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Research Article

On the Fractional Nagumo Equation with Nonlinear Diffusion and Convection

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We presented the Nagumo equation using the concept of fractional calculus. With the help of two analytical techniques including the homotopy decomposition method (HDM) and the new development of variational iteration method (NDVIM), we derived an approximate solution. Both methods use a basic idea of integral transform and are very simple to be used.

1. Introduction

The Nagumo equation with linear or nonlinear diffusion and convection has broadly been useful to population dynamics, ecology, neurophysiology, chemical reactions, and flame propagation [1–3]. In particular, the case where the equation involves degenerate nonlinear diffusion is of considerable interest [4–8]. In this case, a travelling wave front solution of sharp type is known to exist for exactly one value of the wave speed. Such wave fronts, for instance, represent collective motion of populations in particular collective cell spreading, invasion in ecology, and concentration in chemical reactions [9–15]. However, it has been showing that the real world problems describe via fractional order derivative gives better prediction [16–20]. It is, therefore, important; further extend the nonlinear Nagumo equation using the concept of fractional derivative order.

It is something very difficult to obtain the exact solution of nonlinear equation with fractional order derivative. Many scholars sometimes, to avoid this difficulty, solve this class of problem numerically. However, even with numerical scheme it is also difficult to provide a numerical solution for nonlinear equations. Thus, many scholars, to access the behaviour of the solution of the real problem under study, present an approximate solution of this type of equations.

In the literature, there exist several analytical techniques [21–25] to deal with nonlinear equations including fractional type. The purpose of this work is to present an approximate solution for the generalized nonlinear Nagumo equation withnonlinear diffusion and convection via the well-known variational iteration method (VIM) and the homotopy decomposition method (HDM). The nonlinear fractional Nagumo equation considered here is given below as

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta u^n \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\alpha u^n \frac{\partial u}{\partial x}\right] + \gamma u\left(1 - u^m\right)\left(u^m - \delta\right), \quad 0 < \alpha \le 1, \tag{1}$$

$$u(x, 0) = f(x), \tag{2}$$

$$u(0, t) = g(t), \tag{3}$$

where α, β, γ, and δ are constants, and for the sake of simplicity in this paper, we consider $m = n = 1 = \delta$. The concept of fractional order is not well by some scholars; in order to accommodate those, we present in the next section the basic information regarding this concept.

2. Basic Information about Fractional Calculus

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann-Liouville and the Caputo derivatives. For Caputo we have the following.

Definition 1 (see [26–30]). A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2 (partial derivatives of fractional order [31–34]). Assume now that $f(x)$ is a function of n variables x_i, $i = 1, \ldots, n$ also of class C on $D \in \mathbb{R}_n$. We define partial derivative of order α for f with respect to x_i, the function

$$a\partial_{\underline{x}}^{\alpha} f = \frac{1}{\Gamma(m-\alpha)} \int_a^{x_i} (x_i - t)^{m-\alpha-1} \partial_{x_i}^m f(x_j)\Big|_{x_j = t} dt, \quad (4)$$

where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

Definition 3 (see [26–30]). The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$\begin{gathered} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad \alpha > 0, \ x > 0, \\ J^0 f(x) = f(x). \end{gathered} \quad (5)$$

Properties of the operator can be found in [31–38]; we mention only the following.

For

$$\begin{gathered} f \in C_\mu, \qquad \mu \geq -1, \qquad \alpha, \beta \geq 0, \qquad \gamma > -1, \\ J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \\ J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \\ J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{gathered} \quad (6)$$

3. Basic Information of the HDM and VIM

In this section, we shall present the basic information of the chosen analytical methods: the homotopy decomposition method and variational iteration method; we shall start with the homotopy decomposition method.

3.1. Information about the HDM. The interested reader can find the full detail of the methodology of the homotopy decomposition method in [39–42]. This relatively new method was recently used to solve some nonlinear fractional partial differential equations. However, since the new development of the variational iteration method using the Laplace transform was recently introduced, we shall present its methodology in the following subsection.

