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Research Article

The Role of Optimal Intervention Strategies on Controlling Excessive Alcohol Drinking and Its Adverse Health Effects

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We propose and analyze a mathematical model for alcohol drinking problem. The transmission process is modeled as a social "contact" process between "heavy" alcohol drinkers and "light" alcohol drinkers within an unchanging shared drinking environment. The basic reproductive number of the model is computed and the stability of the model steady states is investigated. Further, the model is fitted to data on alcohol drinking for Cape Town and Gauteng, South Africa. In addition, the basic model is extended to incorporate three time dependent intervention strategies. The control functions represent the efforts and policies aimed at weakening the intensity of social interactions between light and heavy drinkers and increase the fraction of treated individuals who permanently quit alcohol drinking. Optimal control results suggest that effective control of high-risk alcohol drinking can be achieved if more resources and efforts are devoted on weakening the intensity of social interactions between light and heavy drinkers.

1. Introduction

Excessive alcohol drinking has been attributed to many types of violence including violence in public settings, sexual violence, domestic violence, and child maltreatment [1, 2]. Prior studies suggest that excessive alcohol drinking results in 2.5 million global deaths annually, and about 88,000 of these deaths occur in United States [3]. In addition, alcohol misuse is thought to play a part in approximately 3 million crimes which occur over world each year [2]. Long-term excessive alcohol drinking can lead to the development of chronic diseases (alcoholic hepatitis, cirrhosis, and so on), neurological impairments, and several social problems [3]. Further, it can interfere with testicular function and male hormone production resulting in impotence, infertility, and reduction of male secondary sex characteristics such as facial and chest hair.

Mathematical modeling of alcohol drinking dynamics and its impact on human health has been an interesting topic for a number of researchers [4–8]. In [4], Thomas and Lungu proposed a mathematical model to investigate the effect of heavy alcohol drinking on the transmission and progression of HIV/AIDS and to assess the impact of heavy drinkers on human immune deficiency virus (HIV) related social and health problems such as tuberculosis (TB) case load and the number of orphans. Their work suggested that there is need for vital counselling and education about the adverse effects of heavy alcohol drinking in order to reduce new HIV cases and the population of orphans in the community.

In Lee et al., [4] an SIR (Susceptible-Infectious-Recovered) type of epidemic model is proposed to study the impact of optimal control intervention strategies in low and high problem drinking populations. The model constituted of three compartments of alcohol drinkers: light or moderate drinkers, heavy drinkers, and recovery class. The proposed model incorporated three optimal control strategies. Their work revealed among other results that control measures may have long-term effect on reducing or eliminating excessive alcohol use in the community only when carried out in conjunction with policies that generated dramatic changes in the population's behavioural norms.

More recently, Wang et al. [7] proposed a compartmental model to explore the impact of optimal control intervention strategies tied on reducing or eliminating excessive alcohol use at Lanzhou University of Technology (LUT) in China. Their work highlighted some specific optimal control measures which can be key to reduce alcoholism within the university community.

Motivated by the aforementioned studies we formulate a model for alcohol drinking that includes incorporating the effects of peer-influence, accounts for multiple intervention strategies, and allows optimal control methods to be used. Alcohol drinking models in [5, 6] provide the starting point for our discussion. The aim of our study is to use a mathematical modeling to gain insights into the transmission dynamics of alcohol drinking and to assess the role of multiple intervention strategies. The model is formulated and analyzed in Section 2; an important threshold parameter which determines the number of new conversions is determined and qualitatively used to investigate the stability of model steady states. Impact of multiple optimal intervention strategies is carried out in Section 3. In Section 4, we give some conclusions and discussions.

2. Model Formulation and Analysis

2.1. Framework. In this section, a new dynamic model for alcohol drinking is presented. We consider the total population of alcohol drinkers denoted by $N(t)$, which includes "light" or "susceptible" $S(t)$, "heavy" $H(t)$, occasional alcohol drinkers $A(t)$ (these are individuals who are in alcohol-related treatment and occasionally drink alcohol, at a frequency less than that of light drinkers), and recovered $R(t)$ (these are individuals who are on treatment and those who have successful completed treatment and have permanently quit alcohol drinking). Thus,

$$N(t) = S(t) + H(t) + A(t) + R(t). \tag{1}$$

We assume a constant size population with a recruitment and non-alcohol-related death rate given by μ. The recruitment of susceptible is proportional to the drinking population and is given by $\mu N(t)$. Initiation into heavy alcohol drinking "transmission" occurs when a susceptible individual has contact with someone in the heavy alcohol compartment. The force of infection describing this mechanism is given by

$$g(H) = \beta H(1 + \alpha H), \tag{2}$$

where β is the transmission rate and α is a positive constant. Here, we adopt the approach in Buonomo and Lacitignola [6] which incorporates the effects of peer-influence on the spread of high-risk alcohol consumption behavior. Our model is based on the following equations:

$$\begin{aligned} \dot{S} &= \mu N - g(H)\frac{S}{N} - \mu S + (1-p)\psi A, \\ \dot{H} &= g(H)\frac{S}{N} - (\phi + \epsilon + \mu) H, \\ \dot{A} &= (1-f)\phi H - (\psi + \mu) A, \\ \dot{R} &= f\phi H + p\psi A - \mu R, \end{aligned} \tag{3}$$

where the upper dot represents the derivative of the component with respect to time. The time spent in the heavy alcohol drinking compartment is $(\phi + \epsilon + \mu)^{-1}$, where ϕ denote the rate at which heavy drinking individuals enter into a treatment program and ϵ represent alcohol-induced death rate. Upon entering a treatment program, a fraction $(1 - f)$ become occasional drinkers and the remainder f is assumed to permanently quit alcohol drinking. Occasional alcohol drinkers successfully complete treatment at rate ψ, with a fraction p of occasional drinkers who complete therapy assumed to permanently quit alcohol drinking while the remainder $(1 - p)$ become light alcohol drinkers.

It is helpful to rescale system (3) so that we have dimensionless variables. We let

$$\begin{aligned} s &= \frac{S}{N}, \\ h &= \frac{H}{N}, \\ a &= \frac{H}{N}, \\ r &= \frac{R}{N}, \end{aligned} \tag{4}$$

so that $g(H)$ becomes $g(h) = \beta(1 + \alpha h)h$. We thus have the following rescaled system:

$$\begin{aligned} \dot{s} &= \mu - g(h)s - \mu s + (1-p)\psi a, \\ \dot{h} &= g(h)s - (\phi + \epsilon + \mu) h, \\ \dot{a} &= (1-f)\phi h - (\psi + \mu) a, \\ \dot{r} &= f\phi h + p\psi a - \mu r. \end{aligned} \tag{5}$$

Observe that the total drinking population (system (5)) satisfies the equation:

$$\dot{s} + \dot{h} + \dot{a} + \dot{r} = \mu - \mu(s + h + a + r) - \epsilon a \le \mu(1 - n). \tag{6}$$

