0 \end{aligned} \tag{14}$$

has a unique solution,

$$u(t) = \int_a^b G(t,s)\,\phi_q\left(\int_a^b H(s,\tau)\,h(\tau)\,d\tau\right)ds, \tag{15}$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)} \cdot \begin{cases} \dfrac{(t-a)^{\beta-1}(b-s)^{\beta-\beta_1-1}}{(b-a)^{\beta-\beta_1-1}}, & a \le t \le s \le b, \\ \dfrac{(t-a)^{\beta-1}(b-s)^{\beta-\beta_1-1}}{(b-a)^{\beta-\beta_1-1}} - (t-s)^{\beta-1}, & a \le s \le t \le b. \end{cases} \tag{16}$$

Here $H(t,s)$ is Green's function for

$$\begin{aligned} -D_{a^+}^{\beta}\left(\phi_p(x(t))\right) &= 0, \quad a < t < b, \\ \phi_p(x(a)) &= 0, \\ D_{a^+}^{\beta_1}\left(\phi_p(x(b))\right) &= 0. \end{aligned} \tag{17}$$

Proof. An equivalent integral equation for (13) is given by

$$\begin{aligned} \phi_p\left(D_{a^+}^{\alpha}u(t)\right) &= \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1} h(\tau)\,d\tau \\ &+ c_1(t-a)^{\beta-1} + c_2(t-a)^{\beta-2}. \end{aligned} \tag{18}$$

By (14), one can determine $c_2 = 0$ and $c_1 = (-1/(b-a)^{\beta-\beta_1-1})(1/\Gamma(\beta))\int_a^b (b-\tau)^{\beta-\beta_1-1}h(\tau)d\tau$.

Thus, the unique solution of (13), (2) is

$$\begin{aligned} &\phi_p\left(D_{a^+}^{\alpha}u(t)\right) \\ &= \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1} h(\tau)\,d\tau \\ &- \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-\beta_1-1}}\frac{1}{\Gamma(\beta)}\int_a^b (b-\tau)^{\beta-\beta_1-1} h(\tau)\,d\tau \\ &= -\int_a^b H(t,\tau)\,h(\tau)\,d\tau. \end{aligned} \tag{19}$$

Therefore, $\phi_p^{-1}(\phi_p(D_{a^+}^{\alpha}u(t))) = -\phi_p^{-1}(\int_a^b H(t,\tau)h(\tau)d\tau)$. Consequently, $D_{a^+}^{\alpha}u(t) + \phi_q(\int_a^b H(t,\tau)h(\tau)d\tau) = 0$.

Hence,

$$u(t) = \int_a^b G(t,s)\,\phi_q\left(\int_a^b H(s,\tau)\,h(\tau)\,d\tau\right)ds. \tag{20}$$

□

Lemma 3. *Green's function $G(t,s)$ satisfies the following inequalities:*

(i) $G(t,s) \le G(b,s)$, *for all* $(t,s) \in [a,b]\times[a,b]$,

(ii) $G(t,s) \ge ((\eta-a)/(b-a))^{\alpha-1}G(b,s)$, *for all* $(t,s) \in [\eta,b]\times[a,b]$.

Proof. Consider Green's function given by (6).

Let $a \le t \le s \le b$. Then, we have

$$\begin{aligned} \frac{\partial}{\partial t}G_1(t,s) &= \frac{1}{\Gamma(\alpha-1)}\left[\frac{(t-a)^{\alpha-2}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\right] \\ &\ge \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left[\frac{(1-a)^{\alpha-2}b^{\alpha-\alpha_1+1}\left(1-(1-\alpha_1)s+O\left(s^2\right)\right)}{(b-a)^{\alpha-\alpha_1-1}}\right] \\ &\cdot (1-s)^{\alpha-2} \ge 0. \end{aligned} \tag{21}$$

Let $a \leq s \leq t \leq b$. Then, we have

$$\begin{aligned}\frac{\partial}{\partial t}G_1(t,s) &= \frac{1}{\Gamma(\alpha-1)}\left[\frac{(t-a)^{\alpha-2}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-2}\right] \\ &\geq \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\left[\frac{(1-a)^{\alpha-2}b^{\alpha-\alpha_1+1}\left(1-(1-\alpha_1)s+O\left(s^2\right)\right)-(b-a)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\right](1-s)^{\alpha-2} \geq 0.\end{aligned} \tag{22}$$

Therefore $G_1(t,s)$ is increasing in t, which implies $G_1(t,s) \leq G_1(b,s)$.