3.2. Some Information about the Variational Iteration Method. In its initial development, the essential nature of the method was to construct the following correction functional for (2) when α is a natural number:

$$\begin{aligned} w_{n+1}(x,t) &= w_n(x,t) \\ &+ \int_0^t \lambda(t,\tau) \Big[-\frac{\partial^m w(x,\tau)}{\partial t^m} + L(w_n(x,\tau)) \\ &+ N(w_n(x,\tau)) + k(x,\tau) \Big] d\tau, \end{aligned} \quad (7)$$

where $\lambda(t,\tau)$ is the so-called Lagrange multiplier [43] and $w_n(x,t)$ is the n-approximate solution. However, this development was not suitable for equations with fractional order derivative [43]. Therefore, in their work, they apply the new development of the VIM proposed in [44] to find the Lagrange multiplier. In this new VIM, the first step of the basic character of the method is to apply the Laplace transform on both sides of (2) to obtain

$$\begin{aligned} & s^m w(x,s) - s^{m-1} w(x,0) - \cdots w^{m-1}(x,0) \\ & = \mathscr{L}[L(w(x,t)) + N(w(x,t)) + k(x,t)]. \end{aligned} \quad (8)$$

The recursive formula of (8) can now be used to put forward the main recursive method connecting the Lagrange multiplier as

$$\begin{aligned} & w_{n+1}(x,s) \\ & = w_n(x,s) \\ & + \lambda(s) \left[s^m w_n(x,s) - s^{m-1} w(x,0) - \cdots w^{m-1}(x,0) \right. \\ & \left. - \mathscr{L}\left[L(w_n(x,t)) + N(w_n(x,t)) + k(x,t) \right] \right]. \end{aligned} \quad (9)$$

Now considering $\mathscr{L}[L(w_n(x,t)) + N(w_n(x,t)) + k(x,t)]$, the restricted term; the Lagrange multiplier can be obtained as [43]

$$\lambda(s) = -\frac{1}{s^m}. \quad (10)$$

Now, applying the inverse Laplace transform on both sides of (9), we obtain the following iteration:

$$\begin{aligned} & w_{n+1}(x,t) \\ & = w_n(x,t) \\ & - \mathscr{L}^{-1} \left[\frac{1}{s^m} \left[s^m w_n(x,s) - s^{m-1} w(x,0) - \cdots w^{m-1}(x,0) \right. \right. \\ & - \mathscr{L} \Big[L(w_n(x,t)) \\ & \left. \left. + N(w_n(x,t)) + k(x,t) \Big] \right] \right]. \end{aligned} \quad (11)$$

4. Application to the Fractional Nagumo Equation

In this section, we present the application of the homotopy decomposition method and the new development of the so-called variational iteration method to the nonlinear fractional Nagumo equation. We shall start with the HDM.

4.1. Application of the HDM. Let us consider the fractional nonlinear Nagumo Equation with the following initial condition:

$$u(x,0) = x^2,$$
$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[au\frac{\partial u}{\partial x}\right] + \gamma u(1-u)(u-1), \quad (12)$$
$$0 < \alpha \le 1.$$

Using the steps involved in the HDM we arrive at the following:

$$u(x,t) = u(x,0) + \frac{1}{\Gamma[1-\alpha]}\int_0^t (t-\tau)^{1-\alpha}\left[-\beta u\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left[au\frac{\partial u}{\partial x}\right] + \gamma u(1-u)(u-1)\right]d\tau. \quad (13)$$

Now, assume the solution of the above equation can be expressed in series form as follows:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \quad (14)$$

