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Research Article

A Note on First Passage Functionals for Lévy Processes with Jumps of Rational Laplace Transforms

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This paper investigates the two-sided first exit problem for a jump process having jumps with rational Laplace transform. The corresponding boundary value problem is solved to obtain an explicit formula for the first passage functional. Also, we derive the distribution of the first passage time to two-sided barriers and the value at the first passage time.

1. Introduction

One-sided and two-sided exit problems for the compound Poisson processes and jump-diffusion processes with two-sided jumps have been applied widely in a variety of fields. For example, in actuarial mathematics, the problem of first exit from a half-line is of fundamental interest with regard to the classical ruin problem and the expected discount penalty function or the Gerber-Shiu function as well as the expected total; see, for example, Mordecki [1], Xing et al. [2], Zhang et al. [3], Lewis and Mordecki [4], and Avram et al. [5]. In mathematical finance, the first passage time plays a crucial role for the pricing of many path-dependent options and American type options; see, for example, Geman and Yor [6], Bertoin [7], Kyprianou [8], Rogers [9], Avram et al. [10], and Alili and Kyprianou [11]. Recently, Cai [12] investigated the first passage time of hyperexponential jump-diffusion process. Cai and Kou [13] proposed a mixed-exponential jump-diffusion process to model the asset return and found an expression for the joint distribution of the first exit problem for a jump and overshoot for a mixed-exponential jump-diffusion process. Chen et al. [14] and Yin et al. [15] discussed the first passage functional for hyperexponential jump-diffusion process.

Motivated by works mentioned above, the main objective of this paper is to study the first exit time of the two-sided first exit problem for jump-diffusion process having jump with rational Laplace transform proposed by Lewis and Mordecki [4]; see also Kuznetsov [16]. This extends recent results obtained in Chen et al. [14, Theorem 2.5] on the hyperexponential jump-diffusion process.

The rest of the paper is organized as follows. In Section 2, we introduce the jump-diffusion process having jumps with rational Laplace transform. Section 3 concentrates on deriving the joint distribution of first exit time and a nonnegative bounded measurable function of the process value at the first exit time to two flat barriers. Section 4 presents the analytical solution to the pricing problem of standard double-barrier options.

2. The Model

A Lévy jump-diffusion process $X = \{X_t,\ t \geq 0\}$ is defined as

$$X_t = X_0 + \mu t + \sigma W_t - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the drift and volatility of the diffusion part, respectively, $W = \{W_t,\ t \geq 0\}$ is a (standard) Brownian motion, $N = \{N_t,\ t \geq 0\}$ is a homogeneous Poisson process with rate λ, and $\{Y_i,\ i = 1, 2, \ldots\}$ are independent and identically distributed random variables supported in $\mathbb{R} \setminus \{0\}$; moreover, $\{W_t,\ t \geq 0\}$, $\{N_t,\ t \geq 0\}$ and

$\{Y_i,\ i = 1,2,\ldots\}$ are mutually independent; finally, the probability density function (pdf) of Y_1 is given by

$$f(y) = \sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij}\frac{(\eta_j)^i y^{i-1}}{(i-1)!}e^{-\eta_j y}\mathbf{1}_{\{y\geq 0\}} + \sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{ij}\frac{(\theta_j)^i (-y)^{i-1}}{(i-1)!}e^{\theta_j y}\mathbf{1}_{\{y<0\}}, \tag{2}$$

where $p_{ij}, q_{ij} \geq 0$, such that $\sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{i,j} = 1$, $\mathrm{Re}(\eta_j)$ and $\mathrm{Re}(\theta_j) > 0$ and that $\eta_i \neq \eta_j$ and $\theta_i \neq \theta_j$ for all $i \neq j$.

Another important tool to establish the key result of the article is the infinitesimal generator of X_t. Note that X_t is a Markovian process and its infinitesimal generator is given by

$$\begin{aligned}\mathscr{L}h(x) &:= \lim_{t\searrow 0}\frac{\mathbb{E}[h(X_t) \mid X_0 = x] - h(x)}{t} \\ &= \mu h'(x) + \frac{\sigma^2}{2}h''(x) \\ &\quad + \lambda\int_{-\infty}^{+\infty}(h(x-y) - h(x))f(y)\,dy,\end{aligned} \tag{3}$$

for any bounded and twice continuously differentiable function h.

Throughout the rest of the paper, the law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$; we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. The Lévy exponent of X is given by

$$\begin{aligned}G(\zeta) &= \frac{\ln\mathbb{E}[\exp(\zeta X_t)]}{t} \\ &= \mu\zeta + \frac{\sigma^2}{2}\zeta^2 + \lambda\left(\mathbb{E}\left[e^{-\zeta Y_1}\right] - 1\right) \\ &= \mu\zeta + \frac{\sigma^2}{2}\zeta^2 \\ &\quad + \lambda\left(\sum_{j=1}^{m}\sum_{i=1}^{m_j}\frac{p_{ij}(\eta_j)^i}{(\eta_j+\zeta)^i} + \sum_{j=1}^{n}\sum_{i=1}^{n_j}\frac{-q_{ij}(\theta_j)^i}{(\zeta-\theta_j)^i} - 1\right).\end{aligned} \tag{4}$$

Accordingly, G is a rational function on $\mathbb{C}$. The equation $G(\zeta) - \alpha = 0$ with $> 0, \sigma > 0$, and $\mu \in \mathbb{R}$ yields $S = M + N + 2$ zeros with $M = \sum_{i=1}^{m} m_i$ and $N = \sum_{j=1}^{n} m_i$; see Kuznetsov [16].