Thus, $n(t) = 1 - (1 - n_0)e^{-\mu t}$ for n_0 a constant. Therefore, the feasible region of system (5) is given by the closed set:

$$\Omega = \{(s, h, a, r) \mid 0 \le s + h + a + r \le 1\}. \tag{7}$$

2.2. Equilibrium Points and Their Stability Analysis. System (5) has a high-level drinking "free" equilibrium $\mathscr{E}^0$ (i.e., $s_0 = 1, h_0 = a_0 = r_0 = 0$), which represents a population with only susceptible individuals. The possibility of the propagation of high levels of drinking within the susceptible community is analyzed through the examination of the impact generated by the introduction of a small number of "typical" heavy alcohol drinkers (an "invasion" process) in a whole of susceptible population. Now, we evaluate the Jacobian matrix of system (5) about $\mathscr{E}^0$; that is,

$$J = \begin{bmatrix} -\mu & -\beta & (1-p)\psi & 0 \\ 0 & \beta - (\phi + \epsilon + \mu) & 0 & 0 \\ 0 & (1-f)\phi & -(\psi + \mu) & 0 \\ 0 & f\phi & p\psi & -\mu \end{bmatrix}. \tag{8}$$

From (8) it is evident that the eigenvalues of system (5) are

$$\begin{aligned}\lambda_1 &= -\mu,\\ \lambda_2 &= \beta - (\phi + \epsilon + \mu),\\ \lambda_3 &= -(\psi + \mu),\\ \lambda_4 &= -\mu.\end{aligned} \tag{9}$$

Based on the eigenvalues of system (5) we have Theorem 1.

Theorem 1. *The equilibrium point $\mathscr{E}^0$ is locally asymptotically stable whenever $\lambda_2 < 0$, that is, $\beta/(\phi+\epsilon+\mu) < 1$, and unstable otherwise.*

Let the average number of conversions ($s \rightarrow h$), $\mathscr{R}_a$, generated by a "typical" heavy alcohol drinker in a population where the proportion of susceptible individuals is approximately one be given by

$$\mathscr{R}_a = \frac{\beta}{(\phi + \epsilon + \mu)}. \tag{10}$$

Since the variable $r(t)$ does not appear in all the first three equations of system (5), it is sufficient to consider the following model on investigating the stability of the model steady states. Consider

$$\begin{aligned}\dot{s} &= \mu - g(h)s - \mu s + (1-p)\psi a,\\ \dot{h} &= g(h)s - (\phi + \epsilon + \mu)h,\\ \dot{a} &= (1-f)\phi h - (\psi + \mu)a.\end{aligned} \tag{11}$$

Theorem 2. *The equilibrium point $\mathscr{E}^0$ is globally asymptotically stable if $\mathscr{R}_a \leq 1$ and unstable otherwise.*

Proof. From the second and third equations of system (11) we have

$$\begin{aligned}\dot{h} &\leq \beta h(1+\alpha h) - (\phi + \epsilon + \mu)h,\\ \dot{a} &\leq (1-f)\phi h - (\psi + \mu)a.\end{aligned} \tag{12}$$

Then, by the comparison principle, $h(t) \rightarrow 0$, and $a(t) \rightarrow 0$, as $t \rightarrow +\infty$ if $\mathscr{R}_a < 1$. Substitution of these into the first equation of (11) gives $s(t) \rightarrow 1$ as $t \rightarrow +\infty$. This implies that $\mathscr{E}^0$ is a global attractor. This completes the proof. □

In order to investigate the long-term dynamics of high-risk alcohol drinking, we conduct an endemic analysis when $\mathscr{R}_a > 1$. The following theorem shows the existence of the endemic equilibrium.

Theorem 3. *When $\mathscr{R}_a > 1$, there exists an endemic equilibrium of system (11).*

Proof. Let us denote the endemic equilibrium of system (11) by $\mathscr{E}^* = (s^*, h^*, a^*)$, where

$$\begin{aligned}s^* &= \frac{\mu(\mu+\psi) + (1-p)(1-f)\phi\psi h^*}{[\mu+\psi][\mu + g(h^*)]},\\ h^* &= \frac{g(h^*)[\mu(\mu+\psi) + (1-p)(1-f)\phi\psi h^*]}{[\mu+\psi][\mu + g(h^*)]},\\ a^* &= \frac{(1-f)\phi h^*}{(\mu+\psi)},\end{aligned} \tag{13}$$

with $g(h^*) = \beta(1+\alpha h^*)h^*$. We then substitute h^* from (13) into $g(h^*)$ and define the function

$$\begin{aligned}F(h^*) &= \frac{\beta[\mu(\mu+\psi) + (1-p)(1-f)\phi\psi h^*]}{[\phi+\epsilon+\mu][\mu+\psi][\mu+\beta(1+\alpha h^*)h^*]}\\ &+ \frac{\beta^2\alpha h^*[1+\alpha h^*][\mu(\mu+\psi) + (1-p)(1-f)\phi\psi h^*]^2}{[\phi+\epsilon+\mu]^2[\mu+\psi]^2[\mu+\beta(1+\alpha h^*)h^*]^2},\end{aligned} \tag{14}$$

such that there exists an endemic equilibrium for model (11) if and only if there is a positive solution to $F(h^*) = 1$. Because

$$\begin{aligned}F(0) &= \frac{\beta}{(\phi+\epsilon+\mu)} = \mathscr{R}_a,\\ \lim_{h^* \rightarrow +\infty} F(h^*) &= 0,\end{aligned} \tag{15}$$

then there exists an endemic equilibrium if $\mathscr{R}_a > 1$. □

Theorem 4. *The equilibrium point $\mathscr{E}^*$ is globally asymptotically stable if $\mathscr{R}_a > 1$ and unstable otherwise.*

Proof. We consider the following Lyapunov functional:

$$\begin{aligned}u[s(t), h(t), a(t)] &= w_1[s(t) - s^*\ln s(t)]\\ &+ w_2[h(t) - h^*\ln h(t)]\\ &+ w_3[a(t) - a^*\ln a(t)],\end{aligned} \tag{16}$$

where w_j, for $j = 1, 2, 3$, are positive constants. Differentiating u along the solution $[s(t), h(t), a(t)]$ of system (11) gives

$$\begin{aligned}\frac{du}{dt} &= w_1\left[1 - \frac{s^*}{s}\right]\frac{ds}{dt} + w_2\left[1 - \frac{h^*}{h}\right]\frac{dh}{dt}\\ &+ w_3\left[1 - \frac{a^*}{a}\right]\frac{da}{dt}\\ &= w_1\left[1 - \frac{s^*}{s}\right][\mu - g(h)s - \mu s + (1-p)\psi a]\\ &+ w_2\left[1 - \frac{h^*}{h}\right][g(h)s - (\phi+\epsilon+\mu)h]\\ &+ w_3\left[1 - \frac{a^*}{a}\right][(1-f)\phi h - (\psi+\mu)a].\end{aligned} \tag{17}$$