Replacing this in (13) and after comparing the term of the same power of p, we obtain the following recursive formulas:

$$u_0(x,t) = u(x,t),$$
$$u_1(x,t) = \frac{1}{\Gamma[\alpha]}\int_0^t (t-\tau)^{\alpha-1}\left[-\beta u_0\frac{\partial u_0}{\partial x} + \frac{\partial}{\partial x}\left[au_0\frac{\partial u_0}{\partial x}\right] + \gamma u_0(1-u_0)(u_0-1)\right]d\tau,$$
$$u_2(x,t) = \frac{1}{\Gamma[\alpha]}\int_0^t (t-\tau)^{\alpha-1}\left[-\beta H_2^1 + aH_2^2 + aH_2^3 + \gamma H_2^4 + \gamma H_2^5 - u_1\right]d\tau. \quad (15)$$

Here,

$$H_2^1 = u_0(x,t)\partial_x u_1(x,t) + u_1(x,t)\partial_x u_0(x,t),$$
$$H_2^2 = 2\partial_x u_0(x,t)\partial_x u_1(x,t),$$
$$H_2^3 = u_0(x,t)\partial_{xx}u_1(x,t) + u_1(x,t)\partial_{xx}u_0(x,t),$$
$$H_2^4 = u_0^2(x,t)u_1(x,t) + u_1^3 + u_1^2(x,t)u_0(x,t),$$
$$H_2^5 = 2u_0(x,t)u_1(x,t). \quad (16)$$

The general recursive formula for $p \ge 3$ is given as

$$u_n(x,t) = \frac{1}{\Gamma[\alpha]}\int_0^t (t-\tau)^{\alpha-1}\left[-\beta H_n^1 + aH_n^2 + aH_n^3 + \gamma H_n^4 + \gamma H_n^5 - u_{n-1}\right]d\tau, \quad (17)$$

with

$$H_n^1(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\partial_x u_{n-j}(x,t),$$
$$H_n^2(x,t) = \sum_{j=0}^{n-1} \partial_x u_j(x,t)\partial_x u_{n-j}(x,t),$$
$$H_n^3(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\partial_{xx} u_{n-j}(x,t), \quad (18)$$
$$H_n^4(x,t) = \sum_{j=0}^{n-1}\sum_{k=0}^{j} u_j(x,t)u_{j-k}(x,t)u_{n-j-1}(x,t),$$
$$H_n^5(x,t) = \sum_{j=0}^{n-1} u_j(x,t)u_{n-j}(x,t),$$

so that, integrating the above set of integral equations, we obtain the following:

$$u_0(x,t) = x^2,$$
$$u_1(x,t) = -\frac{t^\alpha x^2\left(-6a + 2x\beta + \left(-1+x^2\right)^2\gamma\right)}{\Gamma(1+\alpha)},$$
$$u_2(x,t) = t^{2\alpha}x^2\Bigg(\Big(48a^2 + \gamma - 2a\Big(3 + 24x\beta + 4\gamma - 22x^2\gamma + x\Big(10x\beta^2 + x\gamma\left(-2 + x^2 - 2\gamma + 3x^2\gamma - x^6\gamma\right) + 2\beta\left(1 + \left(2 - 8x^2 + 3x^4\right)\gamma\right)\Big)\Big)\Big) \times \frac{1}{\Gamma(1+2\alpha)}$$

$$+\frac{t^{\alpha}x^{4}\gamma\left(-6a+2x\beta+\left(-1+x^{2}\right)^{2}\gamma\right)^{2}\Gamma(1+2\alpha)}{\Gamma^{2}(1+\alpha)\Gamma(1+3\alpha)} - \frac{t^{2\alpha}\left(-6a+2x\beta+\left(-1+x^{2}\right)^{2}\gamma\right)^{3}\Gamma(1+3\alpha)}{\Gamma^{3}(1+\alpha)\Gamma(1+4\alpha)}\Bigg). \tag{19}$$

Here, we have computed only three terms in the series solution. However, using the recursive formula, we can compute the remaining terms, and the approximate solution is given as follows:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots+\cdots. \tag{20}$$

4.2. New Development of the Variational Iteration Method. In this subsection, we test the efficiency of the new development of the variation iteration method by solving the nonlinear fractional equation (1). Therefore, following the methodology of NDVIM, we are at the following.