Let us denote the zeros of $G(\zeta) - \alpha$ in the half-plane $\mathrm{Re}(\zeta) > 0$ $\{\mathrm{Re}(\zeta) < 0\}$ as $\rho_1, \rho_2, \ldots, \rho_{M+1}$ $\{\rho_{M+2}, \rho_{M+3}, \ldots, \rho_{M+N+2}\}$. Also, we assume that all zeros of $G(\zeta) - \alpha$ are simple.

3. Distribution of the First Passage Time to Two Flat Barriers

Define τ to be the first passage time of X_t to two flat barriers h and H with $h < H$; that is,

$$\tau := \inf\{t \geq 0 : X_t \geq h \text{ or } X_t \leq H\}. \tag{5}$$

And let

$$\phi(x) = \mathbb{E}_x\left[e^{-\alpha\tau}g(X_\tau)\right], \tag{6}$$

where $\alpha > 0$ and g is nonnegative bounded measurable function.

Now, by Feynman-Kac formula (see, e.g., Theorem 4.4.2, Karatzas and Shreve [17]) we have that $\phi(x)$ must satisfy

$$\begin{aligned}(\mathscr{L} - \alpha)\phi(x) &= 0 \quad \text{in } (h, H), \\ \phi(x) &= g(x) \quad \text{on } (-\infty, h] \cup [H, +\infty).\end{aligned} \tag{7}$$

Our goal in this section is to solve the boundary problem (7) and find explicit formulae for $\phi(x)$. We first show that ϕ satisfies an integrodifferential equation and then derive an ordinary differential equation for ϕ. Based on the ODE, we show that ϕ can be written as a linear combination of known exponential functions. As a consequence, its explicit form is obtained, for instance, choosing $g(x)$ to be $e^{\gamma x}$; it is easy to derive the joint distribution of the first passage time of X to two flat barriers and the process value at the first passage time.

Now, let $\mathscr{P}_0(\zeta) = \prod_{j=1}^{m}\prod_{i=1}^{m_j}(\zeta+\eta_j)^i\prod_{j=1}^{n}\prod_{i=1}^{n_j}(\zeta-\theta_j)^i$; then $\mathscr{P}_1(\zeta) = \mathscr{P}_0(\zeta)(G(\zeta) - \alpha)$ is a polynomial whose zero coincides with those of $G(\zeta) - \alpha$. Also, denote by D the differential operator such that its characteristic polynomial is $\mathscr{P}_1(\zeta)$.

The following lemma will be needed for our proof of Proposition 2.

Lemma 1. *Let $d^{(k)}$ indicate the kth derivative with respect to x of any differentiable function and define*

$$F(i, \eta_j, x) = \left(\frac{d}{dx} + \eta_j\right)^{(i)} e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}\,dy \tag{8}$$

with $(d/dx + \eta_j)^{(i)}$ being the generalized Leibniz operator such that

$$\left(\frac{d}{dx} + \eta_j\right)^{(i)} := \sum_{k=0}^{i}\binom{i}{k}(\eta_j)^{i-k}d^{(k)}. \tag{9}$$

Then, for all $i \geq 1$,

$$F(i, \eta_j, x) = (i-1)!\,\phi(x). \tag{10}$$

Proof. We proceed by induction on i. For $i = 1$, we have

$$\begin{aligned}F(1, \eta_j, x) &= \left(\frac{d}{dx} + \eta_j\right)e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)e^{\eta_j y}\,dy \\ &= -\eta_j e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)e^{\eta_j y}\,dy + \phi(x) \\ &\quad + \eta_j e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)e^{\eta_j y}\,dy = \phi(x).\end{aligned} \tag{11}$$

Moreover,

$$\begin{aligned}\left(\frac{d}{dx}+\eta_j\right)&e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy\\&=-\eta_j e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy\\&\quad+(i-1)e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}e^{\eta_j y}dy\\&\quad+\eta_j e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy\\&=(i-1)e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}e^{\eta_j y}dy.\end{aligned}\tag{12}$$

It follows inductively that

$$\begin{aligned}F(i,\eta_j,x)&=\left(\frac{d}{dx}+\eta_j\right)^{(i)}\\&\cdot e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy\\&=\left(\frac{d}{dx}+\eta_j\right)^{(i-1)}\left(\frac{d}{dx}+\eta_j\right)\\&\cdot e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy=(i-1)\\&\cdot\left(\frac{d}{dx}+\eta_j\right)^{(i-1)}e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-2}e^{\eta_j y}dy\\&=(i-1)F(i-1,\eta_j,x)=(i-1)!F(1,\eta_j,x)\\&=(i-1)!\phi(x),\end{aligned}\tag{13}$$

which is the desired result. □

We may now state the following.

Proposition 2. *Suppose a bounded solution ϕ defined on $\mathbb{R}$ to the boundary value problem (7) exists. Then on $\mathbb{R}\setminus\{h,H\}$, ϕ is infinitely differentiable and satisfies the ODE,*

$$D\phi\equiv 0,\quad \text{on } (h,H).\tag{14}$$

Hence, on (h,H), $\phi(x)=\sum_{k=1}^{S}Q_k e^{\rho_k x}$ for some constants Q_k.

