

## Research Article

# A Method to Construct Generalized Fibonacci Sequences

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The main purpose of this paper is to study the convergence properties of Generalized Fibonacci Sequences and the series of partial sums associated with them. When the proper values of an  $s \times s$  real matrix  $A$  are real and different, we give a necessary and sufficient condition for the convergence of the matrix sequence  $A, A^2, A^3, \dots$  to a matrix  $B$ .

## 1. Introduction

The Fibonacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots \quad (1)$$

is an interesting numerical sequence that occurs quite frequently in many parts of nature. This sequence has a special feature; every element of this sequence, starting from the third, is the sum of its two predecessors and can be generated recursively by the formula

$$x_{n+2} = x_n + x_{n+1}. \quad (2)$$

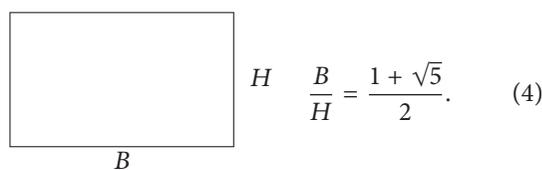
It is clear that we need the first two terms  $x_0 = 0$ ,  $x_1 = 1$  and the recursive formula to define the sequence.

If we want to know the term  $x_k$  without constructing the previous terms, we can use the unexplainable formula (see [1]):

$$x_k = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right]. \quad (3)$$

What do the irrational numbers  $\sqrt{5}$  have to do with the original sequence?

The so-called *Golden ratio*  $\lambda = (1 + \sqrt{5})/2$  appears in nature very frequently. It is also considered the most esthetic ratio between the basis and height of a rectangle:



$$\frac{H}{B} = \frac{1 + \sqrt{5}}{2}. \quad (4)$$

If we replace the recursive formula by

$$2x_{n+2} = x_n + x_{n+1}, \quad (5)$$

we obtain a new sequence  $0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, \dots$  and this sequence is no longer divergent; in fact, it converges to  $2/3$ .

To define a *Generalized Fibonacci Sequence*, we fix a natural number  $s$  and two elements

$$\begin{aligned} &(x_0, x_1, \dots, x_{s-1}), \\ &(a_0, a_1, \dots, a_{s-1}) \end{aligned} \quad (6)$$

in the Euclidean space  $\mathbb{R}^s$ . The recursive formula is

$$x_{k+s} = a_0 x_k + a_1 x_{k+1} + \dots + a_{s-1} x_{k+s-1}. \quad (7)$$

The main purpose of this paper is to study the convergence properties of Generalized Fibonacci Sequences and the series of partial sums associated with them. When the proper values of an  $s \times s$  real matrix  $A$  are real and different, we give a necessary and sufficient condition for the convergence of the matrix sequence  $I, A, A^2, A^3, \dots$  to a matrix  $B$ : we say  $A^k \rightarrow B$  if for every ordered pair  $(i, j)$ , where  $i, j \in \{1, 2, \dots, s\}$ , the sequence of the  $(i, j)$ -entries of  $A^k$  converges to the  $(i, j)$ -entry of  $B$ . As a particular case, we study when do we have the convergence of the powers  $\varphi, \varphi^2, \varphi^3, \dots$  of a *Moebius transformation* to a constant function.

Then we would like to list the four published monographs about generalized Fibonacci sequences [1-4] and several more specialized articles [5-8].