Now we prove

$$G_2(\eta,s) \geq 0, \quad s \in [a,b]. \tag{23}$$

In fact, if $s \geq \eta$, obviously, (23) holds. If $s \leq \eta$, one has

$$\begin{aligned}G_2(\eta,s) &= \frac{1}{\Gamma(\alpha)}\left[\frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (\eta-s)^{\alpha-\alpha_1-1}\right] \\ &\geq \frac{1}{\Gamma(\alpha)}\left[\frac{(\eta-a)^{\alpha-\alpha_1-1}(b-s)^{\alpha-\alpha_1-1}-(\eta-\eta s)^{\alpha-\alpha_1-1}(b-a)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\right] \geq 0.\end{aligned} \tag{24}$$

That implies that (23) is also true. Therefore, by (6), (21), and (23), we find

$$\begin{aligned}\frac{\partial}{\partial t}G(t,s) &= \frac{\partial}{\partial t}G_1(t,s) + \frac{(\alpha-1)\xi(t-a)^{\alpha-2}}{d}G_2(\eta,s) \\ &\geq 0.\end{aligned} \tag{25}$$

Therefore $G(t,s)$ is increasing with respect to $t \in [a,b]$. Hence the inequality (i) is proved. Now, we establish the inequality (ii).

On the other hand, if $a \leq s \leq t \leq b$, we have

$$\begin{aligned}&G_1(t,s) \\ &\quad= \frac{1}{\Gamma(\alpha)}\left[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (t-s)^{\alpha-1}\right] \\ &\quad\geq \frac{1}{\Gamma(\alpha)}\left(\frac{t-a}{b-a}\right)^{\alpha-1} \\ &\quad\cdot\left[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}} - (b-s)^{\alpha-1}\right] \\ &\quad\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G_1(b,s).\end{aligned} \tag{26}$$

If $a \leq t \leq s \leq b$, we have

$$\begin{aligned}G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\left[\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\right] \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{t-a}{b-a}\right)^{\alpha-1}\left[\frac{(b-a)^{\alpha-1}(b-s)^{\alpha-\alpha_1-1}}{(b-a)^{\alpha-\alpha_1-1}}\right] \\ &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G_1(b,s).\end{aligned} \tag{27}$$

Therefore

$$G_1(t,s) \geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G_1(b,s). \tag{28}$$

From (6) and (28) we have

$$\begin{aligned}G(t,s) &= G_1(t,s) + \frac{\xi(t-a)^{\alpha-1}}{d}G_2(\eta,s) \\ &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G_1(t,s) + \frac{\xi(t-a)^{\alpha-1}}{d}G_2(\eta,s) \\ &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G(b,s).\end{aligned} \tag{29}$$

Therefore

$$\begin{aligned}G(t,s) &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}G(b,s) \\ &\qquad \forall (t,s) \in [\eta,b]\times[a,b].\end{aligned} \tag{30}$$

Hence the inequality (ii) is proved. □

Lemma 4. *Green's function $H(t,s)$ satisfies the following inequalities:*

(i) $H(t,s) \leq H(s,s)$, *for all* $(t,s) \in [a,b] \times [a,b]$,

(ii) $H(t,s) \geq \gamma H(s,s)$, *for all* $(t,s) \in I \times [a,b]$,

where $I = [(3a+b)/4, (a+3b)/4]$ *and* $\gamma = (1/4)^{\beta-1}$.

The method of proof is similar to that [20], and we omit it here.

Theorem 5 (Leggett-Williams [3]). *Let $T : \overline{P}_c \rightarrow \overline{P}_c$ be completely continuous and let ϕ be a nonnegative continuous concave functional on P such that $\phi(y) \leq \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist a, b, c, and d with $0 < a < b < d \leq c$ such that*

$(A1)$ *$\{y \in P(\phi,b,d) : \phi(y) > b\} \neq \emptyset$ and $\phi(Ty) > b$ for $y \in P(\phi,b,d)$,*

$(A2)$ *$\|Ty\| < a$ for $\|y\| \leq a$,*

$(A3)$ *$\phi(Ty) > b$ for $y \in P(\phi,b,c)$ with $\|Ty\| > d$. Then T has at least three fixed points y_1, y_2, and y_3 in $\overline{P}_c$ satisfying $\|y_1\| < a$, $b < \phi(y_2)$, $\|y_3\| > a$, and $\phi(y_3) < b$.*

Theorem 6 (see [3]). *Let $T : \overline{P}_c \rightarrow P$ be a completely continuous operator and let ϕ be a nonnegative continuous concave functional on P such that $\phi(y) \leq \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist a, b, and c with $0 < a < b < c$ such that*

$(B1)$ *$\{y \in P(\phi,b,c) : \phi(y) > b\} \neq \emptyset$ and $\phi(Ty) > b$ for $y \in P(\phi,b,c)$,*

