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*Research Article*

# Stochastically Ultimate Boundedness and Global Attraction of Positive Solution for a Stochastic Competitive System

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A stochastic competitive system is investigated. We first show that the positive solution of the above system does not explode to infinity in a finite time, and the existence and uniqueness of positive solution are discussed. Later, sufficient conditions for the stochastically ultimate boundedness of positive solution are derived. Also, with the help of Lyapunov function, sufficient conditions for the global attraction of positive solution are established. Finally, numerical simulations are presented to justify our theoretical results.

## 1. Introduction

In recent years, many researches have been done on the dynamics of many types of Lotka-Volterra competitive systems. Owing to their theoretical and practical significance, these competitive systems have been investigated extensively and there exists a large volume of literature relevant to many good results (see [1–7]). Particularly, in [8], Gopalsamy introduced the following autonomous two-species competitive system:

$$\begin{aligned} y_1'(t) &= y_1(t)\left[a_1 - b_1 y_1(t) - c_1 y_2(t) - d_1 y_1^2(t)\right], \\ y_2'(t) &= y_2(t)\left[a_2 - b_2 y_2(t) - c_2 y_1(t) - d_2 y_2^2(t)\right], \end{aligned} \tag{1}$$

where $y_1(t)$ and $y_2(t)$ can be interpreted as the population size of two competing species at time $t$, respectively. All parameters involved with the above model are positive constants and can be interpreted in more detail; $a_1$ and $a_2$ are the intrinsic growth rates of two species; $b_1$, $d_1$, $b_2$, and $d_2$ represent the effects of intraspecific competition; $c_1$ and $c_2$ are the effects of interspecific competition. Recently, Tan et al. [9] have considered the effect of impulsive perturbations and discussed the uniformly asymptotic stability of almost periodic solutions for a corresponding nonautonomous impulsive version of (1). It has also been noticed that the ecological systems, in the real world, are often perturbed by various types of environmental noise. May [10] also pointed out that, due to environmental fluctuation, the birth rate, the death rate, and other parameters usually show random fluctuation to a certain extent. To accurately describe such systems, it is necessary to use stochastic differential equations. Recently, various stochastic dynamical models have been introduced extensively in [11–17] and many interesting and valuable results including extinction, persistence, and stability can be found in [18–20].

Motivated by the above works, in this contribution, we assume that the environmental noise affects mainly the intrinsic growth rate $a_i$ with

$$a_i \longrightarrow a_i + \sigma_i \dot{w}_i(t), \quad i = 1, 2, \tag{2}$$

where $\dot{w}_i(t)$ are independent white noises, $w_i(t)$ are standard Brownian motions defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and $\sigma_i^2$ represent the intensities of the white noises. Then the stochastically perturbed sytem (1) can be Itô's equations

$$\begin{aligned} dy_1(t) &= y_1(t)\left[a_1 - b_1 y_1(t) - c_1 y_2(t) - d_1 y_1^2(t)\right]dt \\ &\quad + \sigma_1 y_1(t)\,dw_1(t), \\ dy_2(t) &= y_2(t)\left[a_2 - b_2 y_2(t) - c_2 y_1(t) - d_2 y_2^2(t)\right]dt \\ &\quad + \sigma_2 y_2(t)\,dw_2(t) \end{aligned} \tag{3}$$

with the initial values $y_i(0) > 0$.

In this paper, we will focus on the stochastically ultimate boundedness and global attraction of positive solutions of system (3). To the best of our knowledge, there are few published papers concerning system (3). The rest of this paper is organized as follows. In Section 2, we present some assumptions, definitions, and lemmas. In Section 3, we investigate the existence and uniqueness of positive solution, and then, we discuss the stochastically ultimate boundedness of positive solutions. In Section 4, we discuss the global attraction of positive solutions. Finally, we conclude and present a specific example to justify the analytical results.

## 2. Preliminaries

Throughout this paper, we give the notation $R_+^2 = \{y_1 > 0, y_2 > 0\}$ and assumptions.

$(S_1)$ $b_1 > d_1, b_2 > d_2$.

$(S_2)$ For any initial value $(y_1(0), y_2(0)) \in R_+^2$, there exists $p > 1$ such that

$$y_i(0) \le \frac{a_i + (1/2)(p-1)\sigma_i^2}{b_i}, \quad i = 1, 2. \tag{4}$$

$(S_3)$ $b_1 > c_2, b_2 > c_1$.