### 2. Main Results

Consider a Generalized Fibonacci Sequence (GFS) with initial terms  $x_0, x_1, \dots, x_{s-1}$  and recursive formula (7). Define the matrices  $s \times s$ :

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{s-1} \end{pmatrix}, \tag{8}$$

$$D_k = \begin{pmatrix} x_k & x_{k+1} & \cdots & x_{k+s-1} \\ x_{k+1} & x_{k+2} & \cdots & x_{k+s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k+s-1} & x_{k+s} & \cdots & x_{k+2s-2} \end{pmatrix}, \quad k \in \mathbb{N} \cup \{0\}.$$

The characteristic polynomial of the matrix  $C$  is

$$\varphi(x) = x^s - a_{s-1}x^{s-1} - a_{s-2}x^{s-2} - \dots - a_0. \tag{9}$$

Suppose all the roots  $\lambda_1, \lambda_2, \dots, \lambda_s$  of  $\varphi(x)$  are real and pairwise different; that is,  $\lambda_i \neq \lambda_j$ , for  $i \neq j$ . Consider *Vandermonde's matrix*:

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \cdots & \lambda_s^{s-1} \end{pmatrix}. \tag{10}$$

Since  $|V| = \pm \prod_{1 \leq i < j \leq s} (\lambda_i - \lambda_j)$ , we deduce that  $|V| \neq 0$  and, hence,  $V$  is invertible. We need the following matrix relation.

**Theorem 1.**  $V$  and  $C$  are related by the following formula:

$$VEV^{-1} = C, \tag{11}$$

where  $E$  is the diagonal matrix:

$$E = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_s \end{pmatrix}. \tag{12}$$

*Proof.* Let  $v_1, v_2, \dots, v_s \in \mathbb{R}^s$  be the proper vectors of the matrix  $C$ . We have then  $v_i C = \lambda_i v_i$  for each  $i = 1, 2, \dots, s$ . If  $w = v_1 + v_2 + \dots + v_s$  and  $k \in \mathbb{N}$ , we have  $wC^k = \lambda_1^k v_1 + \lambda_2^k v_2 + \dots + \lambda_s^k v_s$ ,  $k = 0, 1, \dots, s-1$ . Since  $|V| \neq 0$ , we deduce that the vectors  $w, wC, wC^2, \dots, wC^{s-1}$  are linearly independent and hence they constitute a basis for  $\mathbb{R}^s$ . Calling  $\mathcal{B} = \{v_1, v_2, \dots, v_s\}$ ,  $\mathcal{B}' = \{w, wC, \dots, wC^{s-1}\}$ , we have  $V = m(\text{Id}, \mathcal{B}', \mathcal{B})$ . On the other hand, consider the linear transformation  $T : \mathbb{R}^s \rightarrow \mathbb{R}^s$  defined by the formula  $T(v) = vC$ . Clearly,

$$C = m(T, \mathcal{B}', \mathcal{B}'). \tag{13}$$

Therefore,

$$\begin{aligned} C &= m(T, \mathcal{B}', \mathcal{B}') \\ &= m(\text{Id}, \mathcal{B}', \mathcal{B}) m(T, \mathcal{B}, \mathcal{B}) m(\text{Id}, \mathcal{B}, \mathcal{B}') \\ &= VEV^{-1}. \end{aligned} \tag{14}$$

□

In the next theorem, we relate  $V$  and  $C$  with  $D_k$ .

**Theorem 2.** One has the following formulas:

- (a)  $CD_k = D_{k+1}$  and  $C^k D_0 = D_k$  for every  $k \in \mathbb{N}$ .
- (b)  $D_k = VE^k V^{-1} D_0$  for every  $k \in \mathbb{N}$ .

*Proof.* (a) It is straightforward.

(b) Using (36), we have

$$D_k = C^k D_0 = (VEV^{-1})^k D_0 = VE^k V^{-1} D_0. \tag{15}$$

□

Using the formula  $D_k = VE^k V^{-1} D_0$ , we obtain any member of the corresponding Generalized Fibonacci Sequence.