$(B2)$ *$\|Ty\| < a$ for $\|y\| \leq a$,*

$(B3)$ *$\phi(Ty) > (b/c)\|Ty\|$ for $y \in \overline{P}_c$ with $\|Ty\| > c$. Then T has at least two fixed points y_1 and y_2 in $\overline{P}_c$ satisfying $\|y_1\| < a$, $\|y_2\| > a$ and $\phi(y_2) < b$.*

Theorem 7 (see [21]). *Let P be a closed convex set in a Banach space E and let Ω be a bounded open set such that $\Omega_p := \Omega \cap P \neq \emptyset$. Let $T : \overline{\Omega}_p \rightarrow P$ be a compact map. Suppose that $x \neq Tx$ for all $x \in \partial_p$:*

(C1) *Existence: if $i(T,\Omega_p,P) \neq \emptyset$, then T has a fixed point in Ω_p.*

(C2) *Normalization: if $u \in \Omega_p$, then $i(\widehat{u},\Omega_p,P) = 1$, where $\widehat{u}(x) = u$ for $x \in \overline{\Omega}_p$.*

(C3) *Homotopy: let $v : [0,1] \times \overline{\Omega}_p \rightarrow P$ be a compact map such that $x \neq v(t,x)$ for $x \in \partial\Omega_p$ and $t \in [0,1]$. Then $i(v(0,\cdot),\Omega_p,P) = i(v(1,\cdot),\Omega_p,P)$.*

(C4) *Additivity: if $U1, U2$ are disjoint relatively open subsets of Ω_p such that $x \neq Tx$ for $x \in \overline{\Omega}_p \setminus (U1 \cup U2)$, then $i(T,\Omega_p,P) = i(T,U1,P) + i(T,U2,P)$, where $i(T,Uj,P) = i(T \mid \overline{u}j, Uj, P)$ $(j=1,2)$.*

Theorem 8 (see [22]). *Let P be a cone in a Banach space E. For $q > 0$, define $\Omega_q = \{x \in P : \|x\| < q\}$. Assume that $T : \overline{\Omega}_q \rightarrow P$ is a compact map such that $x \neq Tx$ for $x \in \partial\Omega_q$. Thus, one has the following conclusions:*

$(D1)$ *If $\|x\| < \|Tx\|$ for $x \in \partial\Omega_q$, then $i(T,\Omega_q,P) = 0$.*

$(D2)$ *If $\|x\| \geq \|Tx\|$ for $x \in \partial\Omega_q$, then $i(T,\Omega_q,P) = 1$.*

3. Main Results

In this section, the existence of at least two or at least three positive solutions for fractional differential equation with p-Laplacian operator BVP (1)-(2) is established by using fixed point index theory and Leggett-Williams fixed point theorems.

Let $E = \{u : u \in C[a,b]\}$ be the real Banach space equipped with the norm $\|u\| = \max_{t\in[a,b]}|u(t)|$. Define the cone $P \subset E$ by

$$P = \left\{u \in E : u(t) \geq 0,\ \forall t \in [a,b],\ \min_{t\in[\eta,b]} u(t) \geq \left(\frac{\eta - a}{b - a}\right)^{\alpha-1} \|u\|\right\}. \quad (31)$$

Let $T : P \rightarrow E$ be the operator defined by

$$Tu(t) = \int_a^b G(t,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds. \quad (32)$$

If $u \in P$ is a fixed point of T, then u satisfies (32) and hence u is a positive solution of p-Laplacian fractional order BVP (1)-(2).

Lemma 9. *The operator T defined by (32) is a self-map on P.*

Proof. Let $u \in P$. Clearly, $Tu(t) \geq 0$, for all $t \in [a,b]$ and

$$\begin{aligned} Tu(t) &= \int_a^b G(t,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds \\ &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds \end{aligned} \quad (33)$$

so that

$$\|Tu\| \leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds. \quad (34)$$

On the other hand, by Lemma 3, we have

$$\begin{aligned} \min_{t\in[\eta,b]} Tu(t) &= \min_{t\in[\eta,b]} \int_a^b G(t,s) \cdot \phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds \\ &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1} \int_a^b G(b,s) \cdot \phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds \\ &\geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1} \|Tu\|. \end{aligned} \quad (35)$$

Hence $Tu \in P$ and so $T : P \to P$. Standard argument involving the Arzela-Ascoli theorem shows that T is completely continuous. □

For convenience of the reader, we denote

$$A = \frac{2\left((b-a)/(\eta-a)\right)^{\alpha-1}}{\int_\eta^b G(\eta,s)\,\phi_q\left(\int_{\tau\in I}\gamma H(\tau,\tau)\,d\tau\right)ds},$$
$$B = \frac{1}{\int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right)ds},$$
$$f_0 = \lim_{u\to 0^+}\min_{t\in[a,b]}\frac{f(t,u)}{\phi_p(u)},$$
$$f_\infty = \lim_{u\to\infty}\max_{t\in[a,b]}\frac{f(t,u)}{\phi_p(u)}. \quad (36)$$