In the following, let us briefly review several basic definitions and lemmas which will be useful for establishing our main results.

*Definition 1.* The solution $(y_1(t), y_2(t))$ of system (3) is stochastically ultimately bounded a.s. if for arbitrary $\varepsilon_i \in (0,1)$, there exists a positive constant $\varphi_i = \varphi(\varepsilon_i)$ such that

$$\limsup_{t\to+\infty} P\left\{\left|y_i(t)\right| > \varphi_i\right\} < \varepsilon_i, \quad i = 1, 2. \tag{5}$$

*Definition 2.* Let $(y_1(t), y_2(t))$ be a positive solution of system (3). Then $(y_1(t), y_2(t))$ is said to be globally attractive provided that any other solution $(y_1^*(t), y_2^*(t))$ of system (3) satisfies

$$\begin{aligned} \lim_{t\to+\infty}\left|y_1(t) - y_1^*(t)\right| &= 0, \\ \lim_{t\to+\infty}\left|y_2(t) - y_2^*(t)\right| &= 0 \text{ a.s.} \end{aligned} \tag{6}$$

**Lemma 3** (see [21]). *Let $p > 2$, $g \in \mathcal{M}^p([0,T]; R^m)$ such that*

$$E\int_0^T |g(s)|^p ds < \infty, \tag{7}$$

*where $\mathcal{M}^p([0,T]; R^m)$ is the family of processes $\{h(t)\}_{0\le t\le T}$ such that*

$$E\int_0^T |h(t)|^p dt < \infty. \tag{8}$$

*Then*

$$E\left|\int_0^T g(s)\,dw(s)\right|^p \le \left(\frac{p(p-1)}{2}\right)^{p/2} T^{(p-2)/2} E\int_0^T |g(s)|^p ds. \tag{9}$$

**Lemma 4** (see [22]). *Suppose that $a_1, a_2, \ldots, a_n$ are real numbers; then*

$$\left|a_1 + a_2 + \cdots + a_n\right|^p \le C_p\left(\left|a_1\right|^p + \left|a_2\right|^p + \cdots + \left|a_n\right|^p\right), \tag{10}$$

*where $p > 0$ and*

$$C_p = \begin{cases} 1, & 0 < p \le 1; \\ n^{p-1}, & p > 1. \end{cases} \tag{11}$$

**Lemma 5** (see [23]). *Assume that an n-dimensional stochastic process $X(t)$ on $t \ge 0$ satisfies the condition*

$$E|X(t) - X(s)|^\alpha \le \lambda|t-s|^{1+\beta}, \quad 0 \le s,\ t < \infty, \tag{12}$$

*for positive constants $\alpha$, $\beta$, and $\lambda$. Then there exists a continuous version $\widetilde{X}(t)$ of $X(t)$ which has the property that, for every $\vartheta \in (0, \beta/\alpha)$, there is a positive random variable $\psi(\omega)$ such that*

$$P\left\{\omega : \sup_{0<|t-s|<\psi(\omega), 0\le s,t<\infty} \frac{\left|\widetilde{X}(t,\omega) - X(t,\omega)\right|}{|t-s|^\vartheta} \le \frac{2}{1-2^{-\vartheta}}\right\} = 1. \tag{13}$$

*In other words, almost every sample path of $\widetilde{X}(t)$ is locally but uniformly Hölder continuous with exponent $\vartheta$.*

**Lemma 6** (see [24]). *Let $f(t)$ be a nonnegative function on $t \ge 0$ such that $f(t)$ is integrable on $t \ge 0$ and is uniformly continuous on $t \ge 0$. Then $\lim_{t\to+\infty} f(t) = 0$.*

## 3. Stochastically Ultimate Boundedness

In this section, we first show, under the assumption $(S_1)$, that the positive solution of system (3) will not explode to infinity at any finite time.