At endemic point we have the following identities:

$$\mu = g(h^*) s^* + \mu s^* + (1-p)\psi a^*,$$
$$(\phi + \epsilon + \mu) = g(h^*) \frac{s^*}{h^*}, \quad (18)$$
$$(\mu + \psi) = (1-f)\phi \frac{h^*}{a^*}.$$

Set $w_1 = w_2 = 1$, and $w_3 = g(h^*)s^*/(1-f)\phi h^*$, so that

$$\begin{aligned}\frac{du}{dt} &= \mu s^* \left[2 - \frac{s}{s^*} - \frac{s^*}{s}\right] \\ &+ g(h^*) s^* \left[3 - \frac{sh^* a}{s^* h a^*} - \frac{h a^*}{h^* a} - \frac{s^*}{s}\right] \\ &+ g(h^*) s h^* \frac{a}{a^*}\left[1 - \frac{s^* h}{s h^*}\right] \\ &+ g(h) s^* \left[1 - \frac{s h^*}{s^* h}\right] \\ &+ (1-p)\psi (a^* + a) \frac{s^*}{s}\left[1 - \frac{a^*}{a}\right].\end{aligned} \quad (19)$$

Therefore, $\dot{u} \leq 0$ for all $s, h, a, s^*, h^*, a^* \geq 0$, since the arithmetic mean is greater than or equal to the geometric mean. Further, the equality is satisfied if and only if $s = s^*$, $h = h^*, a = a^*$. Therefore, by Lyapunov-Lasalle asymptotic stability theorem [9, 10] the positive equilibrium point $\mathscr{E}^*$ is globally asymptotically stable whenever $\mathscr{R}_a > 1$. This completes the proof. □

2.3. Data Fitting. In this section, we estimate the model parameters used in our numerical simulations. Due to lack of data regarding the alcohol drinking parameters, in this section we estimate the values through fitting our model with data in [11], for Cape Town and Gauteng, South Africa. The model fits the reported data of excessive alcohol use in Gauteng and Cape Town, South Africa (Figure 1). A comparison between the fitted model and the real data demonstrates that our proposed model could be useful in predicting future trend on excessive alcohol use in different communities. The data in [11] was obtained from South African Community Epidemiology Network on Drug Use (SACENDU). SACENDU, established in 1996, is a network of researchers, practitioners, and policy makers from various sentinel areas in South Africa. Members of SACENDU meet every six months to provide community-level public health surveillance of alcohol and other drug (AOD) use trends and associated consequences through the presentation and discussion of quantitative and qualitative research data. The data in Tables 1 and 2 represent the proportion of patients with alcohol as the primary substance of abuse in Cape Town and Gauteng community, respectively. These are patients who had reported at various specialist treatment centres across all sites in Cape Town and Gauteng between 1996 and 2008.

Assuming that these patients are heavy alcohol drinkers, we now fit our model with the observed data. The fitting process involves the use of the least squares-curve fitting method. To fit our model we will use a Matlab code, and the unknown parameters will be assigned lower and upper bound from which a set of parameter values that produce the best fit will be obtained.

One measure of the severity of high-risk alcohol drinking is determined by the average number of new heavy alcohol drinkers generated by an individual heavy alcohol drinker during his or her time as a high-risk alcohol drinker, $\mathscr{R}_a$. Recall that

$$\mathscr{R}_a = \frac{\beta}{(\phi + \mu + \epsilon)}. \quad (20)$$

Now, using the fitted values in Table 3 it follows that the reproductive number for Gauteng and Cape Town is 3.71 and 3.96, respectively. Thus, one heavy alcohol drinker in either Gauteng or Cape Town is likely to influence approximately four light alcohol drinkers to become heavy alcohol drinkers during his or her drinking period as a heavy alcohol drinker.

Since the reproductive number of our proposed model is comprised of four model parameters we now investigate which of the four parameters have the greatest influence on increasing or decreasing the magnitude of the reproductive number. In computing the sensitivity analysis, we adopt the approach described by Arriola and Hyman [12]. The normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter. When the variable is a differentiable function of the parameter, the sensitivity index may be alternatively defined using partial derivatives.

Definition 5. The normalized forward sensitivity index of a variable, u, that depends differentiably on a parameter, p, is defined as

$$\Gamma_p^u := \frac{\partial u}{\partial p} \times \frac{p}{u}. \quad (21)$$

Table 4 illustrates the sensitivity indices of $\mathscr{R}_a$, evaluated at the baseline parameter values given in Table 3. Model parameters whose sensitivity index values are near −1 or +1 suggest that a change in their magnitude has a significant impact on either increasing or decreasing the size of $\mathscr{R}_a$. From Table 4 it is clear that $\mathscr{R}_a$ is most sensitive to β, such that an increase in β by 10% would increase $\mathscr{R}_a$ by 10% in all the communities, that is, Gauteng and Cape Town. Further, an increase in ϕ by 10% would decrease $\mathscr{R}_a$ 9,3% for Gauteng community and 5.8% for Cape Town community. Thus, alcohol-related treatment is likely to have more impact in Gauteng than in Cape Town.

In view of these results, one can suggest that policy makers should introduce control measures aimed at minimizing or eliminating excessive alcohol use in these communities. If control measures are to decrease endemicity, consideration of $\mathscr{R}_a$ echoes the importance of weakening the intensity of social interactions between the light and heavy drinkers, modelled by parameter β, since it is a parameter which has the greatest influence on changing the size of $\mathscr{R}_a$.

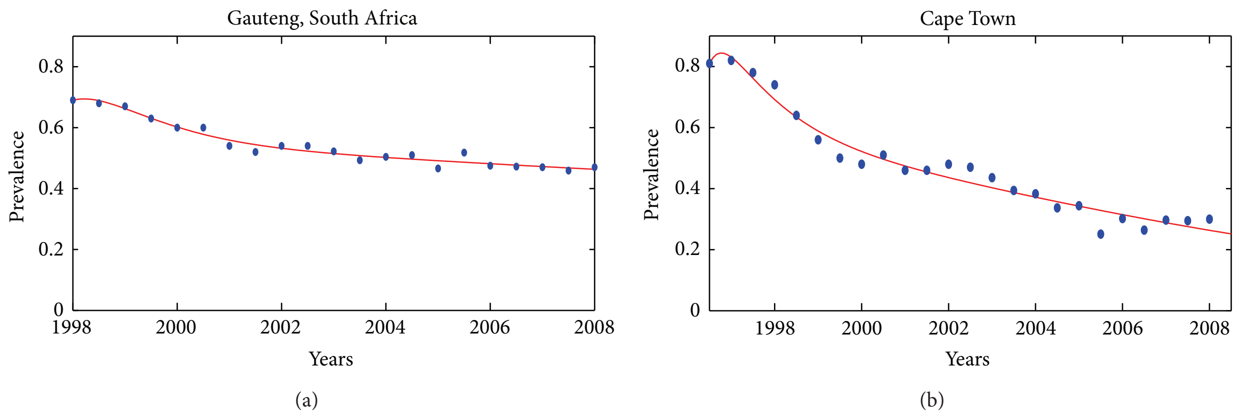


Figure 1: Model system (3) fitted to data for alcohol users who visited specialist treatment centres in (a) Gauteng and (b) Cape Town. The blue circles indicate the actual data and the solid line indicates the model fit to the data.