Theorem 10. *Let $f(t,u)$ be nonnegative continuous on $[a,b]\times[0,\infty)$. Assume that there exist constants a', b' with $b' > a' > 0$ such that the following conditions are satisfied:*

(*H1*) *$f(t,u(t)) \geq \phi_p(Ab')$ for all $(t,u) \in [\eta,b]\times[b', b'((b-a)/(\eta-a))^{2(\alpha-1)}]$.*

(*H2*) *$f(t,u(t)) < \phi_p(Ba')$ for all $(t,u)\in[a,b]\times[0,a']$.*

Then fractional order BVP (1)-(2) has at least two positive solutions u_1 and u_2 satisfying $\|u_1\| < a'$, $\min_{t\in[\eta,b]}u_2(t) < b'$, and $\|u_2\| > a'$.

Proof. Let $\theta : P \to [0,\infty)$ be the nonnegative continuous concave functional defined by $\theta(u) = \min_{t\in[\eta,b]}u(t)$, $u \in P$. Evidently, for each $u\in P$, we have $\theta(u)\leq\|u\|$.

It is easy to see that $T : \overline{P}_{b'((b-a)/(\eta-a))^{2(\alpha-1)}} \to P$ is completely continuous and $b'((b-a)/(\eta-a))^{2(\alpha-1)} > b' > a' > 0$. We choose $u(t) = b'((b-a)/(\eta-a))^{2(\alpha-1)}$; then

$$u \in P\left(\theta, b', b'\left(\frac{b-a}{\eta-a}\right)^{2(\alpha-1)}\right),$$
$$\theta(u) = b'\left(\frac{b-a}{\eta-a}\right)^{2(\alpha-1)} > b'. \quad (37)$$

So $\{u \in P(\theta, b', b'((b-a)/(\eta-a))^{2(\alpha-1)}) \mid \theta(u) > b'\} \neq \emptyset$. Hence, if $u \in P(\theta, b', b'((b-a)/(\eta-a))^{2(\alpha-1)})$, then $b' \leq u(t) \leq b'((b-a)/(\eta-a))^{2(\alpha-1)}$ for $t\in[\eta,b]$. Thus for $t\in[\eta,b]$, from assumption $(H1)$, we have

$$Tu(\eta)$$
$$= \int_a^b G(\eta,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds$$
$$\geq \int_\eta^b G(\eta,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds$$
$$\geq Ab'\int_\eta^b G(\eta,s)\,\phi_q\left(\int_{\tau\in I}\gamma H(\tau,\tau)\,d\tau\right)ds$$
$$= 2b'\left(\frac{b-a}{\eta-a}\right)^{\alpha-1} > b'\left(\frac{b-a}{\eta-a}\right)^{\alpha-1}. \quad (38)$$

Consequently,

$$\min_{t\in[\eta,b]}Tu(t) \geq \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}\|Tu\|$$
$$> \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}\times b'\left(\frac{b-a}{\eta-a}\right)^{\alpha-1} = b' \quad (39)$$

for $\eta\leq t\leq b$, $b' \leq u(t) \leq b'((b-a)/(\eta-a))^{2(\alpha-1)}$. That is,

$$\theta(Tu) > b', \quad \forall u \in P\left(\theta, b', b'\left(\frac{b-a}{\eta-a}\right)^{2(\alpha-1)}\right). \quad (40)$$

Therefore, condition $(B1)$ of Theorem 6 is satisfied. Now if $u\in\overline{P}_{a'}$, then $\|u\|\leq a'$. By assumption $(H2)$, we have

$$\|Tu\| = \max_{t\in[a,b]}|Tu(t)| = \max_{t\in[a,b]}\left[\int_a^b G(t,s)\,\phi_q\right.$$
$$\left.\cdot\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds\right] \leq \int_a^b G(b,s)$$
$$\cdot\,\phi_q\left(\int_a^b H(\tau,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds$$
$$< Ba'\int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right)ds = a', \quad (41)$$

which shows that $T : \overline{P}_{a'} \to P_{a'}$, that is, $\|Tu\| < a'$ for $u\in\overline{P}_{a'}$. This shows that condition $(B2)$ of Theorem 6 is satisfied. Finally, we show that $(B3)$ of Theorem 6 also holds. Assume that $u\in\overline{P}_{b'((b-a)/(\eta-a))^{2(\alpha-1)}}$ with $\|Tu\| > b'((b-a)/(\eta-a))^{2(\alpha-1)}$; then by the definition of cone P, we have

$$\theta(Tu) = \min_{t\in[\eta,b]}Tu(t) \geq \left(\frac{\eta-a}{b-a}\right)^{(\alpha-1)}\|Tu\|$$
$$> \left(\frac{\eta-a}{b-a}\right)^{2(\alpha-1)}\|Tu\|$$
$$= \frac{b'}{b'\left((b-a)/(\eta-a)\right)^{2(\alpha-1)}}\|Tu\|. \quad (42)$$