**Lemma 7.** *Let $(S_1)$ hold and the initial value $(y_1(0), y_2(0)) \in R_+^2$. Then system (3) has a unique solution $(y_1(t), y_2(t))$ on $t \ge 0$, which will remain in $R_+^2$ with probability one.*

*Proof.* It is easy to see that the coefficients of system (3) satisfy the local Lipschitz condition. Then for any given initial value $(y_1(0), y_2(0)) \in R_+^2$, there exists a unique local solution $(y_1(t), y_2(t))$ on $[0, \tau_e)$, where $\tau_e$ is the explosion time. To show that the positive solution is global, we only need to show that $\tau_e = +\infty$ a.s. Let $n_0$ be sufficiently large for every component of $(y_1(0), y_2(0))$ which remains in the interval $[1/n_0, n_0]$. For each integer $n \geq n_0$, one can define the stopping time

$$\tau_n = \inf\left\{t \in [0,\tau_e) : y_1(t) \notin \left(\frac{1}{n}, n\right) \text{ or } y_2(t) \notin \left(\frac{1}{n}, n\right)\right\}. \tag{14}$$

Clearly, $\tau_n$ is increasing as $n \to +\infty$. Assign $\tau_{+\infty} = \lim_{n\to+\infty}\tau_n$, whence $\tau_{+\infty} \leq \tau_e$ a.s. If we can show that $\tau_{+\infty} = +\infty$ a.s., then $\tau_e = +\infty$ a.s. and $(y_1(t), y_2(t)) \in R_+^2$ a.s. for all $t \geq 0$. In other words, to complete the proof, we just need to show that $\tau_{+\infty} = +\infty$ a.s.

By reduction to absurdity, we suppose that $\tau_{+\infty} \neq +\infty$; then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_{+\infty} \leq T\} > \varepsilon. \tag{15}$$

As a result, there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1$

$$P\{\tau_n \leq T\} \geq \varepsilon. \tag{16}$$

Define a $C^2$-function $V : R_+^2 \to R_+$ as

$$V(y_1, y_2) = y_1 - 1 - \ln y_1 + y_2 - 1 - \ln y_2. \tag{17}$$

Obviously, $V(y_1, y_2)$ is a nonnegative function. If $(y_1(t), y_2(t)) \in R_+^2$, then using Itô's formula, one can derive that

$$\begin{aligned}
dV(y_1, y_2) &= \left(1 - \frac{1}{y_1}\right)dy_1 + 0.5\frac{1}{y_1^2}(dy_1)^2 + \left(1 - \frac{1}{y_2}\right)dy_2 \\
&\quad + 0.5\frac{1}{y_2^2}(dy_2)^2 \\
&= \{-d_1y_1^3(t) - b_1y_1^2(t) + d_1y_1^2(t) \\
&\quad + a_1y_1(t) + b_1y_1(t) + c_2y_1(t) \\
&\quad - c_1y_1(t)y_2(t) - c_2y_1(t)y_2(t) - d_2y_2^3(t) \\
&\quad - b_2y_2^2(t) + d_2y_2^2(t) + a_2y_2(t) \\
&\quad + b_2y_2(t) + c_1y_2(t) - a_1 - a_2 \\
&\quad + 0.5\sigma_1^2 + 0.5\sigma_2^2\}\,dt \\
&\quad + (y_1(t) - 1)\sigma_1 dw_1(t) + (y_2(t) - 1)\sigma_2 dw_2(t) \\
&= F(y_1, y_2)\,dt + (y_1(t) - 1)\sigma_1 dw_1(t) \\
&\quad + (y_2(t) - 1)\sigma_2 dw_2(t),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
F(y_1, y_2) &= -d_1y_1^3(t) - b_1y_1^2(t) + d_1y_1^2(t) \\
&\quad + a_1y_1(t) + b_1y_1(t) + c_2y_1(t) \\
&\quad - c_1y_1(t)y_2(t) - c_2y_1(t)y_2(t) - d_2y_2^3(t) \\
&\quad - b_2y_2^2(t) + d_2y_2^2(t) + a_2y_2(t) + b_2y_2(t) \\
&\quad + c_1y_2(t) - a_1 - a_2 + 0.5\sigma_1^2 + 0.5\sigma_2^2.
\end{aligned} \tag{19}$$

A simple calculation shows that

$$\begin{aligned}
F(y_1, y_2) &\leq -(b_1 - d_1)y_1^2(t) + (a_1 + b_1 + c_2)y_1(t) \\
&\quad - (b_2 - d_2)y_2^2(t) \\
&\quad + (a_2 + b_2 + c_1)y_2(t) - a_1 - a_2 \\
&\quad + 0.5\sigma_1^2 + 0.5\sigma_2^2.
\end{aligned} \tag{20}$$

