Research Article

Numerical Solution of Pantograph-Type Delay Differential Equations Using Perturbation-Iteration Algorithms

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The pantograph equation is a special type of functional differential equations with proportional delay. The present study introduces a compound technique incorporating the perturbation method with an iteration algorithm to solve numerically the delay differential equations of pantograph type. We put forward two types of algorithms, depending upon the order of derivatives in the Taylor series expansion. The crucial convenience of this method when compared with other perturbation methods is that this method does not require a small perturbation parameter. Furthermore, a relatively fast convergence of the iterations to the exact solutions and more accurate results can be achieved. Several illustrative examples are given to demonstrate the efficiency and reliability of the technique, even for nonlinear cases.

1. Introduction and Preliminaries

The pantograph equation is a special type of functional differential equations with proportional delay. It arises in rather different fields of pure and applied mathematics, such as electrodynamics, control systems, number theory, probability, and quantum mechanics. Many researchers have studied the pantograph-type delay differential equation using analytical and numerical techniques [1–8]. The second-order pantograph-type delay differential equation is given as [7, 8]

$$\frac{d^2 u(x)}{dx^2} = f(x, u(x), u(\alpha x)), \quad x \in (0, T) \qquad (1)$$

with the boundary conditions

$$
\begin{aligned}
u(0) &= b, \\
u(T) &= c,
\end{aligned}
\qquad (2)
$$

where $T > 0$ and $0 < \alpha < 1$.

A brief review of recent literature on the methods of solution for pantograph-type delay differential equations can be found in the paper by Trif [9].

On the other hand, perturbation methods [10] have been among the most common approximate methods used for solving algebraic equations, differential equations, integrod-ifferential equations, and difference equations. The primary limitation of these methods is the necessity of a small parameter. This parameter might come out as an original physical parameter of the given equation or, alternatively, it may be inserted as an artificial parameter. Several different methods have been established in order to deal with this limitation.

In this study, we present the application of a hybrid technique combining the perturbation method with an iteration algorithm to find a numerical solution for pantograph-type delay differential equations. Various researchers considered this new perturbation-iteration method [11–16]. Pakdemirli and coworkers, besides others, have studied extensively this method. The first study on algebraic equations was carried out

by Pakdemirli and Boyacı [17] where they proposed new root finding algorithms using the new systematic approach. Later, fourth-order [18] and fifth-order [19] derivative algorithms were presented. Aksoy and Pakdemirli [20] systematically generated the present method for both linear and nonlinear second-order differential equations and applied to it Bratu-type equations. Dolapçi et al. [21] used this method to solve Fredholm and Volterra integral equations and argued that it can be applied to both types of integral equations.

The main purpose of the work presented in this paper is to apply the new perturbation-iteration solution for the pantograph-type delay differential equations. To the authors' knowledge, the application of perturbation-iteration algorithms to pantograph-type delay differential equations is novel. The two types of perturbation-iteration algorithms, PIA$(1, 1)$ and PIA$(1, 2)$, are introduced in the next section. Then, the method is applied to pantograph equations via six examples, two of which are first-order linear, one is first-order nonlinear, and the others are second-order nonlinear; they showed excellent agreement with the published results and verified the accuracy and efficiency of the present method. The results are discussed and commented on.

The general form of pantograph-type delay differential equations of both first- and second-order, including (1), can be stated as

$$F\left(u'', u', u, u_\alpha, \varepsilon, x\right) = 0, \tag{3}$$

where $u = u(x)$, $u_\alpha = u(\alpha x)$, $0 < \alpha < 1$ is a given constant, and ε is the perturbation parameter. In this study, we investigated a solution for (3) which is closed form. Equation (3) covers many different problems studied by various researchers.

2. Perturbation-Iteration Algorithms

Perturbation-iteration algorithms are briefly called PIA(n, m). Here, n denotes the number of correction terms in the perturbation expansion and m denotes the order of derivatives in the Taylor series expansion. Generally, n is smaller than or equal to m; otherwise, the correction terms cannot be calculated. In this study, PIA$(1, 1)$ and PIA$(1, 2)$ algorithms constructed by taking correction term in the perturbation expansion together with first- and second-order derivatives, respectively, are employed.

2.1. Perturbation-Iteration Algorithm PIA(1, 1).
In PIA$(1, 1)$ algorithm, we propose a perturbation-iteration algorithm by taking one correction term in the perturbation expansion and correction terms of only the first derivatives in the Taylor expansion; that is, $n = 1$, $m = 1$. Let us consider a second-order pantograph differential equation written in the form of (3). Taking only one correction term in the perturbation expansion, the straightforward expansion for the solution of each iteration can be written as follows:

$$u_{n+1} = u_n + \varepsilon \left(u_c\right)_n, \tag{4}$$

where $(u_c)_n$ is the nth correction term of the perturbation-iteration algorithm. Then, we substitute (4) into (3) and expand it in a Taylor series with first derivatives to obtain