So condition $(B3)$ of Theorem 6 is satisfied. Thus using Theorem 6, T has at least two fixed points. Consequently, boundary value problem (1)-(2) has at least two positive solutions u_1 and u_2 in $\overline{P}_{b'((b-a)/(\eta-a))^{2(\alpha-1)}}$ satisfying

$$\|u_1\| < a',$$
$$\min_{t\in[\eta,b]}u_2(t) < b',$$
$$\|u_2\| > a'. \quad (43)$$

□

Theorem 11. *Let $f(t,u)$ be nonnegative continuous on $[a,b]\times[0,\infty)$. Assume that there exist constants a', b', c' with $((\eta-a)/(b-a))^{2(\alpha-1)}c' > b' > a' > 0$ such that*

(H3) $f(t,u(t)) < \phi_p(Ba')$ *for all* $(t,u)\in[a,b]\times[0,a']$,

(H4) $f(t,u(t)) \ge \phi_p(Ab')$ *for all* $(t,u)\in[\eta,b]\times[b',b'((b-a)/(\eta-a))^{2(\alpha-1)}]$,

(H5) $f(t,u(t)) \le \phi_p(Bc')$ *for all* $(t,u)\in[a,b]\times[0,c']$.

Then fractional order BVP (1)-(2) has at least three positive solutions u_1, u_2, and u_3 with $\|u_1\| < a'$, $\min_{t\in[\eta,b]}u_2(t) > b'$, $\|u_3\| > a'$, and $\min_{t\in[\eta,b]}u_3(t) < b'$.

Proof. If $u\in\overline{P}_{c'}$, then $\|u\|\le c'$. By assumption $(H5)$, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[a,b]}|Tu(t)| = \max_{t\in[a,b]}\left[\int_a^b G(t,s)\,\phi_q\right.\\ &\quad\cdot\left.\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds\right] \le \int_a^b G(b,s)\\ &\quad\cdot\phi_q\left(\int_a^b H(\tau,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds\\ &\le Bc'\int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right)ds = c'.\end{aligned}\tag{44}$$

This shows that $T:\overline{P}_{c'}\to\overline{P}_{c'}$. Using the same arguments as in the proof of Theorem 10, we can show that $T:\overline{P}_{c'}\to\overline{P}_{c'}$ is a completely continuous operator. It follows from conditions $(H3)$ and $(H4)$ in Theorem 11 that $c' > b'((b-a)/(\eta-a))^{2(\alpha-1)} > b' > a'$. Similarly to the proof of Theorem 10, we have $T:\overline{P}_{a'}\to P_{a'}$ and

$$\begin{gathered}\left\{u\in P\left(\theta,b',b'\left(\frac{b-a}{\eta-a}\right)^{2(\alpha-1)}\right)\mid\theta(u)>b'\right\}\ne\emptyset,\\ \theta(Tu)>b',\end{gathered}\tag{45}$$

for all $u\in P(\theta,b',b'((b-a)/(\eta-a))^{2(\alpha-1)})$.

Moreover, for $u\in P(\theta,b',c')$ and $\|Tu\| > b'((b-a)/(\eta-a))^{2(\alpha-1)}$, we have

$$\begin{aligned}\theta(Tu) &= \min_{t\in[\eta,b]}Tu(t) \ge \left(\frac{\eta-a}{b-a}\right)^{\alpha-1}\|Tu\|\\ &> b'\left(\frac{b-a}{\eta-a}\right)^{\alpha-1} > b'.\end{aligned}\tag{46}$$

So all the conditions of Theorem 5 are satisfied. Thus using Theorem 5, T has at least three fixed points. So, th boundary value problem (1)-(2) has at least three positive solutions u_1, u_2, and u_3 with $\|u_1\| < a'$, $\min_{t\in[\eta,b]}u_2(t) > b'$, $\|u_3\|(t) > a'$, and $\min_{t\in[\eta,b]}\|u_3\|(t) < b'$. □

Theorem 12. *Let $f(t,u)$ be nonnegative continuous on $[a,b]\times[0,\infty)$. If the following assumptions are satisfied:*

(H6) $f_0 = f_\infty = \infty$;