**Theorem 3.** Consider

$$x_k = \frac{(-1)^{s-1}}{|V|} \begin{vmatrix} \lambda_1^k & \lambda_2^k & \cdots & \lambda_s^k & 0 \\ 1 & 1 & \cdots & 1 & x_0 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \cdots & \lambda_s^{s-1} & x_{s-1} \end{vmatrix}. \tag{16}$$

*Proof.* The first row of the matrix  $VE^k$  is the following:  $(\lambda_1^k, \lambda_2^k, \dots, \lambda_s^k)$ . The first column of the matrix  $V^{-1}D_0$  is

$$\frac{1}{|V|} \begin{pmatrix} \sum_{j=1}^s V_{j1} x_{j-1} \\ \sum_{j=1}^s V_{j2} x_{j-1} \\ \vdots \\ \sum_{j=1}^s V_{js} x_{j-1} \end{pmatrix}, \quad (17)$$

where  $V_{ij}$  is the cofactor of the entry of  $V$  in the  $(i, j)$  position.  $x_k$  is the entry in the  $(1, 1)$  position of the matrix  $D_k$ . Therefore,

$$x_k = \frac{1}{|V|} \left[ \lambda_1^k \sum_{j=1}^s V_{j1} x_{j-1} + \lambda_2^k \sum_{j=1}^s V_{j2} x_{j-1} + \dots + \lambda_s^k \sum_{j=1}^s V_{js} x_{j-1} \right]. \quad (18)$$

The expression inside the square brackets coincides with

$$(-1)^{s-1} \begin{vmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_s^k & 0 \\ 1 & 1 & \dots & 1 & x_0 \\ \lambda_1 & \lambda_2 & \dots & \lambda_s & x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} & x_{s-1} \end{vmatrix}. \quad (19)$$

To see this, develop this determinant by the first row and the last column. The coefficient of  $\lambda_i^k x_{j-1}$  is then

$$(-1)^{i-1} (-1)^{s+j} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \lambda_1 & \dots & \lambda_{i-1} \lambda_{i+1} & \dots & \lambda_s \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{j-1} & \dots & \lambda_{i-1}^{j-1} \lambda_{i+1}^{j-1} & \dots & \lambda_s^{j-1} \\ \lambda_1^{j+1} & \dots & \lambda_{i-1}^{j+1} \lambda_{i+1}^{j+1} & \dots & \lambda_s^{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{s-1} & \dots & \lambda_{i-1}^{s-1} \lambda_{i+1}^{s-1} & \dots & \lambda_s^{s-1} \end{vmatrix} \quad (20)$$

$$= (-1)^{s-1} V_{ji}.$$

This completes the proof.  $\square$

In the particular case  $k = 2$ , we obtain

$$x_k = -\frac{1}{\lambda_2 - \lambda_1} \begin{vmatrix} \lambda_1^k & \lambda_2^k & 0 \\ 1 & 1 & x_0 \\ \lambda_1 & \lambda_2 & x_1 \end{vmatrix}. \quad (21)$$

If we further assume that  $x_0 = 0$  and  $x_1 = 1$ , we obtain

$$x_k = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^k - \lambda_2^k). \quad (22)$$

In the original Fibonacci sequence, we have

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (23)$$

and hence  $\varphi(x) = x^2 - x - 1$ . The roots of this polynomial are  $\lambda_1 = (1 + \sqrt{5})/2$  and  $\lambda_2 = (1 - \sqrt{5})/2$ . We justify then the mythical formula  $x_k = (1/\sqrt{5})[(1 + \sqrt{5})/2]^k - ((1 - \sqrt{5})/2)^k$ .

In the case  $a_0 = a_1 = 1/2$ , we obtain

$$C = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (24)$$

$$\varphi(x) = \begin{vmatrix} x & -1 \\ -\frac{1}{2} & x - \frac{1}{2} \end{vmatrix} = x^2 - \frac{1}{2}x - \frac{1}{2}.$$

The roots of  $\varphi(x)$  are  $\lambda_1 = 1$  and  $\lambda_2 = -1/2$ . Hence,  $x_k = (2/3)[1 - (-1/2)^k]$ . It is now clear that this last GFS converges to  $2/3$ .

We give next a sufficient condition for the convergence of the series of a GFS.