The Lagrange multiplier is

$$\lambda(s)=-\frac{1}{s^{\alpha}} \tag{21}$$

and the recursive formula is given as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \mathscr{L}^{-1}\Bigg[\frac{1}{s^{\alpha}}\Bigg[\mathscr{L}\Bigg[-\beta u_n\frac{\partial u_n}{\partial x} + \frac{\partial}{\partial x}\left[a u_n\frac{\partial u_n}{\partial x}\right] + \gamma u_n(1-u_n)(u_n-1)\Bigg]\Bigg]\Bigg] \end{aligned} \tag{22}$$

with the initial term

$$u_0(x,t)=x^2, \tag{23}$$

so that using the iteration formulas we obtain

$$\begin{aligned} u_1(x,t) &= x^2-\frac{t^{\alpha}}{\Gamma(1+\alpha)} \times\left(6ax^2-2x^3\beta+x^2\left(1-x^2\right)\left(x^2-1\right)\gamma\right), \\ u_2(x,t) &= \Big(x^2\Big(t^{3\alpha}\Big(-144a^3 + 12a^2 \times\left(18x\beta+4\gamma-22x^2\gamma+21x^4\gamma\right) \\ &\quad - 4a\Big(12x\beta\gamma-54x^3\beta\gamma + 48x^5\beta\gamma+\gamma^2 + 43x^4\gamma^2-48x^6\gamma^2+18x^8\gamma^2 + 2x^2\left(12\beta^2-7\gamma^2\right)\Big) \\ &\quad + x\Big(2\beta\gamma^2+52x^4\beta\gamma^2-52x^6\beta\gamma^2 + 18x^8\beta\gamma^2+26x^7\gamma^3-14x^9\gamma^3 + 3x^{11}\gamma^3+4x^2\left(3\beta^3-5\beta\gamma^2\right) \\ &\quad + 6x^5\left(5\beta^2\gamma-4\gamma^3\right)+2x\left(5\beta^2\gamma-\gamma^3\right) + x^3\left(-36\beta^2\gamma+11\gamma^3\right)\Big)\Big) \\ &\quad \times\Gamma(1+\alpha)\Gamma^2(1+2\alpha)\Gamma(1+4\alpha) \\ &\quad + t^{2\alpha}\Big(180a^2+6x\beta\gamma-20x^3\beta\gamma+14x^5\beta\gamma + \gamma^2+12x^4\gamma^2-10x^6\gamma^2+3x^8\gamma^2 \\ &\quad - 4a\left(23x\beta+9\gamma - 25x^2\gamma+16x^4\gamma\right) + 2x^2\left(5\beta^2-3\gamma^2\right)\Big) \\ &\quad \times\Gamma(1+3\alpha)\Gamma^3(1+\alpha)\Gamma(1+4\alpha) + \Gamma(1+2\alpha)\Gamma(1+3\alpha) \\ &\quad \times\Big(t^{4\alpha}x^4\gamma \times\left(-6a+2x\beta+\gamma-2x^2\gamma+x^4\gamma\right)^3 \times\Gamma(1+3\alpha) \\ &\quad -\Big(2t^{\alpha}\left(16a-2x\beta-\gamma+2x^2\gamma-x^4\gamma\right) -\Gamma(1+\alpha)\Big)\Gamma(1+\alpha) \times\Gamma^3(1+\alpha)\Gamma(1+4\alpha)\Big)\Big)\Big) \\ &\quad \times\Big(\Gamma^3(1+\alpha)\Gamma(1+2\alpha) \times\Gamma(1+3\alpha)\Gamma(1+4\alpha)\Big)^{-1}. \end{aligned} \tag{24, 25}$$

Using the recursive formula, the remaining term can be obtained but here, due to the length of this term, we computed only three terms and the approximate solution case given as

$$u(x,t)=u_2(x,t). \tag{26}$$

In the following section, we compare the approximate solution via HDM and NDVIM.