(H7) *there exists a constant $\mu_1 > 0$ such that $f(t,u) < \phi_p(B\mu_1)$, for $(t,u)\in[a,b]\times[0,\mu_1]$, then boundary value problem (1)-(2) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < \mu_1 < \|u_2\|$.*

Proof. From Lemma 1, we obtain $T:P\to P$ being completely continuous. In view of $f_0=\infty$, there exists $\sigma_1\in(0,\mu_1)$ such that $f(t,u)\ge\phi_p(\eta_1 u)$, for $a\le t\le b$, $0<u\le\sigma_1$, where $\eta_1\in(A/2,\infty)$. Let $\Omega_{\sigma_1}=\{u\in P\mid\|u\|<\sigma_1\}$. Then, for any $u\in\partial\Omega_{\sigma_1}$, we have

$$\begin{aligned}Tu(\eta) &= \int_a^b G(\eta,s)\,\phi_q\left(\int_a^b H(s,\tau)\right.\\ &\quad\cdot\left. f(\tau,u(\tau))\,d\tau\right)ds \ge \int_\eta^b G(\eta,s)\\ &\quad\cdot\phi_q\left(\int_a^b H(s,\tau)\,\phi_p(\eta_1 u)\,d\tau\right)ds \ge \int_\eta^b G(\eta,\\ &\quad s)\,\phi_q\left(\int_{\tau\in I}\gamma H(\tau,\tau)\right.\\ &\quad\cdot\left.\phi_p\left(\eta_1\left(\frac{\eta-a}{b-a}\right)^{\alpha-1}\|u\|\right)d\tau\right)ds\\ &= \eta_1\left(\frac{\eta-a}{b-a}\right)^{\alpha-1}\|u\|\int_\eta^b G(\eta,s)\\ &\quad\cdot\phi_q\left(\int_{\tau\in I}\gamma H(\tau,\tau)\,d\tau\right)ds = \frac{2\eta_1}{A}\|u\| > \|u\|,\end{aligned}\tag{47}$$

which implies $\|Tu\| > \|u\|$ for $u\in\partial\Omega_{\sigma_1}$. Hence, Theorem 8 implies

$$i\left(T,\Omega_{\sigma_1},P\right)=0.\tag{48}$$

On the other hand, since $f_\infty=\infty$, there exists $\sigma_3>\mu_1$ such that $f(t,u)\ge\phi_p(\eta_2 u)$, for $u\ge\sigma_3$, where $\eta_2\in(A/2,\infty)$. Let $\sigma_2>\max\{\sigma_3((b-a)/(\eta-a))^{\alpha-1},\mu_1\}$ and $\Omega_{\sigma_2}=\{u\in P\mid\|u\|<\sigma_2\}$. Then $\min_{t\in[\eta,b]}u(t)\ge((b-a)/(\eta-a))^{\alpha-1}\|u\|>\sigma_3$, for any $u\in\partial\Omega_{\sigma_2}$. By using the method to get (48), we obtain $Tu(\eta) > (2\eta_1/A)\|u\| > \|u\|$, which implies $\|Tu\|>\|u\|$ for $u\in\partial\Omega_2$. Thus, from Theorem 8, we have

$$i\left(T,\Omega_{\sigma_2},P\right)=0.\tag{49}$$

Finally, let $\Omega_{\mu_1}=\{u\in P\mid\|u\|<\mu_1\}$. Then, for any $u\in\partial\Omega_{\mu_1}$, by $(H7)$, we then get

$$\begin{aligned}Tu(t) &= \int_a^b G(t,s)\,\phi_q\left(\int_a^b H(s,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds\\ &\le \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,f(\tau,u(\tau))\,d\tau\right)ds\\ &< \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,\phi_p(B\mu_1)\,d\tau\right)ds\end{aligned}$$

$$= B\mu_1 \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right) ds = \mu_1 = \|u\|, \quad (50)$$

which implies $\|Tu\| < \|u\|$ for $u \in \partial\Omega_{\mu_1}$. Using Theorem 8 again, we get

$$i\left(T, \Omega_{\mu_1}, P\right) = 1. \quad (51)$$

Note that $\sigma_1 < \mu_1 < \sigma_2$, by the additivity of fixed point index and (48)–(51); we obtain

$$\begin{aligned} i\left(T, \Omega_{\mu_1} \setminus \overline{\Omega}_{\sigma_1}, P\right) &= i\left(T, \Omega_{\mu_1}, P\right) - i\left(T, \Omega_{\sigma_1}, P\right) = 1, \\ i\left(T, \Omega_{\sigma_2} \setminus \overline{\Omega}_{\mu_1}, P\right) &= i\left(T, \Omega_{\sigma_2}, P\right) - i\left(T, \Omega_{\mu_1}, P\right) \\ &= -1. \end{aligned} \quad (52)$$