Proof. Applying the infinitesimal generator $\mathscr{L}$ to the function ϕ, we obtain

$$\begin{aligned}\mathscr{L}\phi(x)&=\frac{\sigma^2}{2}\phi''(x)+\mu\phi'(x)+\lambda\sum_{i=1}^{m}\sum_{j=1}^{m_i}p_{ij}\\&\quad\cdot\frac{(\eta_j)^i}{(i-1)!}e^{-\eta_j x}\int_{-\infty}^{x}\phi(y)(x-y)^{i-1}e^{\eta_j y}dy\\&\quad+\lambda\sum_{j=1}^{m}\sum_{i=1}^{m_j}q_{ij}\\&\quad\cdot\frac{(\theta_j)^i}{(i-1)!}e^{\theta_j x}\int_{x}^{+\infty}\phi(y)(y-x)^{i-1}e^{-\theta_j y}dy\\&\quad-\lambda\phi(x).\end{aligned}\tag{15}$$

Next, ϕ will be shown to satisfy an ODE. Using Lemma 1, we get, for $j=1,2,\ldots,m$, $i=1,2,\ldots,m_j$,

$$\begin{aligned}\left(\frac{d}{dx}+\eta_j\right)^{(i)}&e^{-\eta_j x}\int_{-\infty}^{x}(x-y)^{i-1}\phi(y)e^{\eta_j y}dy\\&=(i-1)!\phi(x).\end{aligned}\tag{16}$$

The same computation will yield, for $j=1,2,\ldots,n$, $i=1,2,\ldots,n_j$,

$$\begin{aligned}\left(\frac{d}{dx}-\theta_j\right)^{(i)}&e^{\theta_j x}\int_{x}^{+\infty}(y-x)^{i-1}\phi(y)e^{-\theta_j y}dy\\&=-(i-1)!\phi(x).\end{aligned}\tag{17}$$

Now, since $\sigma>0$ and $(\mathscr{L}-\alpha)\phi\equiv 0$ then, thanks to Proposition 3.3 in the work of Chen et al. [18], ϕ is infinitely differentiable on (h,H) and

$$\begin{aligned}0&=\prod_{j=1}^{m}\prod_{i=1}^{m_j}\left(\frac{d}{dx}+\eta_j\right)^{(i)}\prod_{j=1}^{n}\prod_{i=1}^{n_j}\left(\frac{d}{dx}-\theta_j\right)^{(i)}(\mathscr{L}-\alpha)\phi(x)\\&=\prod_{j=1}^{m}\prod_{i=1}^{m_j}\left(\frac{d}{dx}+\eta_j\right)^{(i)}\\&\quad\cdot\prod_{j=1}^{n}\prod_{i=1}^{n_j}\left(\frac{d}{dx}-\theta_j\right)^{(i)}\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+\mu\frac{d}{dx}-\lambda-\alpha\right)\phi(x)\\&\quad+\lambda\sum_{j=1}^{m}\sum_{i=1}^{m_j}\prod_{k=1,k\neq j}^{m}\prod_{i=1}^{m_j}\left(\frac{d}{dx}+\eta_k\right)^{(i)}p_{ij}\frac{(\eta_j)^i}{(i-1)!}(i-1)!\phi(x)\\&\quad-\lambda\sum_{j=1}^{n}\sum_{i=1}^{n_j}\prod_{k=1,k\neq j}^{n}\prod_{i=1}^{n_j}\left(\frac{d}{dx}-\theta_k\right)^{(i)}q_{ij}\frac{(\theta_j)^i}{(i-1)!}(i-1)!\phi(x).\end{aligned}\tag{18}$$

Hence, (18) transforms the integrodifferential equation $(\mathscr{L}-\alpha)\phi\equiv 0$ into an ODE : $\widehat{D}\phi\equiv 0$, where $\widehat{D}$ is high order differential operator.

To complete the proof, $\widehat{D}$ must be shown to coincide with D. To show that the characteristic polynomials of D and $\widehat{D}$ will suffice, write $\widehat{\mathscr{P}}(\zeta)$ as the characteristic polynomial of $\widehat{D}$. Then, by (18), $\widehat{\mathscr{P}}$ is given by

$$\begin{aligned}\widehat{\mathscr{P}}(\zeta)&=\prod_{j=1}^{m}\prod_{i=1}^{m_j}(\zeta+\eta_j)^i\prod_{j=1}^{n}\prod_{i=1}^{n_j}(\zeta-\theta_j)^i\Bigg[\mu\zeta+\frac{\sigma^2}{2}\zeta^2\\&\quad+\lambda\left(\sum_{j=1}^{m}\sum_{i=1}^{m_j}\frac{p_{ij}(\eta_j)^i}{(\zeta+\eta_j)^i}+\sum_{j=1}^{n}\sum_{i=1}^{n_j}\frac{-q_{ij}(\theta_j)^i}{(\zeta-\theta_i)^i}-1\right)\\&\quad-\alpha\Bigg]=\mathscr{P}_0(\zeta)(G(\zeta)-\alpha).\end{aligned}\tag{19}$$

This equation reveals that the characteristic polynomial $\mathscr{P}_1(\zeta)$ of D equals that, $\widehat{\mathscr{P}}(\zeta)$, of $\widehat{D}$, which completes the proof. □