Table 1: Alcohol related treatment proportions in Cape Town, South Africa, 1996–2008.

Year	%
96b	81
97a	82
97b	78
98a	74
98b	64
99a	56
99b	50
00a	48
00b	51
01a	46
01b	46
02a	48
02b	47
03a	43.6
03b	39.4
04a	38.3
04b	33.7
05a	34.4
05b	25.1
06a	30.2
06b	26.4
07a	29.5
07b	29.7
08a	30.0

Table 2: Alcohol related treatment proportions in Gauteng, South Africa, 1996–2008.

Year	%
96b	—
97a	—
97b	—
98a	69
98b	68
99a	67
99b	63
00a	60
00b	60
01a	54
01b	52
02a	54
02b	54
03a	52.2
03b	49.3
04a	50.4
04b	51
05a	46.6
05b	51.8
06a	47.5
06b	47.2
07a	45.9
07b	47
08a	47

3. The Optimal Control Problem and Its Analysis

In this section, we aim at exploring the role of optimal intervention strategies on reducing or eliminating excessive alcohol drinking and its related health effects in the community. We extend system (3) to introduce three intervention methods, called controls. Controls are represented as functions of time and assigned reasonable upper and lower bounds. The first control $u_1(t)$ attempts to weaken the intensity of social interactions between the light and heavy drinkers. This control might be implemented through educational efforts or an increase of price or tax on alcohol beverages. For example, the tax hike on beer in California in 1991 resulted in gonorrhoea rates dropping by 30% the following year [8], and from this observation we can infer that alcohol drinking is correlated to gonorrhoea epidemic [8]. Thus, an increase on the price or levy of alcohol beverages

Table 3: Model parameters and their baseline values.

Model parameter	Units	Gauteng	Cape Town	Reference
α	—	0.0018	0.0018	[6]
μ	yr^{-1}	0.020	0.020	[13]
β	yr^{-1}	1.2884	2.000	Fitted
ψ	yr^{-1}	0.9600	0.6997	Fitted
ϕ	yr^{-1}	0.3238	0.2866	Fitted
p	—	0.0375	0.0071	Fitted
f	—	0.0489	0.0677	Fitted
ϵ	yr^{-1}	0.0039	0.1985	Fitted

Table 4: Sensitivity indices of $\mathscr{R}_a$.

Parameter	Gauteng	Cape Town
β	+1	+1
ϕ	−0.93	−0.58
μ	−0.058	−0.040
ϵ	−0.011	−0.39

can have a positive impact on reducing the intensity of the influence caused by heavy drinkers. The second control $u_2(t)$ is tied on increasing the proportion of treated individuals who permanently quit alcohol and the third control $u_3(t)$ is tied on the reduction of the relapsing population.

The new model incorporating time dependent intervention strategies is given by

$$\begin{aligned}\dot{s} &= \mu - (1-u_1)\, g(h)\, s - \mu s + (1-u_3)\,\psi a,\\ \dot{h} &= (1-u_1)\, g(h)\, s - (\phi + \epsilon + \mu)\, h,\\ \dot{a} &= (1-u_2)\,\phi h - (\psi + \mu)\, a.\end{aligned} \tag{22}$$

A successful mitigation scheme is one which reduces or eliminates excessive alcohol drinking and its related health effects at minimal cost. A control scheme is assumed to be optimal if it minimizes the objective functional:

$$J(u_1, u_2, u_3) = \int_0^T \left[h + \frac{A_1}{2}u_1^2 + \frac{A_2}{2}u_2^2 + \frac{A_3}{2}u_3^2\right] dt, \tag{23}$$

where A_1, A_2, A_3 are balancing coefficients transforming the integral into dollars expended over a finite time period of T years. The balancing coefficients account for the relative size and importance preassigned by the modelers to the contributing terms in the objective functional. The existence of optimal control follows from standard results in optimal control theory [14, 15]. The necessary conditions that optimal controls must satisfy are derived using Pontryagin's Maximum Principle (PMP) [15]. Thus, system (22) is converted into an equivalent problem, namely, the problem of minimizing the Hamiltonian H_a given by

$$\begin{aligned}H_a = h &+ \frac{A_1}{2}u_1^2 + \frac{A_2}{2}u_2^2 + \frac{A_3}{2}u_3^2\\ &+ \lambda_1\left[\mu - (1-u_1)\, g(h)\, s - \mu s + (1-u_3)\,\psi a\right]\\ &+ \lambda_2\left[(1-u_1)\, g(h)\, s - (\phi + \epsilon + \mu)\, h\right]\\ &+ \lambda_3\left[(1-u_2)\,\phi h - (\psi + \mu)\, a\right],\end{aligned} \tag{24}$$

where λ_i, for $i = 1, 2, 3$, are the adjoint functions associated with states s, h, and a, respectively. Note that in (24) each adjoint function multiplies the right-hand side of the differential equation of its corresponding state function. The first terms in H_a come from the integrand of the objective functional.

Given an optimal control u_j^{**} (for $j = 1, 2, 3$) and corresponding states (s^{**}, h^{**}, a^{**}), there exist adjoint functions satisfying