$$
\begin{aligned}
F&\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \\
&+ F_{u''}\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \varepsilon \left(u_c''\right)_n \\
&+ F_{u'}\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \varepsilon \left(u_c'\right)_n \\
&+ F_u\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \varepsilon \left(u_c\right)_n \\
&+ F_{u_\alpha}\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \varepsilon \left((u_\alpha)_c\right)_n \\
&+ F_\varepsilon\left(u_n'', u_n', u_n, (u_\alpha)_n, 0\right) \varepsilon = 0,
\end{aligned}
\tag{5}
$$

where $(\ )'$ denotes differentiation with respect to the independent variable and

$$
\begin{aligned}
F_{u''} &= \frac{\partial F}{\partial u''}, \\[4pt]
F_{u'} &= \frac{\partial F}{\partial u'}, \\[4pt]
F_u &= \frac{\partial F}{\partial u}, \\[4pt]
F_{u_\alpha} &= \frac{\partial F}{\partial u_\alpha}, \\[4pt]
F_\varepsilon &= \frac{\partial F}{\partial \varepsilon}.
\end{aligned}
\tag{6}
$$

Bearing in mind that all derivatives are evaluated at $\varepsilon = 0$ and rewriting the equation in the following more suitable form

$$
\begin{aligned}
F_{u''}\left(u_c''\right)_n &+ F_{u'}\left(u_c'\right)_n + F_u \left(u_c\right)_n + F_{u_\alpha}\left((u_\alpha)_c\right)_n \\
&= -F_\varepsilon - \frac{F}{\varepsilon},
\end{aligned}
\tag{7}
$$

one may easily notice that (7) is a variable coefficient pantograph equation. Starting with an initial guess u_0, first, we determine $(u_c)_0$ from (7) and then substitute it into (4) for calculating u_1. This iteration procedure is repeated using (4) and (7) until the approximation is sufficient within a user-defined threshold.

2.2. Perturbation-Iteration Algorithm PIA(1, 2).
In PIA$(1, 2)$ algorithm, we put forward a perturbation-iteration algorithm by taking one correction term in the perturbation expansion and correction terms of up to second derivatives in the Taylor expansion; that is, $n = 1$, $m = 2$. Only one correction term perturbation expansion was given before in (4). Now,

we substitute (4) into (3) and expand in a Taylor series of up to second-order derivatives to obtain

$$F + F_{u''}\varepsilon \left(u_c''\right)_n + F_{u'}\varepsilon \left(u_c'\right)_n + F_u\varepsilon \left(u_c\right)_n$$

$$+ F_{u_\alpha}\varepsilon \left((u_\alpha)_c\right)_n + F_\varepsilon \varepsilon + \frac{1}{2}F_{u''u''}\varepsilon^2 \left(u_c''\right)_n^2$$

$$+ \frac{1}{2}F_{u'u'}\varepsilon^2 \left(u_c'\right)_n^2 + \frac{1}{2}F_{uu}\varepsilon^2 \left(u_c\right)_n^2$$

$$+ \frac{1}{2}F_{u_\alpha u_\alpha}\varepsilon^2 \left((u_\alpha)_c\right)_n^2 + \frac{1}{2}\varepsilon^2 F_{\varepsilon\varepsilon}$$

$$+ F_{u''u'}\varepsilon^2 \left(u_c''\right)_n \left(u_c'\right)_n + F_{u''u}\varepsilon^2 \left(u_c''\right)_n \left(u_c\right)_n \quad (8)$$

$$+ F_{u''u_\alpha}\varepsilon^2 \left(u_c''\right)_n \left((u_\alpha)_c\right)_n + F_{u''\varepsilon}\varepsilon^2 \left(u_c''\right)_n$$

$$+ F_{u'u}\varepsilon^2 \left(u_c'\right)_n \left(u_c\right)_n + F_{u'u_\alpha}\varepsilon^2 \left(u_c'\right)_n \left((u_\alpha)_c\right)_n$$

$$+ F_{u'\varepsilon}\varepsilon^2 \left(u_c'\right)_n + F_{uu_\alpha}\varepsilon^2 \left(u_c\right)_n \left((u_\alpha)_c\right)_n$$

$$+ F_{u\varepsilon}\varepsilon^2 \left(u_c\right)_n + F_{u_\alpha\varepsilon}\varepsilon^2 \left((u_\alpha)_c\right)_n = 0.$$

Then, bearing in mind that all derivatives are evaluated at $\varepsilon = 0$, we rewrite this equation in the following more suitable form:

$$\left(\varepsilon F_{u''} + \varepsilon^2 F_{u''\varepsilon}\right)\left(u_c''\right)_n + \left(\varepsilon F_{u'} + \varepsilon^2 F_{u'\varepsilon}\right)\left(u_c'\right)_n$$

$$+ \left(\varepsilon F_u + \varepsilon^2 F_{u\varepsilon}\right)\left(u_c\right)_n + \left(\varepsilon F_{u_\alpha} + \varepsilon^2 F_{u_\alpha\varepsilon}\right)\left((u_\alpha)_c\right)_n$$

$$+ \frac{1}{2}\varepsilon^2 F_{u''u''}\left(u_c''\right)_n^2 + \frac{1}{2}\varepsilon^2 F_{u'u'}\left(u_c'\right)_n^2$$