**Theorem 4** (main theorem). *Suppose the roots of the characteristic polynomial  $\varphi(x)$  of a GFS  $\{x_k\}$  are pairwise different and all of them lie in the open interval  $(-1, 1)$ . Then the series of  $\{x_k\}$  converges to*

$$\frac{(-1)^{s-1}}{|V|} \begin{vmatrix} 1 & 1 & \dots & 1 & 0 \\ \frac{1}{1-\lambda_1} & \frac{1}{1-\lambda_2} & \dots & \frac{1}{1-\lambda_s} & x_0 \\ \lambda_1 & \lambda_2 & \dots & \lambda_s & x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} & x_{s-1} \end{vmatrix}. \quad (25)$$

*Proof.* This is a consequence of the identity

$$x_0 + x_1 + \dots + x_k = \frac{(-1)^{s-1}}{|V|} \begin{vmatrix} 1 + \lambda_1 + \lambda_1^2 + \dots + \lambda_1^{k-1} & \dots & 1 + \lambda_s + \lambda_s^2 + \dots + \lambda_s^{k-1} & 0 \\ 1 & \dots & 1 & x_0 \\ \lambda_1 & \dots & \lambda_s & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{s-1} & \dots & \lambda_s^{s-1} & x_{s-1} \end{vmatrix} \quad (26)$$

and the convergence  $\sum_{j=0}^{\infty} \lambda_i^j$  to  $1/(1 - \lambda_i)$ .  $\square$

We give now two examples.

*Example 5.* One has  $s = 3, x_0 = x_1 = 0, x_2 = 1, a_0 = 1/6, a_1 = -1,$  and  $a_2 = 11/6.$

The characteristic polynomial  $\varphi(x)$  of the corresponding GFS is

$$\begin{aligned} \varphi(x) &= \begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ -\frac{1}{6} & 1 & x - \frac{11}{6} \end{vmatrix} = x^3 - \frac{11}{6}x^2 + x - \frac{1}{6} \\ &= (x-1)\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right). \end{aligned} \tag{27}$$

Therefore,  $\lambda_1 = 1, \lambda_2 = 1/2, \lambda_3 = 1/3,$  and

$$|V| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{4} & \frac{1}{9} \end{vmatrix} = -\frac{1}{18}. \tag{28}$$

Hence

$$\begin{aligned} x_k &= (-1)^{3-1}(-18) \begin{vmatrix} 1 & 2^{-k} & 3^{-k} & 0 \\ 1 & 1 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & 0 \\ 1 & \frac{1}{4} & \frac{1}{9} & 1 \end{vmatrix} \\ &= -18 \begin{vmatrix} 1 & 2^{-k} & 3^{-k} \\ 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{vmatrix}. \end{aligned} \tag{29}$$

Clearly

$$x_k \longrightarrow -18 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{vmatrix} = 3. \tag{30}$$

*Example 6.* One has  $s = 3, x_0 = x_1 = 0, x_2 = 1, a_0 = 1/18, a_1 = -11/8,$  and  $a_2 = 3/2.$  In this case,  $\varphi(x) = x^3 - (3/2)x^2 + (11/18)x - 1/18 = (x - 1/2)(x - 1/3)(x - 2/3).$  Therefore,  $\lambda_1 = 1/2, \lambda_2 = 1/3, \lambda_3 = 2/3,$  and  $|V| = -1/108.$  Hence,

$$x_k = -108 \begin{vmatrix} \left(\frac{1}{2}\right)^k & \left(\frac{1}{3}\right)^k & \left(\frac{2}{3}\right)^k & 0 \\ 1 & 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{4} & \frac{1}{9} & \frac{4}{9} & 1 \end{vmatrix}. \tag{31}$$