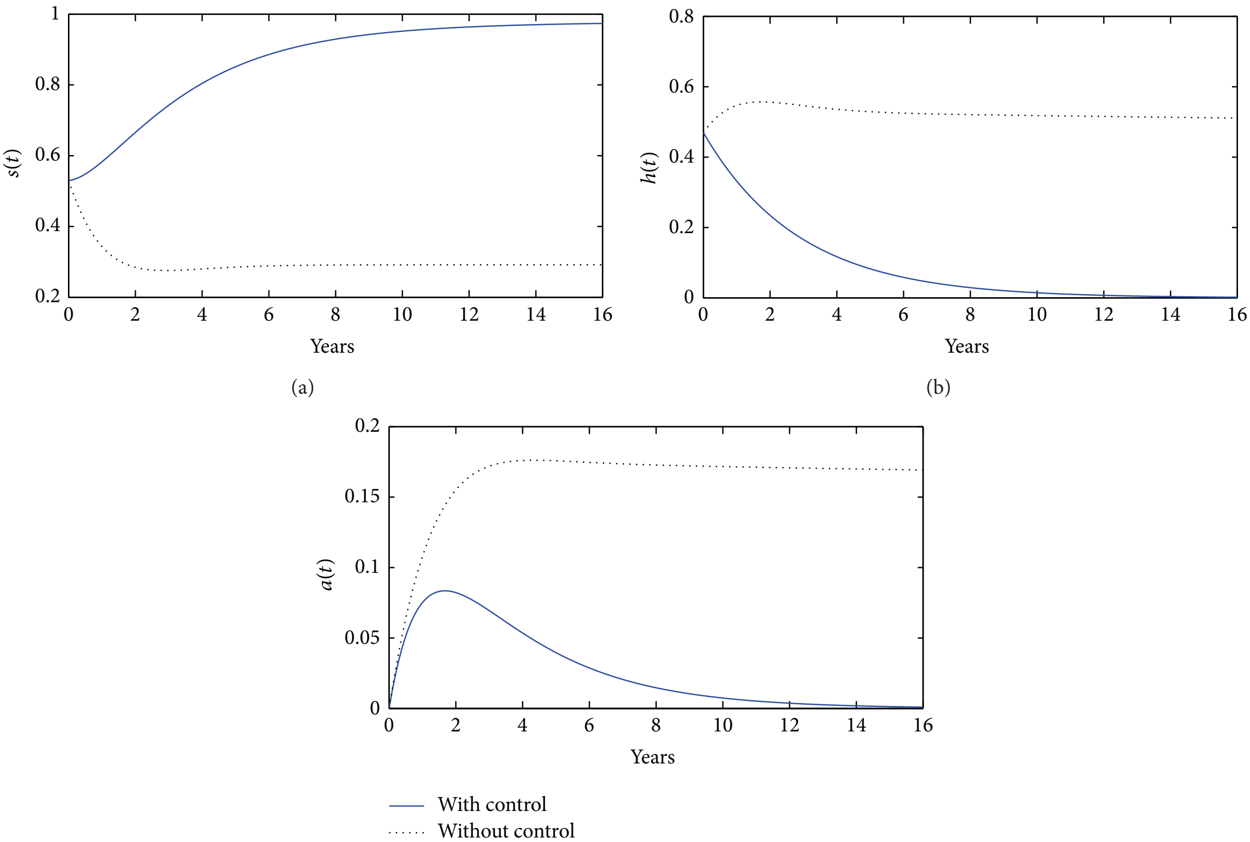
$$\begin{aligned}\dot{\lambda}_1(t) &= \mu\lambda_1 + g(h^{**})(1-u_1)(\lambda_1 - \lambda_2),\\ \dot{\lambda}_2(t) &= -1 + (\mu + \epsilon)\lambda_2 + \phi(\lambda_2 - (1-u_2)\lambda_3)\\ &\quad + g(h^{**})(\lambda_1 - \lambda_2)\, s^{**}\\ &\quad + \beta\alpha s^{**} h^{**}(\lambda_1 - \lambda_2),\\ \dot{\lambda}_3(t) &= \mu\lambda_3 + \psi(\lambda_3 - (1-u_3)\lambda_1),\end{aligned} \tag{25}$$

with transversality conditions, $\lambda_i(T) = 0$, for $i = 1, 2, 3$. Observe that the right-hand side of the first differential $\dot{\lambda}_1(t) = -\partial H_a/\partial s^{**}$ and similarly for the other adjoint functions. The final time boundary conditions (transversality conditions) are zero since there is no dependence on the states at the final time in the objective functional. Furthermore, the optimal controls are characterized by

$$\begin{aligned}u_1^{**}(t) &= \min\left\{\max\left\{0, \frac{g(h^{**})(\lambda_2 - \lambda_1)\, s^{**}}{A_1}\right\}, 1\right\},\\ u_2^{**}(t) &= \min\left\{\max\left\{0, \frac{\phi\lambda_3 h^{**}}{A_2}\right\}, 1\right\},\\ u_3^{**}(t) &= \min\left\{\max\left\{0, \frac{\psi\lambda_1 a^{**}}{A_3}\right\}, 1\right\}.\end{aligned} \tag{26}$$

The control characterization for $u_j^{**}(t)$, $j = 1, 2, 3$, is obtained on $\partial H_a/\partial u_j^{**} = 0$ whenever $0 < u_j^{**}(t) < 1$, taking bounds into account, and similarly for the other controls.

3.1. Optimal Control Numerical Results. In this section, we numerically explore the effects of the three controls on reducing or eliminating excessive alcohol use in the community. The state system (22) and the adjoint system of differential equations together with the control characterization above



(a) (b) (c)

FIGURE 2: Dynamics of system (22) showing the effects of optimal control strategies on eliminating or reducing excessive alcohol drinking in the Gauteng community. Recall that $\mathscr{R}_a = 3.71$.

from the optimality system are solved numerically using the "forward-backward sweep method" [16]. Cost coefficients are fixed within the integral expression (23) and the optimal schedule of the three controls over $T = 16$ years is simulated for (a) Gauteng and (b) Cape Town community. For the control we set $u_1 = 0.7$ while $u_2 = u_3 = 0.75$ and the weights are set to $A_1 = 0.0005$ and $A_2 = A_3 = 0.00003$. Here, we assume that control $u_1(t)$ is more costly compared to all the other controls, due to the fact that its implementation involves a large volume of people than any other control; thus $A_1 \geq (A_2, A_3)$.

(i) Case (a) Gauteng. Here, we utilize the parameter values for Gauteng (Table 3) together with the associated data (Table 2), to investigate the effects of time dependent intervention strategies on controlling high-risk alcohol drinking in Gauteng community. We set $h(0) = 0.47$ (the last recording on Gauteng data), $s_0 = 0.53$, and $a_0 = 0$.

Graphical results in Figure 2 demonstrate the effects of optimal intervention strategies on reducing or eliminating the problem of excessive alcohol drinking in Gauteng community. It is clear from these simulations that time dependent interventions can take 14 years to eliminate excessive alcohol use in Gauteng community.

In Figure 3 we examine the feasibility of the three optimal controls. An optimal control is regarded to be feasible if it starts at the upper bound and remains there till the end point (final time). A control which is feasible or attainable is essential on problem solving; hence more effort and resources should be devoted to such an intervention strategy for effective problem solving. Here, we note that control $u_1(t)$ is more feasible compared to control $u_2(t)$ and $u_3(t)$. Thus, for effective control of excessive alcohol use and alcohol-related violence more effort should be devoted to strategies and policies that weaken the interaction between light and heavy drinkers.

The total relative costs for all the strategies are estimated by computing the approximate area under control functions (Figure 3) with daily costs computed by dividing the total costs by the number of years where nonzero controls are applied [5]. In this case, the total relative cost for the 16-year strategy: $A_1 \cdot (\int_0^T u_1(t)dt = 16)$, $A_2 \cdot (\int_0^T u_2(t)dt = 3.19)$, and $A_3 \cdot (\int_0^T u_3(t)dt = 0.212)$, is 0.0081 with a yearly cost of 0.0005.