Proposition 3. *Suppose that ϕ is a bounded solution to the boundary value problem (7) and, on (h,H), $\phi(x) = \sum_{k=1}^{S} Q_k e^{\rho_k x}$ for some constants Q_k. Then the constant vector Q satisfies the equation*

$$AQ = V_g, \tag{20}$$

where A is $S\times S$ nonsingular matrix given by

$$A = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{21}$$

$$Z_1 = \begin{pmatrix} \frac{\eta_1}{\rho_1+\eta_1} e^{\rho_1 h} & \cdots & \frac{\eta_1}{\rho_S+\eta_1} e^{\rho_S h} \\ \vdots & \ddots & \vdots \\ \left(\frac{\eta_1}{\rho_1+\eta_1}\right)^{m_1} e^{\rho_1 h} & \cdots & \left(\frac{\eta_1}{\rho_S+\eta_1}\right)^{m_1} e^{\rho_S h} \\ \vdots & \ddots & \vdots \\ \frac{\eta_m}{\rho_1+\eta_m} e^{\rho_1 h} & \cdots & \frac{\eta_m}{\rho_S+\eta_m} e^{\rho_S h} \\ \vdots & \ddots & \vdots \\ \left(\frac{\eta_m}{\rho_1+\eta_m}\right)^{m_m} e^{\rho_1 h} & \cdots & \left(\frac{\eta_m}{\rho_S+\eta_m}\right)^{m_m} e^{\rho_S h} \\ e^{\rho_1 h} & \cdots & e^{\rho_S h} \end{pmatrix} \tag{22}$$

$$Z_2 = \begin{pmatrix} \frac{\theta_1}{\rho_1-\theta_1} e^{\rho_1 H} & \cdots & \frac{\theta_1}{\rho_S-\theta_1} e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \left(\frac{\theta_1}{\rho_1-\theta_1}\right)^{n_1} e^{\rho_1 H} & \cdots & \left(\frac{\theta_1}{\rho_S-\theta_1}\right)^{n_1} e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \frac{\theta_n}{\rho_1-\theta_n} e^{\rho_1 H} & \cdots & \frac{\theta_n}{\rho_S-\theta_n} e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \left(\frac{\theta_n}{\rho_1-\theta_n}\right)^{n_n} e^{\rho_1 H} & \cdots & \left(\frac{\theta_n}{\rho_S-\theta_n}\right)^{n_n} e^{\rho_S H} \\ e^{\rho_1 H} & \cdots & e^{\rho_S H} \end{pmatrix} \tag{23}$$

and $V_g = (V_{g,1}(j,i),\ j = 1,2\ldots,m,\ i = 1,2\ldots,m_j, g(h), V_{g,2}(j,i),\ j = 1,2\ldots,n,\ i = 1,2\ldots,n_j, g(H))$,

$$V_{g,1}(j,i) = \int_{-\infty}^{0} g(y+h)\frac{(\eta_j)^i(-y)^{i-1}e^{\eta_j y}}{(i-1)!}dy, \tag{24}$$

$$V_{g,2}(j,i) = \int_{0}^{+\infty} g(y+H)\frac{(\theta_j)^i y^{i-1}e^{-\theta_j y}}{(i-1)!}dy. \tag{25}$$

Proof. Since $(\mathscr{L}-\alpha)\phi = 0$ and $\phi(x) = \sum_{k=1}^{S} Q_k e^{\rho_k x}$ on (h,H), which entails

$$\begin{aligned} 0 &= (\mathscr{L}-\alpha)\phi(x) \\ &= \frac{\sigma^2}{2}\phi''(x) + \mu\phi'(x) + \lambda\int_{-\infty}^{+\infty}\phi(x-y)f(y)\,dy \\ &\quad - (\lambda+\alpha)\phi(x) \\ &= \sum_{k=1}^{S} Q_k e^{\rho_k x}\left(\frac{\sigma^2}{2}\rho_k^2 + \mu\rho_k - (\lambda+\alpha)\right) \\ &\quad + \lambda\int_{-\infty}^{+\infty}\phi(x-y)f(y)\,dy, \end{aligned} \tag{26}$$

furthermore

$$\begin{aligned} &\int_{-\infty}^{+\infty}\phi(x-y)f(y)\,dy = \left(\int_{-\infty}^{h} + \int_{H}^{+\infty}\right) g(y)f(x \\ &\quad - y)\,dy + \int_{x-H}^{x-h}\phi(x-y)f(y)\,dy \\ &= \sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij}e^{-\eta_j(x-h)}\int_{-\infty}^{0}\frac{(\eta_j)^i}{(i-1)!}g(y+h) \\ &\quad \cdot (x-h-y)^{i-1}e^{\eta_j y}dy \\ &\quad + \sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{ij}e^{\theta_j(x-H)}\int_{0}^{+\infty}\frac{(\theta_j)^i}{(i-1)!}g(y+H) \\ &\quad \cdot (y+H-x)^{i-1}e^{-\theta_j y}dy + \sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij}\sum_{k=1}^{S} Q_k e^{\rho_k x} \\ &\quad \cdot \frac{(\eta_j)^i}{(i-1)!}\int_{0}^{x-h} y^{i-1}e^{-(\eta_j+\rho_k)y}dy \\ &\quad + \sum_{j=1}^{m}\sum_{i=1}^{m_j} q_{ij}\sum_{k=1}^{S} Q_k e^{\rho_k x}\frac{(\theta_j)^i}{(i-1)!}\int_{x-H}^{0} y^{i-1}e^{(\theta_j-\rho_k)y}dy. \end{aligned} \tag{27}$$

Now, since $(a\pm y)^{i-1} = \sum_{l=0}^{i-1}\binom{i-1}{l}(\pm y)^l(a)^{i-1-l}$,

$$\begin{aligned} \int_{0}^{a} y^{i-1}e^{-\beta y}dy &= \beta^{-i}\Gamma(i, a\beta) \\ &= \beta^{-i}(i-1)!\left(1 - e^{-a\beta}\sum_{l=0}^{i-1}\frac{(a\beta)^l}{l!}\right) \end{aligned} \tag{28}$$

with $\Gamma(i,x)$ being the incomplete gamma function (see Gradshteyn and Ryzhik [19, page 342]).