$$+ \frac{1}{2}\varepsilon^2 F_{uu}\left(u_c\right)_n^2 + \frac{1}{2}\varepsilon^2 F_{u_\alpha u_\alpha}\left((u_\alpha)_c\right)_n^2 \quad (9)$$

$$+ \varepsilon^2 F_{u''u'}\left(u_c''\right)_n\left(u_c'\right)_n + \varepsilon^2 F_{u''u}\left(u_c''\right)_n\left(u_c\right)_n$$

$$+ \varepsilon^2 F_{u''u_\alpha}\left(u_c''\right)_n\left((u_\alpha)_c\right)_n + \varepsilon^2 F_{u'u}\left(u_c'\right)_n\left(u_c\right)_n$$

$$+ \varepsilon^2 F_{u'u_\alpha}\left(u_c'\right)_n\left((u_\alpha)_c\right)_n + \varepsilon^2 F_{uu_\alpha}\left(u_c\right)_n\left((u_\alpha)_c\right)_n$$

$$= -\varepsilon F_\varepsilon - F - \frac{1}{2}\varepsilon^2 F_{\varepsilon\varepsilon}.$$

Equation (9) is a variable coefficient pantograph equation of order two. We repeat the previously mentioned iteration procedure until the approximation is sufficient within a user-defined threshold.

3. Application of the Method to the Pantograph-Type Delay Differential Equations

In order to illustrate the accuracy and applicability of the presented method, the new perturbation-iteration algorithm is applied to six pantograph-type delay differential equations of different types. For comparison purposes, the solution intervals of problems are chosen generally the same as those in the references. Yet, the solution intervals are enlarged in illustrative examples 3 and 5. For a further wider interval, the method should be modified such that the Taylor series expansions are made around points other than zero.

3.1. Illustrative Example 1. Consider the first-order linear pantograph differential equation [25]:

$$u'(x) + u(x) - \frac{1}{10}u\left(\frac{x}{5}\right) = -\frac{1}{10}e^{-(1/5)x}, \quad 0 \le x \le 1 \quad (10)$$

with the initial condition

$$u(0) = 1 \quad (11)$$

for which the exact solution is $u(x) = e^{-x}$ [25].

Equation (10) can be written in the following form:

$$u'(x) + u(x) - \varepsilon u\left(\frac{x}{5}\right) + \frac{1}{10}e^{-(1/5)x} = 0, \quad (12)$$

where $\varepsilon = 0.1$ is assumed to be the perturbation parameter. The nonzero terms of (7) in PIA(1, 1) algorithm are

$$F_{u'} = 1,$$

$$F_u = 1,$$

$$F_\varepsilon = -u_n\left(\frac{x}{5}\right), \quad (13)$$

$$F = u_n'(x) + u_n(x) + \frac{1}{10}e^{-(1/5)x}.$$

Upon substituting the terms of (13), (7) reduces to

$$\left(u_c'\right)_n(x) + \left(u_c\right)_n(x)$$

$$= u_n\left(\frac{x}{5}\right) - \frac{u_n'(x) + u_n(x) + (1/10)e^{-(1/5)x}}{0.1}. \quad (14)$$

When applying the iteration formula (4), we select an initially assumed function. Here, we start with the following trivial solution which satisfies the given initial condition:

$$u_0 = 1. \quad (15)$$

TABLE 1: Comparison of the absolute errors of the PIA$(1, 1)$ method with other numerical methods for example 1.

x	VIM [22] $n = 4$	Taylor method [23]	Continuous method [24]	JRC method $\alpha = \beta = 0.5$ [25] $N = 64$	PIA$(1, 1)$ $n = 4$
2^{-1}	$2.40e - 15$	$1.24e - 10$	$3.33e - 11$	$3.93e - 14$	$1.43e - 15$
2^{-2}	$7.91e - 17$	$9.74e - 11$	$4.13e - 11$	$2.16e - 14$	$7.91e - 17$
2^{-3}	$2.53e - 18$	$7.00e - 11$	$4.62e - 11$	$1.66e - 15$	$2.53e - 18$
2^{-4}	$8.03e - 20$	$9.14e - 11$	$4.89e - 11$	$6.21e - 15$	$8.03e - 20$
2^{-5}	$2.52e - 21$	$5.28e - 11$	$5.05e - 11$	$3.68e - 14$	$2.52e - 21$
2^{-6}	$7.00e - 23$	$1.95e - 11$	$4.74e - 11$	$4.54e - 14$	$7.92e - 23$