(ii) Case (b) Cape Town. We utilize the parameter values for Cape Town (Table 3) together with the associated data, to investigate the effects of time dependent intervention

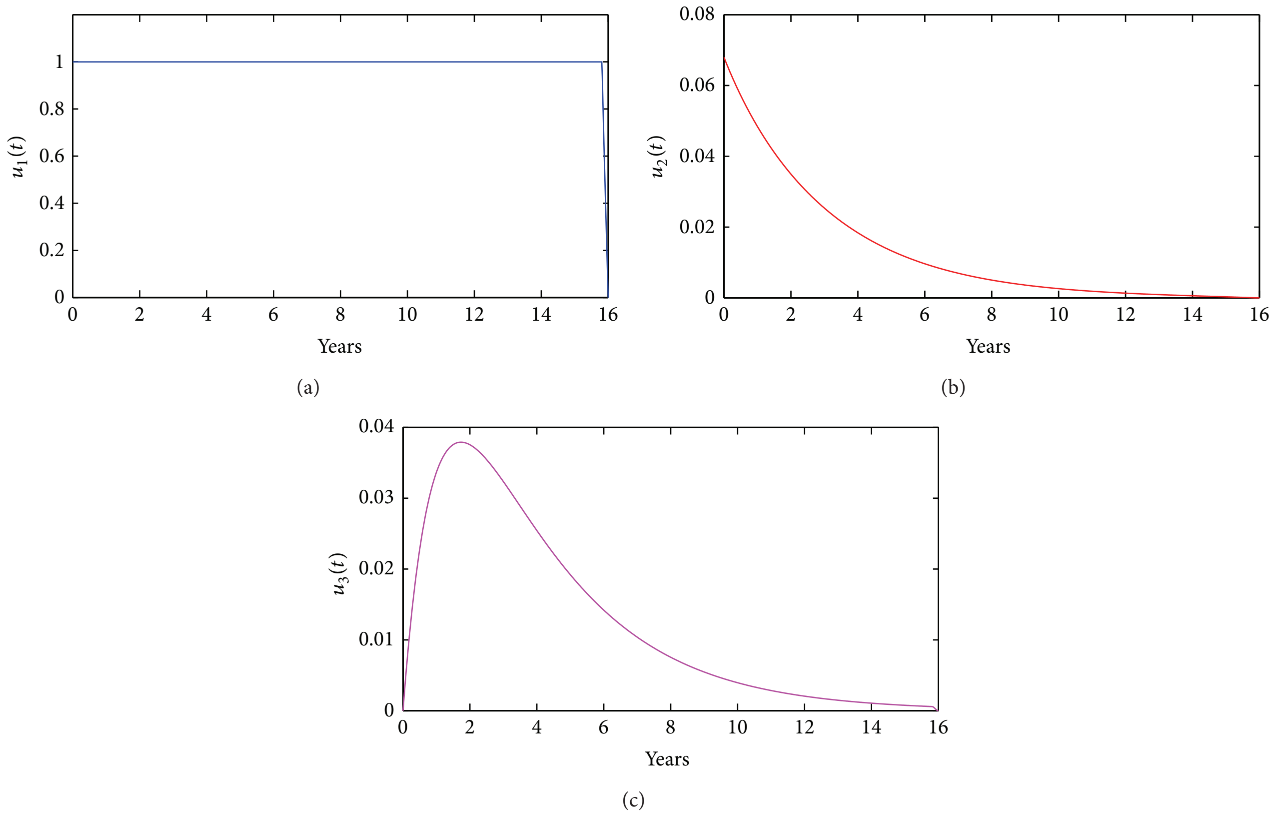


FIGURE 3: Time series plots illustrating the control profiles for controls: (a) $u_1(t)$, (b) $u_2(t)$, and (c) $u_3(t)$, and recall that $\mathcal{R}_a = 3.71$.

strategies on controlling high-risk alcohol drinking in Cape Town. We set $h(0) = 0.3$ (the last recording on Cape Town data), $s_0 = 0.7$, and $a_0 = 0$.

Based on our simulations (Figure 4) we can deduce that high-risk alcohol drinking problem can be eliminated in Cape Town, after about 10 years of implementing the aforementioned time dependent intervention strategies. Further, we observe that eliminating alcohol drinking problem in Gauteng will require an additional 4 years compared to the Cape Town scenario.

Figure 5 highlights the feasibility of controls for the Cape Town scenario. The results support earlier findings that more effort should be devoted to strategies and policies that weaken the interaction between light and heavy drinkers for effective control of alcohol drinking problem. The total relative cost for the 16-year strategy for Cape Town: $A_1 \cdot (\int_0^T u_1(t)dt = 15.5)$, $A_2 \cdot (\int_0^T u_2(t)dt = 0.46)$, and $A_3 \cdot (\int_0^T u_3(t)dt = 0.411)$, is 0.0078 with a yearly cost of 0.00049. Here, we observe that the relative costs for Cape Town are slightly more compared to the Gauteng case.

3.2. Efficacy of Optimal Intervention Strategies. In this section we explore the efficacy of optimal intervention strategies of reducing excessive alcohol drinking in the community. We define the efficacy function $E(t)$

$$E(t) = \frac{h(0) - h^*(t)}{h(0)} = 1 - \frac{h^*(t)}{h(0)}, \tag{27}$$

where $h^*(t)$ represent the optimal solutions associated with the optimal control of the corresponding variable and $h(0)$ denote the corresponding initial condition. Function (27) measures the proportional decrease in the number of heavy alcohol drinkers imposed by the intervention with controls (u_1, u_2, u_3), by comparing the number of heavy alcohol drinkers $h^*(t)$ with the initial conditions for which there are no controls implemented, $u_1 = u_2 = u_3 = 0$. By construction, $E(t) \in [0, 1]$ for all time t. Thus, the upper bound of $E(t)$ is one.

Figure 6 illustrates the efficacy of control in Gauteng and Cape Town for a period of 16 years. The results reaffirm earlier findings that alcohol drinking problem can be eliminated after 10 years of implementing these controls in Cape Town and 14 years in Gauteng. Further, it evident that after 4 years of implementing the controls in Cape Town, the efficacy level will take a value above 80% while in Gauteng it will be less than 80% and this re-affirms that less time can be required for elimination of alcohol drinking problem in Cape Town than in Gauteng.