Hence, T has a fixed point u_1 in $\Omega_{\mu_1} \setminus \overline{\Omega}_{\sigma_1}$, and it has a fixed point u_2 in $\Omega_{\sigma_2} \setminus \overline{\Omega}_{\mu_1}$. Clearly, u_1 and u_2 are positive solutions of boundary value problem (1)-(2) and $0 < \|u_1\| < \mu_1 < \|u_2\|$. □

Theorem 13. *Let $f(t,u)$ be nonnegative continuous on $[a,b] \times [0,\infty)$. If the following assumptions are satisfied:*

($H8$) $f_0 = f_\infty = 0$;

($H9$) *there exists a constant $\mu_2 > 0$ such that $f(t,u) > \phi_p(A\mu_2)$, for $(t,u) \in [\eta, b] \times [((\eta-a)/(b-a))^{\alpha-1}\mu_2, \mu_2]$, then boundary value problem (1)-(2) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < \mu_2 < \|u_2\|$.*

Proof. From Lemma 9, we obtain $T : P \to P$ being completely continuous. In view of $f_0 = 0$, there exists $\delta_1 \in (0, \mu_2)$ such that $f(t,u) \leq \phi_p(\eta_2 u)$, for $a \leq t \leq b$, $0 < u \leq \delta_1$, where $\eta_2 \in (0, B)$. Let $\Omega_{\delta_1} = \{u \in P \mid \|u\| < \delta_1\}$. Then, for any $u \in \partial\Omega_{\delta_1}$, we have

$$\begin{aligned} Tu(t) &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\, f(\tau, u(\tau))\,d\tau\right) ds \\ &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,\phi_p(\eta_2 u)\,d\tau\right) ds \\ &\leq \eta_2 \|u\| \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right) ds \\ &= \frac{\eta_2}{B}\|u\| < \|u\|, \end{aligned} \quad (53)$$

which implies $\|Tu\| < \|u\|$ for $u \in \partial\Omega_{\delta_1}$. Hence, Theorem 8 implies

$$i\left(T, \Omega_{\delta_1}, P\right) = 0. \quad (54)$$

Next, since $f_\infty = 0$, there exists $\delta_3 > \mu_2$ such that $f(t,u) \leq \phi_p(\eta_3 u)$, for $u \geq \delta_3$, where $\eta_3 \in (0, B)$. We consider two cases.

Case (i). Suppose that f is bounded, which implies that there exists $N > 0$ such that $f(t,u) \leq \phi_p(N)$ for all $t \in [a,b]$ and $u \in [0,\infty)$. Take $\delta_4 > \max\{N/B, \delta_3\}$. Then, for $u \in P$ with $\|u\| = \delta_4$, we get

$$\begin{aligned} Tu(t) &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\, f(\tau, u(\tau))\,d\tau\right) ds \\ &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,\phi_p(N)\,d\tau\right) ds \\ &= N \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right) ds = \frac{N}{B} < \delta_4 \\ &= \|u\|. \end{aligned} \quad (55)$$

Case (ii). Suppose that f is unbounded. In view of $f : [a,b] \times [0,\infty) \to [0,\infty)$ being continuous, there exist $t^* \in [a,b]$ and $\delta_5 > \max\{((b-a)/(\eta-a))^{\alpha-1}\delta_3, \mu_2\}$ such that $f(t,u) \leq f(t^*, \delta_5)$, for $a \leq t \leq b$, $0 \leq u \leq \delta_5$. Then, for $u \in P$ with $\|u\| = \delta_5$, we obtain

$$\begin{aligned} Tu(t) &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\, f(\tau, u(\tau))\,d\tau\right) ds \\ &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\, f(t^*, \delta_5)\,d\tau\right) ds \\ &\leq \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,\phi_p(\eta_3\|u\|)\,d\tau\right) ds \\ &\leq \eta_3\delta_5 \int_a^b G(b,s)\,\phi_q\left(\int_a^b H(\tau,\tau)\,d\tau\right) ds = \frac{\eta_3\delta_5}{B} \\ &< \delta_5 = \|u\|. \end{aligned} \quad (56)$$