Consequently, by combining (26) and (27) and taking into account the fact that $G(\rho_k)-\alpha=0$ for all $k=1,2,\dots,S$, we obtain

$$\begin{aligned}
0 &= \sum_{j=1}^{m}\sum_{i=1}^{m_j} p_{ij}e^{-\eta_j(x-h)}\int_{-\infty}^{0}\frac{(\eta_j)^i}{(i-1)!}g(y+h)\\
&\quad\cdot\sum_{l=0}^{i-1}\binom{i-1}{l}(x-h)^l(-y)^{i-1-l}e^{\eta_j y}dy\\
&\quad+\sum_{j=1}^{n}\sum_{i=1}^{n_j} q_{ij}e^{\theta_j(x-H)}\int_{0}^{+\infty}\frac{(\theta_j)^i}{(i-1)!}g(y+H)\\
&\quad\cdot\sum_{l=0}^{i-1}\binom{i-1}{l}(H-x)^l(y)^{i-1-l}e^{-\theta_j y}dy\\
&\quad+\sum_{k=1}^{S}Q_k e^{\rho_k h}\sum_{j=1}^{m}\sum_{i=1}^{m_j}-p_{ij}e^{-\eta_j(x-h)}\\
&\quad\cdot\frac{(\eta_j)^i}{(\eta_j+\rho_k)^i}\sum_{l=0}^{i-1}\frac{[(\eta_j+\rho_k)(x-h)]^l}{l!}\\
&\quad+\sum_{k=1}^{S}Q_k e^{\rho_k H}\sum_{j=1}^{m}\sum_{i=1}^{m_j}-q_{ij}e^{\theta_j(x-H)}\\
&\quad\cdot\frac{(\theta_j)^i}{(\rho_k-\theta_j)^i}\sum_{l=0}^{i-1}\frac{[(\theta_j-\rho_k)(x-H)]^l}{l!}.
\end{aligned}\tag{29}$$

Comparing $e^{-\eta_j(x-h)}$ and $e^{\theta_j(x-H)}$ yields (20), which entails the desired result. □

Lemma 4. *For a given value of $\alpha>0$ the matrix A given by (21) is invertible.*

Proof. Assume that $AC=0$ for some vector $C=(C_1,C_2,\dots,C_S)^T$. Consider the function $V(x)=\sum_{k=1}^{S}C_k e^{\rho_k x}$ for $x\in(h,H)$ and $V(x)=0$, otherwise, with $\rho_1,\dots,\rho_S$ to be the distinct zeros of the equation $G(x)-\alpha=0$. Since $AC=0$ and $V(x)$ is a solution to the boundary value problem,

$$\begin{aligned}
(\mathscr{L}-\alpha)\phi(x)&=0, \quad \text{in } (h,H),\\
\phi(x)&=0, \quad \text{on } (-\infty,h]\cup[H,+\infty).
\end{aligned}\tag{30}$$

From the uniqueness of solutions to the boundary value problem (30), $V(x)\equiv 0$ in (h,H). Now, since $\{e^{\rho_k x},\ 1\le k\le S\}$ are linearly independent then $C=0$ and A is invertible. □

In the following, $\mathbf{y}\cdot\mathbf{z}$ is written for the usual inner product of the vectors $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{C}^S$ and for every real value x, $\mathbf{e}^{\rho}_{\alpha}(x)=[e^{\rho_1 x},e^{\rho_2 x},\dots,e^{\rho_S x}]$, where $\rho_1,\rho_2,\dots,\rho_S$ are the $S=N+M+2$ roots of the equation $G(\zeta)=\alpha$. Our main result is the following.

Theorem 5. *For any $\alpha\ge 0$ and a nonnegative bounded measurable function g on $(h,H)^c$, the following assertions are equivalent:*

(a) $\phi(x)=\mathbb{E}_x[e^{-\alpha\tau}g(X_\tau)]$, *where* $\tau:=\inf\{t\ge 0: X_t\ge h \text{ or } X_t\le H\}$.

(b) *The function $\phi(x)$ solves the boundary problem (7).*

(c) *The function $\phi(x)$ is given by the formula*

$$\phi(x)=\begin{cases}\mathbf{Q}(g)\cdot\mathbf{e}^{\rho}_{\alpha}(x), & \text{if } x\in(h,H),\\ g(x), & \text{if } x\notin(h,H),\end{cases}\tag{31}$$

with $\mathbf{Q}(g)=A^{-1}V_g$ and A and V_g are given by formulas (21) and (24), respectively.