4. Concluding Remarks

Excessive alcohol consumption remains a major health challenge in both developed and developing nations. In this paper, two mathematical models for alcohol drinking incorporating peer-influence have been proposed and qualitatively analyzed. The transmission process is modeled as

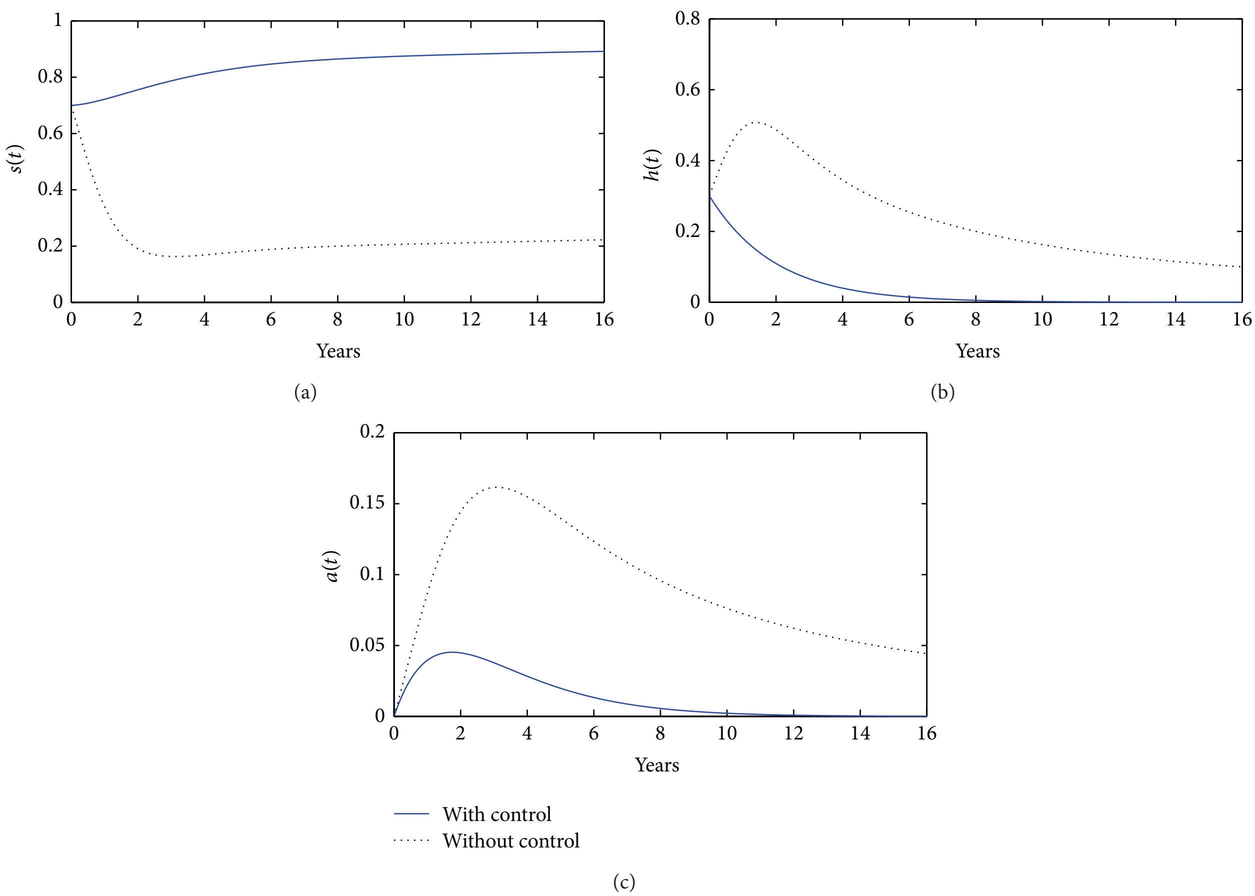


FIGURE 4: Dynamics of system (22) showing the effects of optimal control strategies on eliminating or reducing excessive alcohol drinking in Cape Town. Recall that $\mathcal{R}_a = 3.96$.

a social "contact" process between "heavy" alcohol drinkers and "light" alcohol drinkers within an unchanging shared drinking environment. The first model is an autonomous system with constant parameters that incorporates the relevant alcohol drinking components and alcohol-related treatment. The model reproductive number $\mathcal{R}_a$ was derived and proven to be a sharp threshold for disease dynamics. Particularly, whenever the reproductive number is less than unity, alcohol drinking dies out, and when the reproductive number is greater than unity, there exists a unique endemic equilibrium that is globally asymptotically stable. Due to lack of data regarding the parameter values, we used the observed data on alcohol drinking problem (the data reflect the proportion of alcohol users who visited specialist treatment centres) in Cape Town and Gauteng, South Africa, for a period ranging from 1996 to 2008 [11]. Numerical illustrations demonstrate that policy makers should introduce control measures aimed at minimizing or eliminating excessive alcohol use in these communities. If control measures are to decrease endemicity, consideration of $\mathcal{R}_a$ echoes the importance of weakening the intensity of social interactions between the light and heavy drinkers, modelled by parameter β, since it is a parameter which has the greatest influence on changing the size of $\mathcal{R}_a$.

In the second model, we extended the autonomous system to incorporate time dependent intervention strategies. Three intervention methods, called controls, have been introduced into our earlier model. Controls are represented as functions of time and assigned reasonable upper and lower bounds. The first control $u_1(t)$ attempts to weaken the intensity of social interactions between the light and heavy drinkers. The second control aims at increasing the number of treated individuals who permanently quit alcohol use, and the third control is tied on reducing the number of relapsing individuals. Among other important results our study suggests that for effective control of excessive alcohol use in the community more attention should be devoted to the first control than any other control. In addition, our analysis demonstrates that time dependent interventions have the potential to eliminate the problem of excessive alcohol use in Gauteng community in a period of 14 years and in 10 years for Cape Town community.

Despite a number of simplifying assumptions and the unavailability of a large data, we hope that the model will allow public health personnel and policy makers to plan effectively so as to reduce alcohol drinking problem in different communities.