It then follows from $(S_1)$ that the upper bound of $F(y_1, y_2)$, noted by $K$, exists. We therefore have

$$\begin{aligned}
dV(y_1, y_2) &\leq K\,dt + (y_1(t) - 1)\sigma_1 dw_1(t) \\
&\quad + (y_2(t) - 1)\sigma_2 dw_2(t).
\end{aligned} \tag{21}$$

Integrating both sides from 0 to $\tau_k \wedge T$ yields that

$$\begin{aligned}
&\int_0^{\tau_k\wedge T} dV(y_1, y_2) \\
&\quad \leq \int_0^{\tau_k\wedge T} K\,dt + \int_0^{\tau_k\wedge T}(y_1(t) - 1)\sigma_1 dw_1(t) \\
&\qquad + \int_0^{\tau_k\wedge T}(y_2(t) - 1)\sigma_2 dw_2(t),
\end{aligned} \tag{22}$$

whence taking expectations leads to

$$\begin{aligned}
&EV(y_1(\tau_k\wedge T), y_2(\tau_k\wedge T)) \\
&\quad \leq V(y_1(0), y_2(0)) + KE(\tau_k\wedge T) \\
&\quad \leq V(y_1(0), y_2(0)) + KT.
\end{aligned} \tag{23}$$

Set $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$, and then $P(\Omega_n) \geq \varepsilon$. Note that arbitrary $\omega \in \Omega_n$, there exist some $i$ such that $y_i(\tau_n, \omega)$ equals either $n$ or $1/n$. Then $V(y_1(\tau_n, \omega), y_2(\tau_n, \omega))$ is not less than

$$\min\left\{(n - 1 - \ln n), \left(\frac{1}{n} - 1 + \ln n\right)\right\}. \tag{24}$$

As a consequence,

$$\begin{aligned}
&V(y_1(0), y_2(0)) + KT \\
&\quad \geq E\left[1_{\Omega_n}(\omega)V(y_1(\tau_n, \omega), y_2(\tau_n, \omega))\right] \\
&\quad \geq \varepsilon\min\left\{(n - 1 - \ln n), \left(\frac{1}{n} - 1 + \ln n\right)\right\},
\end{aligned} \tag{25}$$

where $1_{\Omega_n}$ is the indicator function of $\Omega_n$. Let $n \to +\infty$ lead to the contradiction

$$+\infty > V(y_1(0), y_2(0)) + KT = +\infty. \tag{26}$$

So we must have $\tau_{+\infty} = +\infty$ a.s. This completes the proof of Lemma 7. □

Lemma 7 is fundamental in this paper. In the following, we will show that the $p$th moment of the positive solution of system (3) is upper bounded and then discuss the stochastically ultimate boundedness.

**Theorem 8.** *Let* $(S_1)$ *and* $(S_2)$ *hold; then the positive solution* $(y_1(t), y_2(t))$ *of system* (3) *with initial value* $(y_1(0), y_2(0)) \in R_+^2$ *is stochastically ultimately bounded.*

*Proof.* From Lemma 7, we can see that system (3) has a unique positive solution under assumption $(S_1)$. Assign $p > 1$ arbitrarily; then by Itô's formula we can show that

$$\begin{aligned} dy_i^p(t) &= py_i^{p-1}(t)\,dy_i(t) + \frac{1}{2}p(p-1)\,y_i^{p-2}(dy_i(t))^2 \\ &= py_i^p(t)\Big(a_i - b_iy_i(t) - c_iy_j(t) - d_iy_i^2(t) \\ &\qquad + \frac{1}{2}(p-1)\sigma_i^2\Big)dt + p\sigma_i y_i^p dw_i(t). \end{aligned} \tag{27}$$

Integrating and taking expectations on both sides yield that

$$\begin{aligned} &E\left(y_i^p(t)\right) - E\left(y_i^p(0)\right) \\ &\quad = \int_0^t pE\Big\{y_i^p(s)\Big(a_i - b_iy_i(s) - c_iy_j(s) \\ &\qquad - d_iy_i^2(s) + \frac{p-1}{2}\sigma_i^2\Big)\Big\}ds. \end{aligned} \tag{28}$$