So, in either case, if we always choose $\Omega_{\delta_2} = \{u \in E \mid \|u\| < \delta_2 = \max\{\delta_4, \delta_5\}\}$, then we have $\|Tu\| < \|u\|$, for $u \in \partial\Omega_2$. Thus, from Theorem 8, we have

$$i\left(T, \Omega_{\delta_2}, P\right) = 1. \quad (57)$$

Finally, let $\Omega_{\mu_2} = \{u \in P \mid \|u\| < \mu_2\}$. Then, for any $u \in \partial\Omega_{\mu_2}$, $\min_{t\in[\eta,b]} u(t) \geq ((\eta-a)/(b-a))^{\alpha-1}\|u\| = ((\eta-a)/(b-a))^{\alpha-1}\mu_2$, by ($H9$), and we then obtain

$$\begin{aligned} Tu(\eta) &= \int_a^b G(\eta,s)\,\phi_q\left(\int_a^b H(s,\tau)\, f(\tau, u(\tau))\,d\tau\right) ds \\ &\geq \int_\eta^b G(\eta,s)\,\phi_q\left(\int_a^b H(s,\tau)\,\phi_p(A\mu_2)\,d\tau\right) ds \end{aligned}$$

$$\geq A\mu_2 \int_{\eta}^{b} G(\eta, s)\,\phi_q\left(\int_{\tau\in I} \gamma H(\tau,\tau)\,d\tau\right) ds$$
$$= 2\left(\frac{b-a}{\eta-a}\right)^{\alpha-1} \mu_2 > \mu_2 = \|u\|, \quad (58)$$

which implies $\|Tu\| > \|u\|$ for $u \in \partial\Omega_{\mu_2}$. An application of Theorem 8 again shows that

$$i\left(T, \Omega_{\mu_2}, P\right) = 0. \quad (59)$$

Note that $\delta_1 < \mu_2 < \delta_2$ by the additivity of fixed point index and (54)–(59); we obtain

$$i\left(T, \Omega_{\mu_2} \setminus \overline{\Omega}_{\delta_1}, P\right) = i\left(T, \Omega_{\mu_2}, P\right) - i\left(T, \Omega_{\delta_1}, P\right)$$
$$= -1, \quad (60)$$
$$i\left(T, \Omega_{\delta_2} \setminus \overline{\Omega}_{\mu_2}, P\right) = i\left(T, \Omega_{\delta_2}, P\right) - i\left(T, \Omega_{\mu_2}, P\right) = 1.$$

Hence, T has a fixed point u_1 in $\Omega_{\mu_2} \setminus \overline{\Omega}_{\delta_1}$, and it has a fixed point u_2 in $\Omega_{\delta_2} \setminus \overline{\Omega}_{\mu_2}$. Consequently, u_1 and u_2 are positive solutions of boundary value problem (1)-(2) and $0 < \|u_1\| < \mu_2 < \|u_2\|$. □

4. Example

In this section, we consider boundary value problem of the fractional differential equation

$$D_{0^+}^{1.5}\left(\phi_p\left(D_{0^+}^{2.5} u(t)\right)\right) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = 0,$$
$$u'(0) = 0,$$
$$u'(1) = \frac{1}{3} u'\left(\frac{1}{2}\right), \quad (61)$$
$$\phi_p\left(D_{0^+}^{2.5} u(0)\right) = 0,$$
$$D_{0^+}^{0.5}\left(\phi_p\left(D_{0^+}^{2.5} u(1)\right)\right) = 0,$$

where

$$f(t,u) = \begin{cases} \dfrac{t}{100} + 12u^5, & 0 \leq u \leq 2, \\ \dfrac{t}{100} + u + 58, & u \geq 2. \end{cases} \quad (62)$$

Let $p = 1/2$. We note that $a = 0$, $b = 1$, $\eta = 1/2$, $\xi = 1/3$, $\beta = 3/2$, $\beta_1 = 1/2$, $n = 3$, $\alpha = 5/2$, and $\alpha_1 = 1$. By a simple calculation, we obtain $((\eta - a)/(b - a))^{\alpha-1} = 0.3535$, $\gamma = 0.5$, $B = 1.0452$, and $A = 34.1482$. Choosing $a' = 1$, $b' = 2$, and $c' = 100$, then $0 < a' < b' < ((\eta - a)/(b - a))^{2(\alpha-1)} c'$ and f satisfies

(i) $f(t, u(t)) < 1.0452 = \phi_p(Ba')$ for all $(t, u) \in [0, 1] \times [0, 1]$,

(ii) $f(t, u(t)) \geq 68.364 = \phi_p(Ab')$ for all $(t, u) \in [0.5, 2] \times [2, 16.0049]$,

(iii) $f(t, u(t)) \leq 104.52 = \phi_p(Bc')$ for all $(t, u) \in [0, 1] \times [0, 100]$.

Consequently, all of the conditions of Theorem 11 are satisfied. With the use of Theorem 11, boundary value problem (61) has at least three positive solutions u_1, u_2, and u_3 with

$$\|u_1\| < 1,$$
$$\min_{t\in[1/2,1]} u_2(t) > 2,$$
$$\|u_3\| > 1, \quad (63)$$
$$\min_{t\in[1/2,1]} u_3(t) < 2.$$

Competing Interests

The author declares that he has no competing interests regarding the publication of this paper.

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