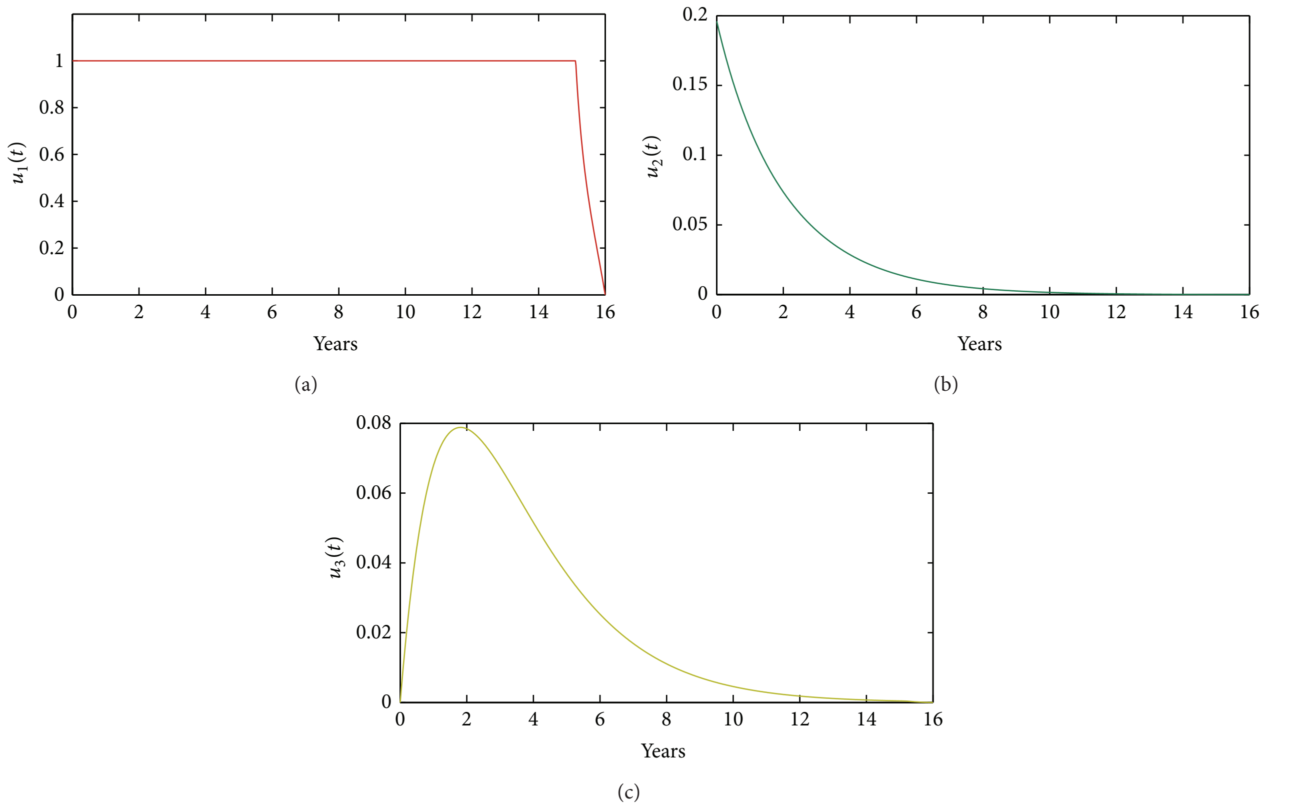


FIGURE 5: Time series plots illustrating the control profiles for controls: (a) $u_1(t)$, (b) $u_2(t)$, and (c) $u_3(t)$, and recall that $\mathcal{R}_a = 3.96$.

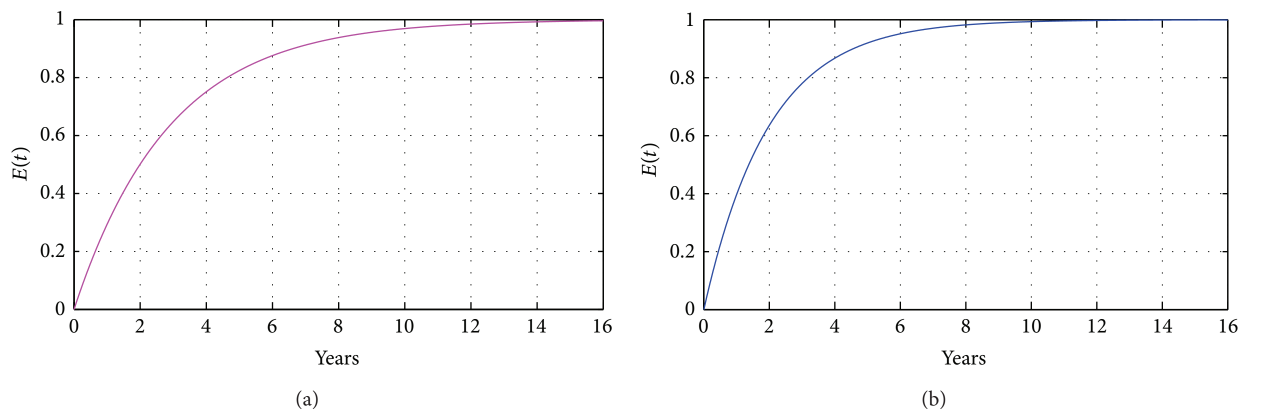


FIGURE 6: Time series plot demonstrating the efficacy of optimal intervention strategies over a period of 16 years: (a) Gauteng case and (b) Cape Town case.

Conflict of Interests

The author declared that there is no conflict of interests.

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References

[1] T. Budd, "Alcohol-related assault: findings from the British Crime Survey," Home Office on-Line Report 35/03, Home Office, London, UK, 2003.

[2] A. L. Greenfeld, *Alcohol and Crime: An Analysis of National Data on the Prevalence of Alcohol Involvement in Crime*, 1998.

[3] Centers for Disease Control and Prevention (CDC), *Alcohol-Related Disease Impact (ARDI)*, CDC, Atlanta, Ga, USA, 2014.

[4] G. Thomas and E. M. Lungu, "The influence of heavy alcohol consumption on HIV infection and progression," *Journal of Biological Systems*, vol. 17, no. 4, pp. 685–712, 2009.

[5] S. Lee, E. Jung, and C. Castillo-Chavez, "Optimal control intervention strategies in low and high risk problem drinking populations," *Socio-Economic Planning Sciences*, vol. 44, pp. 258–265, 2010.

[6] B. Buonomo and D. Lacitignola, "Modeling peer influence effects on the spread of high-risk alcohol consumption behavior," *Ricerche Di Matematica*, vol. 63, no. 1, pp. 101–117, 2014.

[7] X.-Y. Wang, H.-F. Ho, Q.-K. Kong, and W.-X. Shi, "Optimal control strategies in an alcoholism model," *Abstract and Applied Analysis*, vol. 2014, Article ID 954069, 18 pages, 2014.

[8] S. Mushayabasa and C. P. Bhunu, "Modeling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhoea," *Nonlinear Dynamics*, vol. 66, no. 4, pp. 695–706, 2011.

[9] J. S. LaSalle, "The stability of dynamical systems," in *Proceedings of the CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 25, SIAM, Philadelphia, Pa, USA, 1976.

[10] A. Korobeinikov, "Global properties of *SIR* and *SEIR* epidemic models with multiple parallel infectious stages," *Bulletin of Mathematical Biology*, vol. 71, no. 1, pp. 75–83, 2009.

[11] A. Pluddemann, S. Dada, C. Parry, A. Bhana et al., "Monitoring alcohol and drug abuse trends in South Africa, South Africa Community Epidemiology Network on Drug Use (SACENDU)," *SACENDU Research Brief*, vol. 11, no. 2, 2008.

[12] L. Arriola and J. M. Hyman, "Sensitivity analysis for quantifying uncertainty in mathematical models," in *Mathematical and Statistical Estimation Approaches in Epidemiology*, G. Chowell, J. M. Hyman, L. M. A. Bettencourt, and C. Castillo-Chavez, Eds., pp. 195–248, Springer, NewYork, NY, USA, 2009.

[13] F. Nyabadza, J. B. H. Njagarah, and R. J. Smith, "Modelling the dynamics of crystal meth ('Tik') abuse in the presence of dug-supply chains in South Africa.," *Bulletin of Mathematical Biology*, vol. 75, no. 1, pp. 24–48, 2013.

[14] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, NewYork, NY, USA, 1975.

[15] L. S. Pontryagin, V. T. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishcheuko, *The Mathematical Theory of Optimal Processes*, Wiley, New Jersey, NJ, USA, 1962.

[16] S. Lenhart and J. T. Workman, *Optimal Control Applied to Biological Models*, Chapman & Hall, London, UK, 2007.