In this case,  $x_k \rightarrow 0,$  but

$$\sum_{k=1}^{\infty} x_k = -108 \begin{vmatrix} \frac{1}{1-1/2} & \frac{1}{1-1/3} & \frac{1}{1-2/3} & 0 \\ 1 & 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{4} & \frac{1}{9} & \frac{4}{9} & 1 \end{vmatrix} = 9. \tag{32}$$

Given  $s$  different real numbers  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_s,$  we may construct, for every  $(x_0, x_1, \dots, x_{s-1}) \in \mathbb{R}^s,$  a GFS; namely,

$$x_k = \frac{(-1)^{s-1}}{\prod_{1 \leq j < i \leq s} (\lambda_i - \lambda_j)} \begin{vmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_s^k & 0 \\ 1 & 1 & \dots & 1 & x_0 \\ \lambda_1 & \lambda_2 & \dots & \lambda_s & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} & x_{s-1} \end{vmatrix}. \tag{33}$$

The missing sequence  $(a_0, a_1, \dots, a_{s-1})$  may be obtained from the coefficients of the polynomial:

$$\varphi(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_s). \tag{34}$$

For instance,  $a_{s-1} = \lambda_1 + \lambda_2 + \dots + \lambda_s$  and  $a_{s-2} = -\sum_{1 \leq j < i \leq s} \lambda_i \lambda_j.$

With the help of this remark, we prove the following.

**Theorem 7.** Let  $(a_{ij})$  be an invertible  $s \times s$  matrix. Suppose the characteristic polynomial  $\varphi(x)$  of  $A$  factors into the form:

$$\varphi(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_s), \tag{35}$$

where  $\lambda_1, \lambda_2, \dots, \lambda_s$  are pairwise different real numbers. Let  $a_{ij}^{(k)}$  be the entry in the  $(i, j)$  position of the matrix  $A^k.$  Then

$$a_{ij}^{(k)} = \frac{(-1)^{s-1}}{|V|} \begin{vmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_s^k & 0 \\ 1 & 1 & \dots & 1 & a_{ij}^{(0)} \\ \lambda_1 & \lambda_2 & \dots & \lambda_s & a_{ij}^{(1)} \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_s^2 & a_{ij}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} & a_{ij}^{(s-1)} \end{vmatrix}. \tag{36}$$

*Proof.* Let  $(x_k)$  be the GFS determined by (36). It is clear that  $x_k = a_{ij}^{(k)}$  for  $k \leq s - 1.$  Proceeding by induction, suppose  $x_t = a_{ij}^{(t)}$  for every  $t \leq k,$  where  $k \geq s - 1.$  We clearly have

$$\begin{aligned} a_{ij}^{(k+1)} &= (a_{i1}^{(k)}, a_{i2}^{(k)}, \dots, a_{is}^{(k)}) \cdot (a_{1j}^{(1)}, a_{2j}^{(1)}, \dots, a_{sj}^{(1)}) \\ &= a_{i1}^{(k)} a_{1j} + a_{i2}^{(k)} a_{2j} + \dots + a_{is}^{(k)} a_{sj}. \end{aligned} \tag{37}$$

By induction, the terms  $a_{i1}^{(k)}, \dots, a_{is}^{(k)}$  may be obtained using determinants of type (36). We consider the last columns of determinants (36) and we obtain

$$\begin{aligned}
 & a_{1j} \begin{pmatrix} 0 \\ a_{i1}^{(0)} \\ a_{i1}^{(1)} \\ \vdots \\ a_{i1}^{(s-1)} \end{pmatrix} + a_{2j} \begin{pmatrix} 0 \\ a_{i2}^{(0)} \\ a_{i2}^{(1)} \\ \vdots \\ a_{i2}^{(s-1)} \end{pmatrix} + \dots \\
 &= \begin{pmatrix} 0 \\ a_{ij}^{(1)} \\ a_{ij}^{(2)} \\ \vdots \\ a_{ij}^{(s)} \end{pmatrix}.
 \end{aligned} \tag{38}$$