Using (4) and (14), the approximate solutions at each step become

$$u_1 = \frac{1}{10} - \frac{1}{8}e^{-(1/5)x} + \frac{41}{40}e^{-x},$$

$$u_2 = \frac{1}{10^2} + \frac{1}{320}e^{-(1/5)x} - \frac{5}{384}e^{-(1/25)x} + \frac{9599}{9600}e^{-x},$$

$$u_3 = \frac{1}{10^3} - \frac{1}{76800}e^{-(1/5)x} + \frac{1}{3072}e^{-(1/25)x}$$
$$- \frac{125}{95232}e^{-(1/125)x} + \frac{11904001}{11904000}e^{-x},$$

$$u_4 = \frac{1}{10^4} + \frac{1}{95232000}e^{-(1/5)x} - \frac{1}{737280}e^{-(1/25)x}$$
$$+ \frac{25}{761825}e^{-(1/125)x} - \frac{15625}{118849536}e^{-(1/625)x}$$
$$+ \frac{74280959999}{74280960000}e^{-x}$$

$$\vdots$$

Table 1 gives a comparison of the absolute errors of the PIA$(1, 1)$ method with those of other numerical methods. It is apparent from the table that the absolute errors of PIA$(1, 1)$ method are smaller than those of the other methods for $n = 4$, and also, considering the first four iteration solutions given in (16), one may notice that the numerical solution converges to the exact solution. Absolute errors obtained by PIA$(1, 1)$ method are almost the same as those obtained by the Variational Iteration Method (VIM) [22] for $n = 4$.

3.2. Illustrative Example 2.
Consider the first-order linear pantograph differential equation [3]:

$$u'(x) + u(x) + u(0.8x) = 0, \quad 0 \le x \le 1 \tag{17}$$

with the initial condition

$$u(0) = 1 \tag{18}$$

which does not have an exact solution. Equation (17) can be written in the following form:

$$u'(x) + u(x) + \varepsilon u(0.8x) = 0, \tag{19}$$

where ε is the artificially introduced perturbation parameter. The nonzero terms of (7) in PIA$(1, 1)$ algorithm are

$$F_{u'} = 1,$$
$$F_u = 1,$$
$$F_\varepsilon = u_n(0.8x), \tag{20}$$
$$F = u'_n(x) + u_n(x).$$

For this specific example, (7) reduces to

$$\left(u'_c\right)_n(x) + \left(u_c\right)_n(x) = -u_n(0.8x) - u'_n(x) - u_n(x). \tag{21}$$

When applying the iteration formula (4), we start with the following trivial solution as the initial function, which satisfies the given initial conditions:

$$u_0 = 1. \tag{22}$$

Using (4) and (21), the successive iteration results are

$$u_1 = -1 + 2e^{-x},$$

$$u_2 = 1 + 10e^{-x} - 10e^{-(4/5)x},$$

$$u_3 = -1 + \frac{218}{9}e^{-x} - 50e^{-(4/5)x} + \frac{250}{9}e^{-(16/25)x},$$

$$u_4 = 1 + \frac{21490}{549}e^{-x} - \frac{1090}{9}e^{-(4/5)x} + \frac{1250}{9}e^{-(16/25)x}$$
$$- \frac{31250}{549}e^{-(64/125)x}, \tag{23}$$

$$\vdots$$

Table 2 shows the PIA$(1, 1)$ solutions of (17) for iteration numbers $n = 5, 6, 7, 8, 9$. As stated above, we do not know the exact solution; therefore, we estimate a solution and observe a convergence.

A comparison of the results obtained by several different methods is shown in Table 3. The numerical results are consistent.

3.3. Illustrative Example 3.
Consider the first-order nonlinear pantograph differential equation [29]:

$$u'(x) = -2u^2\left(\frac{x}{2}\right) + 1, \quad 0 \le x \le 1 \tag{24}$$

TABLE 2: Convergence of the PIA(1, 1) solution for example 2.

x_i	$u_5(x)$	$u_6(x)$	$u_7(x)$	$u_8(x)$	$u_9(x)$
0	1	1	1	1	1
0.2	0.6646909	0.6646910	0.6646910	0.6646910	0.6646910
0.4	0.4335604	0.4335607	0.4335607	0.4335607	0.4335607
0.6	0.2764790	0.2764824	0.2764823	0.2764823	0.2764823
0.8	0.1714677	0.1714846	0.1714841	0.1714841	0.1714841
1	0.1026142	0.1026723	0.1026700	0.1026701	0.1026701

with the initial condition

$$u(0) = 0 \tag{25}$$

which has the exact solution $u(x) = \sin(x)$.

Equation (24) can be written in the following form:

$$u'(x) + \varepsilon u^2\left(\frac{x}{2}\right) - 1 = 0, \tag{26}$$

where $\varepsilon = 2$ is assumed to be the perturbation parameter. The nonzero terms of (7) in PIA(1, 1) algorithm are

$$F_{u'} = 1,$$

$$F_\varepsilon = u_n^2\left(\frac{x}{2}\right), \tag{27}$$

$$F = u_n'(x) - 1.$$

The nth correction term of the perturbation expansion in (4) is

$$\left(u_c'\right)_n(x) = -u_n^2\left(\frac{x}{2}\right) - \frac{u_n'(x) - 1}{2}. \tag{28}$$