Proof. The fact that (b) implies (c) is straightforward consequence of Proposition 3. Conversely, consider the function $V(x)=\sum_{k=1}^{S}Q_k e^{\rho_k x}$ for $x\in(h,H)$ and $V(x)=g(x)$ otherwise, where g is a bounded function on $(h,H)^c$ and Q_k's are given constants. Then the same reasoning as in Proposition 3 shows that

$$\begin{aligned}
&(\mathscr{L}-\alpha)V(x)\\
&=\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}e^{-\eta_j(x-h)}\int_{-\infty}^{0}\frac{(\eta_j)^i}{(i-1)!}g(y+h)\\
&\quad\cdot(x-h-y)^{i-1}e^{\eta_j y}dy\\
&\quad+\sum_{j=1}^{n}\sum_{i=1}^{n_j}q_{ij}e^{\theta_j(x-H)}\int_{0}^{+\infty}\frac{(\theta_j)^i}{(i-1)!}g(y+H)\\
&\quad\cdot(y+H-x)^{i-1}e^{-\theta_j y}dy+\sum_{j=1}^{m}\sum_{i=1}^{m_j}p_{ij}\sum_{k=1}^{S}Q_k e^{\rho_k x}\\
&\quad\cdot\frac{(\eta_j)^i}{(i-1)!}\int_{0}^{x-h}y^{i-1}e^{-(\eta_j+\rho_k)y}dy\\
&\quad+\sum_{j=1}^{m}\sum_{i=1}^{m_j}q_{ij}\sum_{k=1}^{S}Q_k e^{\rho_k x}\frac{(\theta_j)^i}{(i-1)!}\int_{x-H}^{0}y^{i-1}e^{(\theta_j-\rho_k)y}dy.
\end{aligned}\tag{32}$$

Thanks to (20), we conclude that (c) implies (b).

Let us finally assume that (a) holds. Then by Feynman-Kac formula, the function $\phi(x)$ solves the boundary problem (7); hence (b) holds. Conversely, thanks to Proposition 4.1 in the work of Chen et al. [18], the boundary problem (7) has a unique solution; consequently (b) implies (a). The proof is complete. □

As an illustration of the main result of Theorem 5, we can obtain closed-form expressions for the expectations of a variety of functions with respect to τ and X_τ. For instance, choosing $g(x)=e^{\gamma x}$ in the above theorem, we can derive the joint Laplace transform of (τ,X_τ), which is presented in the following corollary.

Corollary 6. *For any $\alpha > 0$, $\gamma > 0$,*

$$\mathbb{E}_x\left[e^{-\alpha\tau+\gamma X_\tau}\right] = \begin{cases} Q\cdot \mathbf{e}^{\rho}_{\alpha}(x), & \text{if } x \in (h,H), \\ e^{\gamma x}, & \text{if } x \notin (h,H), \end{cases} \tag{33}$$

where $Q = (Q_1, Q_2, \ldots, Q_S)^T = A^{-1}V(\gamma)$, A is given by formula (21), and $V(\gamma)$ is given by

$$\begin{aligned} V(\gamma) = \Bigg(& \frac{e^{\gamma h}}{(\gamma+\eta_j)^i},\ j = 1,2\ldots,m,\ i \\ & = 1,2\ldots,m_j,\ e^{\gamma h},\ \frac{e^{\gamma H}}{(\gamma-\theta_j)^i},\ j = 1,2\ldots,n,\ i \\ & = 1,2\ldots,n_j, e^{\gamma H}\Bigg). \end{aligned} \tag{34}$$

As another consequence of Theorem 5 and Lebesgue's dominated convergence theorem, we get the following for the asymptotic case when $h \to -\infty$ and $H \to +\infty$, respectively.

Corollary 7. *For two flat barriers h and H ($h < H$), define the first downwards passage time under h and the first upwards passage time over H by*

$$\begin{aligned} \tau_h^+ &:= \inf\{t \geq 0 : X_t \geq h\}, \\ \tau_H^- &:= \inf\{t \geq 0 : X_t \leq H\}. \end{aligned} \tag{35}$$

Then for $\alpha > 0$, one has the following:

$$\begin{aligned} \mathbb{E}_x\left[e^{-\alpha\tau_h^+} g\left(X_{\tau_h^+}\right)\right] &= \begin{cases} Q_1\cdot \mathbf{e_1}^{\rho}_{\alpha}(x), & \text{if } x \geq h, \\ g(x), & \text{if } x < h, \end{cases} \\ \mathbb{E}_x\left[e^{-\alpha\tau_H^-} g\left(X_{\tau_H^-}\right)\right] &= \begin{cases} Q_2\cdot \mathbf{e_2}^{\rho}_{\alpha}(x), & \text{if } x \leq H, \\ g(x), & \text{if } x > H, \end{cases} \end{aligned} \tag{36}$$

with

$$\begin{aligned} & Q_1 = (Q_1, Q_2, \ldots, Q_{M+1})^T = A_+^{-1}V_+, \\ & Q_2 = (Q_{M+2}, Q_{M+3}, \ldots, Q_S)^T = A_-^{-1}V_-, \\ & \mathbf{e_1}^{\rho}_{\alpha}(x) = \left[e^{\rho_1 x}, e^{\rho_2 x}, \ldots, e^{\rho_{M+1} x}\right], \\ & \mathbf{e_2}^{\rho}_{\alpha}(x) = \left[e^{\rho_{M+2} x}, e^{\rho_{M+3} x}, \ldots, e^{\rho_S x}\right], \\ & V_+ = \left(V_{g,+}(j,i),\ j = 1,2\ldots,m,\ i = 1,2\ldots,m_j, g(h)\right), \\ & V_- = \left(V_{g,-}(j,i),\ j = 1,2\ldots,n,\ i = 1,2\ldots,n_j, g(H)\right). \\ & V_{g,+}(j,i) = \int_{-\infty}^{0} g(y+h)\frac{(\eta_j)^i(-y)^{i-1}e^{\eta_j y}}{(i-1)!}dy, \\ & V_{g,-}(j,i) = \int_{0}^{+\infty} g(y+H)\frac{(\theta_j)^i y^{i-1}e^{-\theta_j y}}{(i-1)!}dy, \end{aligned}$$