Therefore

$$a_{ij}^{(k+1)} = \frac{(-1)^{s-1}}{|\Delta|} \begin{vmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_s^k & 0 \\ 1 & 1 & \dots & 1 & a_{ij}^{(1)} \\ \lambda_1 & \lambda_2 & \dots & \lambda_s & a_{ij}^{(2)} \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_s^2 & a_{ij}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{s-1} & \lambda_2^{s-1} & \dots & \lambda_s^{s-1} & a_{ij}^{(s)} \end{vmatrix}. \tag{39}$$

The GFS on the right side of this equation starts with  $y_0 = a_{ij}^{(1)}, y_1 = a_{ij}^{(2)}, \dots, y_{s-1} = a_{ij}^{(s)}$  and has the same proper values of the GFS  $(x_k)$ . Hence  $a_{ij}^{(k+1)} = y_k = x_{k+1}$  and the proof is complete.  $\square$

As an exercise, we calculate the powers of a  $2 \times 2$  matrix:

$$A = \begin{pmatrix} a & b \\ h & k \end{pmatrix}, \quad \text{where } \Delta = ak - hb \neq 0. \tag{40}$$

In this case we have  $a_{11}^{(0)} = 1, a_{12}^{(0)} = 0, a_{21}^{(0)} = 0, a_{22}^{(0)} = 1, a_{11}^{(1)} = a, a_{12}^{(1)} = b, a_{21}^{(1)} = h, a_{22}^{(1)} = k$ . Hence

$$a_{ij}^{(t)} = -\frac{1}{\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix}} \begin{vmatrix} \lambda_1^t & \lambda_2^t & 0 \\ 1 & 1 & a_{ij}^{(0)} \\ \lambda_1 & \lambda_2 & a_{ij}^{(1)} \end{vmatrix}. \tag{41}$$

Therefore

$$\begin{aligned}
 a_{11}^{(t)} &= \frac{1}{\lambda_1 - \lambda_2} \begin{vmatrix} \lambda_1^t & \lambda_2^t & 0 \\ 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & a \end{vmatrix}; \\
 a_{12}^{(t)} &= \frac{1}{\lambda_1 - \lambda_2} \begin{vmatrix} \lambda_1^t & \lambda_2^t & 0 \\ 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & b \end{vmatrix}; \\
 a_{21}^{(t)} &= \frac{1}{\lambda_1 - \lambda_2} \begin{vmatrix} \lambda_1^t & \lambda_2^t & 0 \\ 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & h \end{vmatrix}; \\
 a_{22}^{(t)} &= \frac{1}{\lambda_1 - \lambda_2} \begin{vmatrix} \lambda_1^t & \lambda_2^t & 0 \\ 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & k \end{vmatrix},
 \end{aligned} \tag{42}$$

where  $\lambda_1 = (1/2)[(a+k) - \sqrt{(a-k)^2 + 4hb}]$  and  $\lambda_2 = (1/2)[(a+k) + \sqrt{(a-k)^2 + 4hb}]$ . If  $\lambda_1 = 1$  and  $|\lambda_2| < 1$ , we have

$$A^t \rightarrow \frac{1}{1 - \lambda_2} \begin{pmatrix} a - \lambda_2 & b \\ h & k - \lambda_2 \end{pmatrix}. \tag{43}$$

This limit matrix has determinant = 0. So in the case of a Moebius transformation

$$\varphi(z) = \frac{a + bz}{h + kz}, \quad ak \neq hb \tag{44}$$

we have the following corollary.

**Corollary 8.** *If 1 is a proper value of the matrix  $A = \begin{pmatrix} a & b \\ h & k \end{pmatrix}$  and if  $|ak - hb| < 1$ , then the powers  $\varphi^t$  converge to the constant map  $\phi(z) = (a - \lambda_2)/h(1 - \lambda_2) = b/(1 - \lambda_2)(k - \lambda_2)$ .*

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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