In this example, we start with the initial guess:

$$u_0 = x. \tag{29}$$

Thus, the successive approximations are

$$u_1 = x - \frac{1}{6}x^3,$$

$$u_2 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{8064}x^7,$$

$$u_3 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{61}{23224320}x^9 \tag{30}$$

$$- \frac{67}{3406233600}x^{11} + \frac{1}{12881756160}x^{13}$$

$$- \frac{1}{7990652436480}x^{15}.$$

Subsequently, we determine the absolute maximum error for u_n as

$$E_n = \left\|u_n(x) - u(x)\right\|_\infty \tag{31}$$

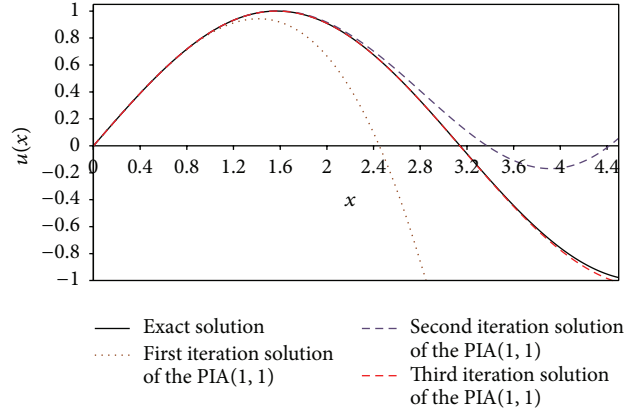$$= \max\left\{\left|u_n(x) - u(x)\right|, \ 0 \le x \le 1\right\}.$$

FIGURE 1: Comparison of the first three iteration solutions of the PIA(1, 1) method with the exact solution for example 3.

The absolute maximum errors E_n for different values of n are given in Table 4. Note that the error decreases continually as n increases.

The same problem is solved in various studies and it is seen that the absolute maximum error for solving this example via Homotopy Asymptotic Method is 1.2×10^{-6} [30] and via Optimal Homotopy Asymptotic Method is 4×10^{-8} [29] while the absolute maximum error obtained by the PIA(1, 1) method is 5.35×10^{-20}. This affirmed that the presented method is more accurate than the other two methods.

In Figure 1, the first three iteration solutions ($n = 1, 2, 3$) of the PIA(1, 1) method contrasted with the exact solution. It is revealed that even when the solution interval of the problem is expanded, PIA(1, 1) method goes on to yield convergent numerical solutions.

3.4. Illustrative Example 4. Consider the initial value problem of second-order differential equation with pantograph delay [31]:

$$u''(x) = -u\left(\frac{x}{2}\right) - u^2(x) + \sin^4(x) + \sin^2(x) + 8, \tag{32}$$

$$x \ge 0$$

with initial conditions

$$u(0) = 2,$$

$$u'(0) = 0 \tag{33}$$

which has the exact solution $u(x) = (5 - \cos 2x)/2$.

3.4.1. PIA(1, 1) Solution. Equation (32) can be written in the following form:

$$u''(x) + \varepsilon\left(u\left(\frac{x}{2}\right) + u^2(x)\right) \tag{34}$$

$$= \sin^4(x) + \sin^2(x) + 8,$$

TABLE 3: Comparison of the PIA(1, 1) solution with other numerical methods for example 2.

x_i	Walsh series method [26]	DUSF series method [27] $m = 100$, $h = 0.01$	Hermite series method [3] $N = 8$	Taylor series method [4] $N = 8$	Laguerre matrix method [28] $N = 8$	PIA(1, 1) $n = 8$
0	1	1	1	1	1	1
0.2	0.665621	0.664677	0.664691	0.664691	0.6646910	0.6646910
0.4	0.432426	0.433540	0.433561	0.433561	0.4335607	0.4335607
0.6	0.275140	0.276460	0.276482	0.276482	0.2764831	0.2764823
0.8	0.170320	0.171464	0.171484	0.171484	0.1714942	0.1714841
1	0.100856	0.102652	0.102670	0.102744	0.1027437	0.1026701

TABLE 4: The absolute maximum errors E_n for example 3.

n	1	2	3	4	5	6
E_n	$8.13e - 3$	$7.16e - 5$	$1.23e - 7$	$4.38e - 11$	$2.91e - 15$	$5.35e - 20$

where $\varepsilon = 1$ is the artificially introduced perturbation parameter. Using (3), (34) returns to

$$F\left(u'', u', u, u_\alpha, \varepsilon\right) = u''(x) + \varepsilon\left(u\left(\frac{x}{2}\right) + u^2(x)\right)$$
$$- \left(\sin^4(x) + \sin^2(x) + 8\right) = 0. \tag{35}$$

The nonzero terms of (7) in PIA(1, 1) algorithm are

$$F_{u''} = 1,$$

$$F_\varepsilon = u_n\left(\frac{x}{2}\right) + u_n^2(x), \tag{36}$$

$$F = u_n''(x) - \sin^4(x) - \sin^2(x) - 8.$$

Substituting (36), (32) reduces to

$$\left(u_c''\right)_n(x) = -u_n\left(\frac{x}{2}\right) - u_n''(x) - u_n^2(x) + \sin^4(x)$$
$$+ \sin^2(x) + 8. \tag{37}$$