5. Numerical Results

We devote this section to the comparison of the numerical solutions obtained via the HDM and the NDVIM for different values of the fractional order derivative. In this case, we chose $\alpha = 1.5$, $\gamma = 1$, and $a = 4$. The following figures show the numerical solution of the time fractional nonlinear

Table 1: Comparison of numerical values for the approximate solutions via HDM and NDVIM.

x	t	HDM $\alpha = 0.25$	NDVIM $\alpha = 0.25$	HDM $\alpha = 0.9$	NDVIM $\alpha = 0.9$
-10	0	100	100	100	100
	1	-1.14275×10^{18}	-1.14319×10^{18}	-3.24381×10^{17}	-3.24545×10^{17}
	2	-2.28554×10^{18}	-2.28628×10^{18}	-3.93359×10^{18}	-3.93465×10^{18}
	6	-6.85676×10^{18}	-6.85845×10^{18}	-2.05326×10^{20}	-2.05347×10^{20}
	10	-1.1428×10^{19}	-1.14305×10^{19}	-1.29153×10^{21}	-1.29161×10^{21}
-5	0	25	25	25	25
	1	-2.98096×10^{12}	-2.99989×10^{12}	-8.45746×10^{11}	-8.52756×10^{11}
	2	-5.9636×10^{12}	-5.99543×10^{12}	-1.02663×10^{13}	-1.03118×10^{13}
	6	-1.78972×10^{13}	-1.79697×10^{13}	-5.36272×10^{14}	-5.37156×10^{14}
	10	-2.98327×10^{13}	-2.99391×10^{13}	-3.37376×10^{15}	-3.37727×10^{15}
5	0	25	25	25	25
	1	-3.50932×10^{12}	-3.53141×10^{12}	-9.95678×10^{11}	-1.00386×10^{12}
	2	-7.02052×10^{12}	-7.05766×10^{12}	-1.20856×10^{13}	-1.21387×10^{13}
	6	-2.10687×10^{13}	-2.11533×10^{13}	-6.31281×10^{14}	-6.32312×10^{14}

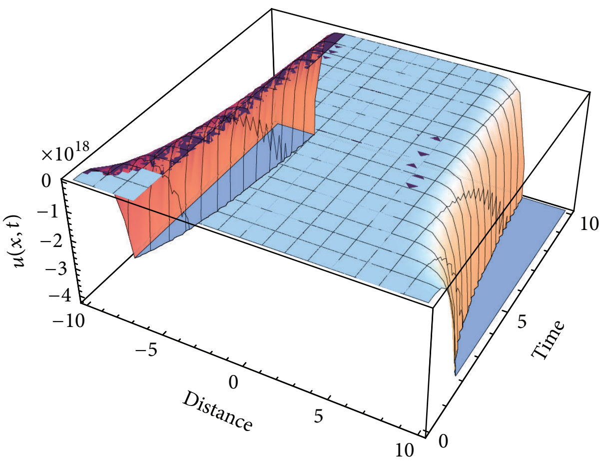

Figure 1: Approximate solution of the time-fractional nonlinear Nagumo equation via the HDM for the value of alpha equal to 0.9.

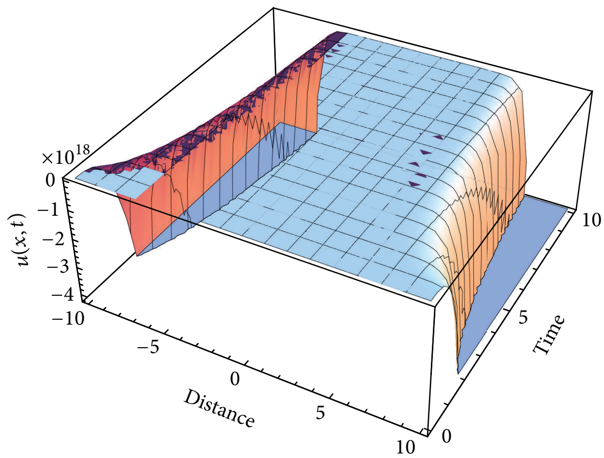

Figure 2: Approximate solution of the time-fractional nonlinear Nagumo equation via the NDVIM for the value of alpha equal to 0.9.