$$\begin{aligned} & A_+ \\ & = \begin{pmatrix} \frac{\eta_1}{\rho_1+\eta_1}e^{\rho_1 h} & \cdots & \frac{\eta_1}{\rho_{M+1}+\eta_1}e^{\rho_{M+1} h} \\ \vdots & \ddots & \vdots \\ \left(\frac{\eta_1}{\rho_1+\eta_1}\right)^{m_1} e^{\rho_1 h} & \cdots & \left(\frac{\eta_1}{\rho_{M+1}+\eta_1}\right)^{m_1} e^{\rho_{M+1} h} \\ \vdots & \ddots & \vdots \\ \frac{\eta_m}{\rho_1+\eta_m}e^{\rho_1 h} & \cdots & \frac{\eta_m}{\rho_{M+1}+\eta_m}e^{\rho_{M+1} h} \\ \vdots & \ddots & \vdots \\ \left(\frac{\eta_m}{\rho_1+\eta_m}\right)^{m_m} e^{\rho_1 h} & \cdots & \left(\frac{\eta_m}{\rho_{M+1}+\eta_m}\right)^{m_m} e^{\rho_{M+1} h} \\ e^{\rho_1 h} & \cdots & e^{\rho_{M+1} h} \end{pmatrix}, \\ & A_- \\ & = \begin{pmatrix} \frac{\theta_1}{\rho_{M+2}-\theta_1}e^{\rho_{M+2} H} & \cdots & \frac{\theta_1}{\rho_S+\theta_1}e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \left(\frac{\theta_1}{\rho_{M+2}-\theta_1}\right)^{n_1} e^{\rho_{M+2} H} & \cdots & \left(\frac{\theta_1}{\rho_S-\theta_1}\right)^{n_1} e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \frac{\theta_n}{\rho_{M+2}-\theta_n}e^{\rho_{M+2} H} & \cdots & \frac{\theta_n}{\rho_S-\theta_n}e^{\rho_S H} \\ \vdots & \ddots & \vdots \\ \left(\frac{\theta_n}{\rho_{M+2}-\theta_n}\right)^{n_n} e^{\rho_{M+2} H} & \cdots & \left(\frac{\theta_n}{\rho_S-\theta_n}\right)^{n_n} e^{\rho_S H} \\ e^{\rho_{M+2} H} & \cdots & e^{\rho_S H} \end{pmatrix}. \end{aligned} \tag{37}$$

4. Pricing Double-Barrier Options

We now show how our theoretical results can be easily applied to derive pricing formulae for standard double-barrier options. We assume the asset price process $\{S_t : t \geq 0\}$ under the risk-neutral probability measure $\mathbb{P}$ is defined as $S_t := e^{X_t}$. The log-return process $\{X_t : t \geq 0\}$ is given by (1) where $X_0 = \log(S_0)$ and $\mu := r - \sigma^2/2 - \lambda\mathbb{E}[e^{Y_1} - 1]$ (i.e., $\mathbb{E}[e^{-r-T}S_T] = S_0$), where $r > 0$ is the risk-free rate. More recently, Cai et al. [20] presented the following.

The payoff of a standard double-barrier option is activated (knocked in) or extinguished (knocked out) when the price of the underlying asset crosses barriers. For example, a knock-out call option will not give the holder the payoff of a European call option unless the underlying price remains within a prespecified range before the option matures. More precisely, consider an interval (L, U) and the initial asset price S_0 is in it. The holder will receive $(S_T - K)^+\mathbf{1}_{\{\tau>T\}}$ at maturity T, where

$$\tau = \inf\{t \geq 0 : S_t \leq U \text{ or } S_t \geq L\}. \tag{38}$$

Under the risk-neutral probability measure, the price of such option with maturity T and strike K is given by

$$e^{-rT}\mathbb{E}\left[\left(S_T-K\right)^{+}\mathbf{1}_{\{\tau>T\}}\mid S_0\right]. \tag{39}$$

Make a change variable $\kappa := -\log K$. Then, the expectation can be represented as

$$C(\kappa,T) := e^{-rT}\mathbb{E}_x\left[\left(S_0e^{X_T}-e^{-\kappa}\right)^{+}\mathbf{1}_{\{\tau>T\}}\right]. \tag{40}$$

Define $\widehat{C}(\alpha,\gamma)$ and $\widehat{\Delta}(\alpha,\gamma)$ as the double Laplace transforms of the price $C(\kappa,T)$ in (40) and the delta $\Delta(\alpha,\gamma) = \partial C(\kappa,T)/\partial S_0$ with respect to T and κ, respectively; that is,

$$\begin{aligned}\widehat{C}(\alpha,\gamma) &= \int_0^\infty\int_{-\infty}^\infty e^{-\gamma\kappa-\alpha T}C(\kappa,T)\,d\kappa\,dT,\\ \widehat{\Delta}(\alpha,\gamma) &= \int_0^\infty\int_{-\infty}^\infty e^{-\gamma\kappa-\alpha T}\Delta(\alpha,\gamma)\,d\kappa\,dT.\end{aligned} \tag{41}$$