The initial trial function is selected as

$$u_0 = 2, \tag{38}$$

and using (37), the approximate first iteration solution is obtained as

$$u_1 = \frac{13}{16} - \frac{1}{16}\sin^4(x) + \frac{3}{16}\cos^2(x) + \frac{23x^2}{16}$$
$$+ \cos^2\left(\frac{x}{2}\right) \tag{39}$$

$$\vdots$$

Even the first iteration of PIA(1, 1) solution, u_1, gave the same result as the first iteration solution obtained by the VIM [31]. Still, the second and third iteration solutions of the problem are better than those of the VIM solutions as observed in Figures 2 and 3.
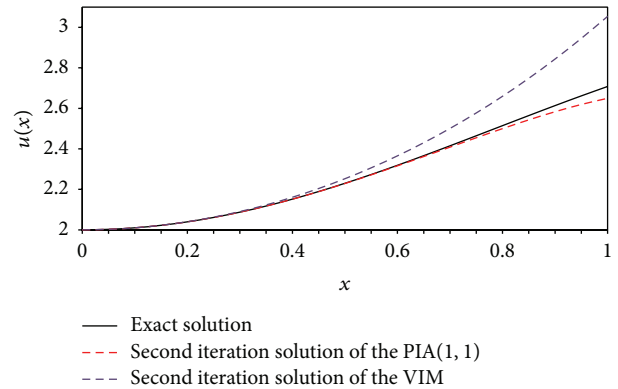

FIGURE 2: Comparison of the second iteration solutions of the VIM and PIA(1, 1) with exact solution for example 4.

— Exact solution
- - - Second iteration solution of the PIA(1, 1)
- - - Second iteration solution of the VIM

3.4.2. PIA(1, 2) Solution. In order to apply the PIA(1, 2) algorithm obtained by (9), we have to write (32) in the following form:

$$F\left(u'', u', u, u_\alpha, \varepsilon\right) = u''(x) + \varepsilon u^2(x) + \varepsilon^2 u\left(\frac{x}{2}\right)$$
$$- \left(\sin^4(x) + \sin^2(x) + 8\right) = 0, \tag{40}$$

where ε is the artificially introduced perturbation parameter. The nonzero terms of (9) in PIA(1, 2) algorithm are

$$F_{u''} = 1,$$

$$F_{u\varepsilon} = 2u_n(x),$$

$$F_\varepsilon = u_n^2(x), \tag{41}$$

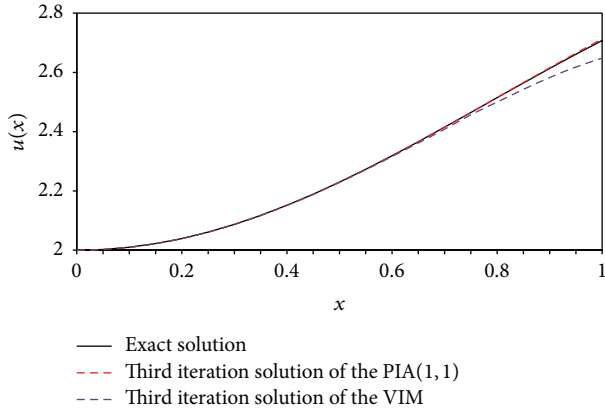$$F_{\varepsilon\varepsilon} = 2u_n\left(\frac{x}{2}\right),$$

$$F = u_n''(x) - \sin^4(x) - \sin^2(x) - 8.$$

FIGURE 3: The comparison of the third iteration solutions of the VIM and PIA$(1, 1)$ with exact solution for example 4.
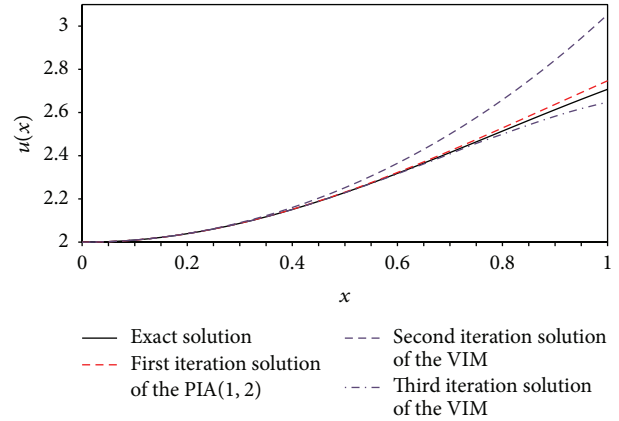

FIGURE 4: The comparison of second and third iteration solutions of the VIM and first iteration solution of the PIA$(1, 2)$ with exact solution of example 4.

Note that introducing the small parameter $\varepsilon = 1$ as a coefficient of the pantograph term simplifies (40) and makes it easily solvable. For this specific example, (40) reads

$$\left(u_c''\right)_n (x) + \left(2u_n (x)\right) \left(u_c\right)_n (x)$$

$$= -u_n \left(\frac{x}{2}\right) - u_n'' (x) - u_n^2 (x) + \sin^4 (x) + \sin^2 (x) \quad (42)$$

$$+ 8.$$

When applying the iteration formula (4), we select the following initially assumed function:

$$u_0 = 2, \quad (43)$$

and using (9), the approximate solution is

$$u_1 = \frac{263}{96} - \frac{49}{96} \cos (2x) - \frac{1}{12} \cos^4 (2x) + \frac{1}{48} \cos^2 (x)$$

$$- \frac{1}{4} x \sin (x) \cos (x) - \frac{1}{6} \cos (x). \quad (44)$$

First iteration of PIA$(1, 2)$ and second and third iterations of VIM solutions are compared with the exact solution. As can be seen from Figure 4, even the first iteration of PIA$(1, 2)$ yields much better results than the first three iteration solutions of the VIM.