We then derive that

$$\begin{aligned} \frac{dE\left(y_i^p(t)\right)}{dt} &= pE\Big\{y_i^p(t)\Big(a_i - b_iy_i(t) - c_iy_j(t) \\ &\qquad - d_iy_i^2(t) + \frac{p-1}{2}\sigma_i^2\Big)\Big\} \\ &\le pa_iE\left(y_i^p(t)\right) - pb_iE\left(y_i^{p+1}(t)\right) \\ &\quad + \frac{1}{2}p(p-1)\sigma_i^2E\left(y_i^p(t)\right) \\ &= pE\left(y_i^p(t)\right)\left[a_i + \frac{p-1}{2}\sigma_i^2\right] - pb_iE\left(y_i^{p+1}(t)\right). \end{aligned} \tag{29}$$

By Hölder inequality one has

$$E\left(y_i^{p+1}(t)\right) \ge \left(E\left(y_i^p(t)\right)\right)^{(p+1)/p}, \tag{30}$$

and moreover

$$\begin{aligned} &\frac{dE\left(y_i^p(t)\right)}{dt} \\ &\quad \le pE\left(y_i^p(t)\right)\left\{a_i + \frac{p-1}{2}\sigma_i^2 - b_i\left[E\left(y_i^p(t)\right)\right]^{1/p}\right\}. \end{aligned} \tag{31}$$

Denote $X_i(t) = E(y_i^p(t))$, $X_i(0) = y_i^p(0)$; then (31) can be rewritten as

$$\frac{dX_i(t)}{dt} \le pX_i(t)\left[a_i + \frac{p-1}{2}\sigma_i^2 - b_iX_i^{1/p}(t)\right]. \tag{32}$$

It follows from $(S_2)$ that

$$0 < b_iX_i^{1/p}(0) = by_i(0) \le a_i + \frac{1}{2}(p-1)\sigma_i^2; \tag{33}$$

that is,

$$0 < X_i(0) \le \left[\frac{a_i + (1/2)(p-1)\sigma_i^2}{b_i}\right]^p := H_i(p). \tag{34}$$

Meanwhile, it is easy to see that

$$\left.\frac{dX_i(t)}{dt}\right|_{X_i(t)=H_i(p)} \le 0. \tag{35}$$

By the standard comparison theorem, we therefore derive that

$$E\left(y_i^p(t)\right) \le H_i(p), \tag{36}$$

which implies that the $p$th moment of positive solution is upper bounded.

Let us now proceed to discuss the stochastically ultimate boundedness of system (3). Setting $\varphi_i = [H_i(p)/\varepsilon_i]^{1/p}$, then by the Chebyshev inequality, we obtain that

$$P\{|y_i(t)| > \varphi_i\} < \frac{E\left(y_i^p(t)\right)}{\varphi_i^p} \le \frac{H_i(p)}{H_i(p)/\varepsilon_i} = \varepsilon_i. \tag{37}$$

This gives that

$$\limsup_{t\to+\infty} P\{|y_i(t)| > \varphi_i\} < \varepsilon_i, \quad i = 1, 2. \tag{38}$$

The proof of Theorem 8 is complete. □

## 4. Global Attraction

In this section, we will establish sufficient conditions for global attraction of system (3).

**Lemma 9.** *Let* $(S_2)$ *hold and let* $(y_1(t), y_2(t))$ *be a solution of* (3) *with initial value* $(y_1(0), y_2(0)) \in R_+^2$*; then almost every sample path of* $(y_1(t), y_2(t))$ *is uniformly continuous for* $t \ge 0$.

*Proof.* We first prove $y_1(t)$. Let us consider the following integral equation:

$$\begin{aligned} y_1(t) &= y_1(0) + \int_0^t f_1(s, y_1(s), y_2(s))\,ds \\ &\quad + \int_0^t g_1(s, y_1(s), y_2(s))\,dw_1(s), \end{aligned} \tag{39}$$

where

$$\begin{aligned} &f_1(s, y_1(s), y_2(s)) \\ &\quad = y_1(s)\left[a_1 - b_1y_1(s) - c_1y_2(s) - d_1y_1^2(s)\right], \\ &g_1(s, y_1(s), y_2(s)) = \sigma_1y_1(s). \end{aligned} \tag{40}$$