Nagumo equation. Figure 1 show, the approximate solution obtained via the HDM for $\alpha = 0.9$, Figure 2 shows the approximate solution via the NDVIM. Figure 3 Show the approximate solution obtained via the HDM for $\alpha = 0.25$ and Figure 4 shows the approximate solution via the NDVIM. Table 1 shows the comparison of the numerical values of the solution obtained via the HDM and the NDVIM, respectively, for different values of alpha.

Both methods used the idea of iteration; the initial components are obtained as the Taylor series of the exact solution. On one hand, the new development of variational iteration method makes use of the Laplace transform, the Lagrange multiplier, and finally the inverse Laplace transform. On the other hand, the HDM uses just a simple integral and the perturbation technique. Both techniques are simple to implement and are very accurate.

6. Conclusion

The Nagumo equation is a very complex equation, for which the exact solution does not exist. The Nagumo equation was extended to the concept of fractional order derivative. The resulting equation was further analyzed within the framework of the homotopy decomposition method and the new development of variational iteration method. Both methods use a simple idea of integral transform. The numerical results

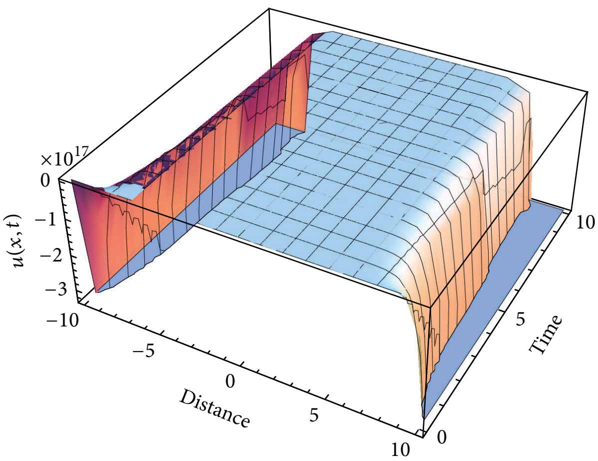

FIGURE 3: Approximate solution of the time-fractional nonlinear Nagumo equation via the HDM for the value of alpha equal to 0.25.

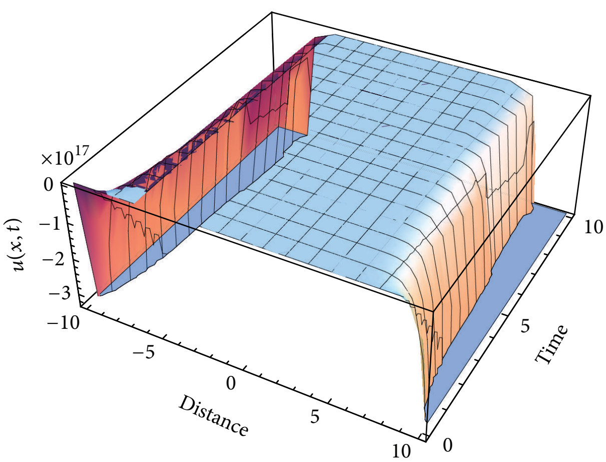

FIGURE 4: We present in Table 1 the numerical values of the approximate solutions obtained via both methods for different values of alpha.

are presented to test the efficiency and the accuracy of both methods. From their iteration formulas, one can conclude that these two methods are simple to be used and are powerful weapons to handle fractional nonlinear equation type.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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