Theorem 8. *For any $\gamma>0$ and $\alpha>\max(G(\gamma+1)-r,0)$*

$$\begin{aligned}\widehat{C}(\alpha,\gamma) &= \frac{S_0^{\gamma+1}}{\gamma(\gamma+1)(G(\gamma+1)-(\alpha+r))}\Big(Q\\ &\quad\cdot\mathbf{e}^{\rho}_{\alpha+r}(x)-e^{(\gamma+1)x}\Big)\mathbf{1}_{(\log(L/S_0),\log(U/S_0))}(x),\end{aligned} \tag{42}$$

$$\begin{aligned}\widehat{\Delta}(\alpha,\gamma) &= \frac{S_0^{\gamma}}{\gamma(G(\gamma+1)-(\alpha+r))}\Big(Q\cdot\mathbf{e}^{\rho}_{\alpha+r}(x)\\ &\quad-e^{(\gamma+1)x}\Big)\mathbf{1}_{(\log(L/S_0),\log(U/S_0))}(x),\end{aligned} \tag{43}$$

where $Q=A^{-1}V(\gamma+1)$ and A associated with $r+a$ is defined as in Theorem 5 and V is given by formula (34).

Proof. Equation (43) is an easy consequence of (42). To prove (43), using an idea of Kou et al. [21] (see also Cai et al. [20]) along with a change of the order of integration and the integral with respect to κ, we obtain

$$\begin{aligned}\widehat{C}(\alpha,\gamma) &= \int_0^\infty dT\int_{-\infty}^\infty e^{-\gamma\kappa-\alpha T}C(\kappa,T)\,d\kappa\\ &= \mathbb{E}_x\left[\int_0^\tau e^{-(\alpha+r)T}dT\int_{-(\log S_0+X_T)}^\infty e^{-\gamma\kappa}\left(S_0e^{X_T}-e^{-\kappa}\right)d\kappa\right]\\ &= \frac{S_0^{\gamma+1}}{\gamma(\gamma+1)}\mathbb{E}_x\left[\int_0^\tau e^{-(\alpha+r)T+(\gamma+1)X_T}dT\right].\end{aligned} \tag{44}$$

Now, we suppose that $G(\gamma+1)<\alpha+r$ and applying Itô's formula to the process $\{e^{-(\alpha+r)t+(\gamma+1)X_t},\ t\ge 0\}$, we obtain

$$\begin{aligned}M_t &:= e^{-(\alpha+r)(t\wedge\tau)+(\gamma+1)X_{t\wedge\tau}}-e^{(\gamma+1)X_0}\\ &\quad-\int_0^{t\wedge\tau}e^{-(\alpha+r)T}\left(-(\alpha+r)e^{(\gamma+1)X_s}+\mathscr{L}e^{(\gamma+1)X_T}\right)dT\\ &= e^{-(\alpha+r)(t\wedge\tau)+(\gamma+1)X_{t\wedge\tau}}-e^{(\gamma+1)X_0}\\ &\quad-(G(\gamma+1)-(\alpha+r))\int_0^{t\wedge\tau}e^{-\alpha T+(\gamma+1)X_T}dT\end{aligned} \tag{45}$$

is a local martingale starting from $M_0=0$. Since $G(\gamma+1)<\alpha+r$, it follows from Fubini's theorem that

$$\begin{aligned}\mathbb{E}&\left[\int_0^t e^{-(\alpha+r)T+(\gamma+1)X_T}dT\right]\\ &= \int_0^t e^{-(\alpha+r)T}\mathbb{E}\left[e^{(\gamma+1)X_T}\right]dT\\ &= \int_0^t e^{(-(\alpha+r)+G(\gamma+1))T}dT\\ &= \frac{1}{(-(\alpha+r)+G(\gamma+1))}\left[e^{(-(\alpha+r)+G(\gamma+1))t}-1\right]\\ &\qquad\forall t\ge 0.\end{aligned} \tag{46}$$

It follows from the dominated convergence theorem that $\{M_t;\ t\ge 0\}$ is actually a martingale. In particular

$$\begin{aligned}\mathbb{E}_x&\left[e^{-(\alpha+r)\tau+(\gamma+1)X_\tau}-e^{(\gamma+1)x}\right]\\ &= (G(\gamma+1)-(\alpha+r))\,\mathbb{E}_x\left[\int_0^\tau e^{-(\alpha+r)T+(\gamma+1)X_T}dT\right].\end{aligned} \tag{47}$$

Combining (44) and (47) and applying Corollary 6 we can therefore conclude that

$$\begin{aligned}\widehat{C}&(\alpha,\gamma)\\ &= \frac{S_0^{\gamma+1}}{\gamma(\gamma+1)(G(\gamma+1)-(\alpha+r))}\Big(\mathbb{E}_x\left[e^{-(\alpha+r)\tau+(\gamma+1)X_\tau}\right]\\ &\quad-e^{(\gamma+1)x}\Big) = \frac{S_0^{\gamma+1}}{\gamma(\gamma+1)(G(\gamma+1)-(\alpha+r))}\Big(Q\\ &\quad\cdot\mathbf{e}^{\rho}_{\alpha+r}(x)-e^{(\gamma+1)x}\Big)\mathbf{1}_{(\log(L/S_0),\log(U/S_0))}(x),\end{aligned} \tag{48}$$

where $Q=A^{-1}V(\gamma+1)$ and A associated with $r+\alpha$ is defined as in Theorem 5 and V is given by formula (34), which ends the proof. □

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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