3.5. Illustrative Example 5.

Consider the initial value problem of nonlinear variable coefficient differential equation with pantograph delay [6]:

$$u'' (x) = u (x) - \frac{8}{x^2} u^2 \left(\frac{x}{2}\right), \quad x \geq 0 \quad (45)$$

with initial conditions

$$u (0) = 0, \quad (46)$$

$$u' (0) = 1$$

which has the exact solution $u(x) = xe^{-x}$.

Equation (45) can be written in the following form:

$$u'' (x) + \varepsilon \left(\frac{8}{x^2} u^2 \left(\frac{x}{2}\right) - u (x)\right) = 0, \quad (47)$$

where $\varepsilon = 1$ is the artificially introduced perturbation parameter. Using (3), (45) returns to

$$F\left(u'', u', u, u_\alpha, \varepsilon\right) = u'' (x) + \varepsilon \left(\frac{8}{x^2} u^2 \left(\frac{x}{2}\right) - u (x)\right)$$

$$= 0. \quad (48)$$

The nonzero terms in PIA$(1, 1)$ algorithm are

$$F_{u''} = 1,$$

$$F_\varepsilon = \frac{8}{x^2} u_n^2 \left(\frac{x}{2}\right) - u_n (x), \quad (49)$$

$$F = u_n'' (x).$$

Substituting (49), (45) reduces to

$$\left(u_c''\right)_n (x) = -\frac{8}{x^2} u_n^2 \left(\frac{x}{2}\right) + u_n (x) - u_n'' (x). \quad (50)$$

The initial trial function is selected as

$$u_0 (x) = x, \quad (51)$$

and using (50), the approximate first iteration solution is obtained as

$$u_1 = x - x^2 + \frac{1}{6} x^3,$$

$$u_2 = x - x^2 + \frac{1}{2} x^3 - \frac{5}{36} x^4 + \frac{1}{80} x^5 - \frac{1}{8640} x^6, \quad (52)$$

$$\vdots$$

The absolute maximum errors E_n for different values of n are given in Table 5, and it is revealed that the error decreases continually as n increases.

TABLE 5: The absolute maximum errors E_n for example 5.

n	1	2	3	4	5	6	7
E_n	$2.01e-1$	$5.61e-3$	$1.59e-5$	$2.75e-7$	$3.68e-9$	$6.76e-11$	$4.94e-13$

— Exact solution
- - - 8th iteration solution of the PIA(1, 1)
- - - 7th iteration solution of the PIA(1, 1)
- · - 6th iteration solution of the PIA(1, 1)
· · · · · 5th iteration solution of the PIA(1, 1)

FIGURE 5: Comparison of the iteration solutions of the PIA(1, 1) method with the exact solution for example 5.

- · - First iteration solution of the PIA(1, 1)
- - - Second iteration solution of the PIA(1, 1)
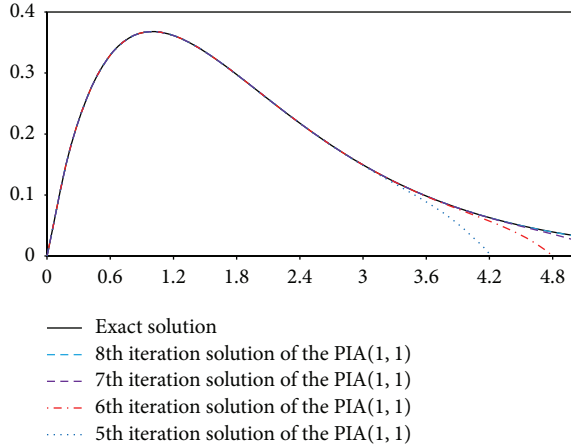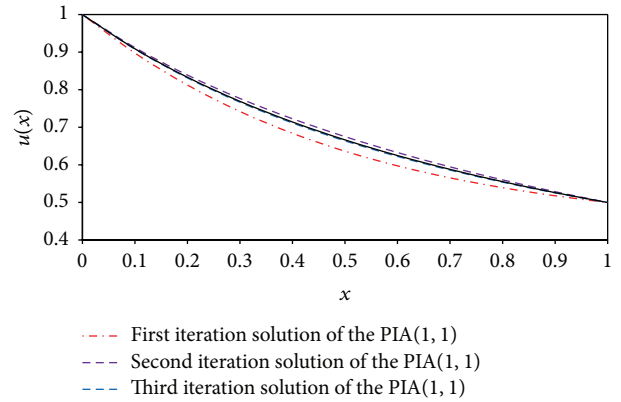- - - Third iteration solution of the PIA(1, 1)

FIGURE 6: Comparison of the first three iteration solutions of the PIA(1, 1) method with the exact solution for example 6.

Figure 5 illustrates the PIA(1, 1) solution of this example in the interval $[0, 5]$. In this example, we enlarge the interval so as to compare our solution more thoroughly with the exact solution. The present method yields pretty good results.

3.6. Illustrative Example 6. Consider the second-order nonlinear pantograph functional differential equation [7, 8]:

$$u''(x) = \left([u(x)]^2 + |u(x)|^3\right) u\left(\frac{x}{2}\right), \quad x \geq 0 \qquad (53)$$

with initial conditions

$$u(0) = 1, \quad u(1) = \frac{1}{2} \qquad (54)$$

which has the exact solution $u(x) = 1/(1+x)$.

Equation (45) can be written in the following form:

$$u''(x) - \varepsilon \left(\left([u(x)]^2 + |u(x)|^3\right) u\left(\frac{x}{2}\right)\right) = 0, \qquad (55)$$

where ε is the artificially introduced small parameter. Using (3), (45) returns to

$$
\begin{aligned}
&F\left(u'', u', u, u_\alpha, \varepsilon\right) \\
&= u''(x) - \varepsilon \left(\left([u(x)]^2 + |u(x)|^3\right) u\left(\frac{x}{2}\right)\right) = 0.
\end{aligned}
\qquad (56)
$$

The nonzero terms in PIA(1, 1) algorithm are

$$F_{u''} = 1,$$

$$F_\varepsilon = -\left(\left([u(x)]^2 + |u_n(x)|^3\right) u_n\left(\frac{x}{2}\right)\right), \qquad (57)$$

$$F = u_n''(x).$$

Substituting (49), (45) reduces to

$$
\begin{aligned}
\left(u_c''\right)_n(x) &= -u_n''(x) \\
&+ \left(\left([u(x)]^2 + |u_n(x)|^3\right) u_n\left(\frac{x}{2}\right)\right).
\end{aligned}
\qquad (58)
$$

The initial trial function is selected as

$$u_0(x) = 1 - \frac{x}{2}, \qquad (59)$$

and using (50), the approximate first iteration solution is obtained as

$$
\begin{aligned}
u_1(x) &= 1 - \frac{1073}{960}x + x^2 - \frac{1}{2}x^3 + \frac{13}{96}x^4 - \frac{3}{160}x^5 \\
&+ \frac{1}{960}x^6,
\end{aligned}
\qquad (60)
$$

$$\vdots$$

In Figure 6, first three iteration solutions obtained by the PIA(1, 1) algorithm are compared with the exact solution. The convergence of iteration solutions to the exact solution is obvious.

Figure 7 illustrates the absolute error functions of the first three iteration solutions of the PIA(1, 1) method. The maximum absolute error decreases from 0,03013 in the first iteration to 0,00262 in the third iteration.

4. Conclusion

The new perturbation-iteration method is employed for the first time to numerically solve the pantograph equations. The comparative results showed that the method is highly
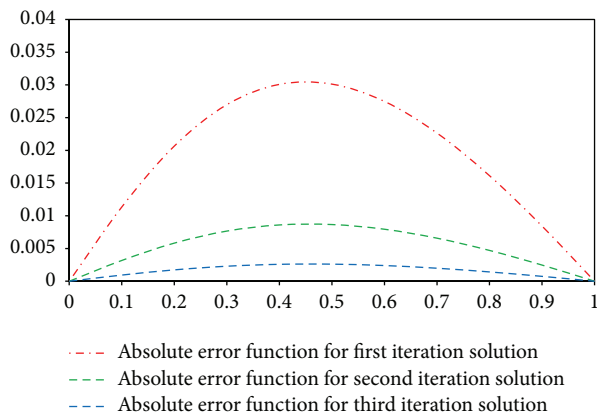
Figure 7: Comparison of the absolute error functions of the first three iteration solutions of the PIA(1, 1) method with the exact solution for example 6.

efficient and reliable in both linear and nonlinear problems. While, for the linear problems studied, PIA(1, 1) algorithm gave almost the same results as Variational Iteration Method, it gave better results for the nonlinear problems. On the other hand, the results obtained in the first iteration of the PIA(1, 2) algorithm are more convergent than those obtained in the third iteration of the Variational Iteration Method; this validates the performance of the present method. Although the equations of PIA(1, 2) algorithm are more entangled than those of PIA(1, 1), faster convergence is achieved with PIA(1, 2) when compared to PIA(1, 1).

It is conceived that the present method can be useful in developing new algorithms since they can be generated in various forms according to the number of correction terms in perturbation expansion and the order of derivation in Taylor expansion.

The disadvantage of the method is that the number of terms in the solution function increases seriously.

In this study, the solution intervals of problems are chosen generally the same as those in the references, for comparison purposes. In three of the examples, the solution intervals are extended moderately; yet, for a wider solution interval, the method should be modified such that the Taylor series expansions are made around points other than zero. The method can also be extended to other types of delay differential equations but some modifications are required.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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