Recalling $(S_2)$, (32), and the standard comparison theorem, we can know that

$$E\left(y_i^p(t)\right) \le H_i(p), \quad i = 1, 2. \tag{41}$$

Then from Lemma 4 and (41) one derives that

$$\begin{aligned}
&E\left(\left|f_1(s, y_1(s), y_2(s))\right|^p\right) \\
&\quad= E\left(y_1^p(s)\left|a_1 - b_1 y_1(s) - c_1 y_2(s) - d_1 y_1^2(s)\right|^p\right) \\
&\quad\le 0.5E\left(y_1^{2p}(s)\right) + 0.5E\left(\left|a_1 - b_1 y_1(s) - c_1 y_2(s) - d_1 y_1^2(s)\right|^{2p}\right) \\
&\quad\le 0.5E\left(y_1^{2p}(s)\right) + 0.5E\left(4^{2p-1}\left(|a_1|^{2p} + |b_1 y_1(s)|^{2p} + |c_1 y_2(s)|^{2p} + \left|d_1 y_1^2(s)\right|^{2p}\right)\right) \\
&\quad\le 0.5E\left(y_1^{2p}(s)\right) + 0.5\cdot 4^{2p-1}|a_1|^{2p} + 0.5\cdot 4^{2p-1}|b_1|^{2p}E\left(y_1^{2p}(s)\right) \\
&\qquad + 0.5\cdot 4^{2p-1}|c_1|^{2p}E\left(y_2^{2p}(s)\right) + 0.5\cdot 4^{2p-1}|d_1|^{2p}E\left(y_1^{4p}(s)\right) \\
&\quad\le 0.5H_1(2p) + 0.5\cdot 4^{2p-1}|a_1|^{2p} + 0.5\cdot 4^{2p-1}|b_1|^{2p}H_1(2p) \\
&\qquad + 0.5\cdot 4^{2p-1}|c_1|^{2p}H_2(2p) + 0.5\cdot 4^{2p-1}|d_1|^{2p}H_1(4p) := \mathscr{L}(p), \\
&E\left(\left|g_1(s, y_1(s), y_2(s))\right|^p\right) \\
&\quad= E\left(\sigma_1^p y_1^p(s)\right) \\
&\quad\le \sigma_1^p E\left(y_1^p(s)\right) \\
&\quad\le \sigma_1^p H_1(p) := \mathscr{R}(p).
\end{aligned} \tag{42}$$

Meanwhile, by Lemma 3, we obtain that, for $0 \le t_1 < t_2 < +\infty$ and $p > 2$,

$$\begin{aligned}
E\left|\int_{t_1}^{t_2} g_1(s, y_1(s), y_2(s))\,dw_1(s)\right|^p
&\le \left[\frac{p(p-1)}{2}\right]^{p/2}(t_2 - t_1)^{(p-2)/2} \\
&\quad\times \int_{t_1}^{t_2} E\left|g_1(s, y_1(s), y_2(s))\right|^p ds.
\end{aligned} \tag{43}$$

Let

$$t_2 - t_1 \le 1, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{44}$$

and then from (42), (43), and Lemma 4, one can derive that

$$\begin{aligned}
&E\left|y_1(t_2) - y_1(t_1)\right|^p \\
&\quad= E\left|\int_{t_1}^{t_2} f_1(s, y_1(s), y_2(s))\,ds + \int_{t_1}^{t_2} g_1(s, y_1(s), y_2(s))\,dw_1(s)\right|^p \\
&\quad\le 2^{p-1}E\left(\int_{t_1}^{t_2}\left|f_1(s, y_1(s), y_2(s))\right|ds\right)^p + 2^{p-1}E\left|\int_{t_1}^{t_2} g_1(s, y_1(s), y_2(s))\,dw_1(s)\right|^p \\
&\quad\le 2^{p-1}\left(\int_{t_1}^{t_2} 1^q ds\right)^{p/q} E\left(\int_{t_1}^{t_2}\left|f_1(s, y_1(s), y_2(s))\right|^p ds\right) \\
&\qquad + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2 - t_1)^{(p-2)/2} \times \int_{t_1}^{t_2} E\left|g_1(s, y_1(s), y_2(s))\right|^p ds \\
&\quad\le 2^{p-1}\left(\int_{t_1}^{t_2} 1^q ds\right)^{p/q}\int_{t_1}^{t_2}\mathscr{L}(p)\,ds + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2 - t_1)^{(p-2)/2}\int_{t_1}^{t_2}\mathscr{R}(p)\,ds \\
&\quad\le 2^{p-1}(t_2 - t_1)^{p/q+1}\mathscr{L}(p) + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2 - t_1)^{p/2}\mathscr{R}(p) \\
&\quad\le 2^{p-1}(t_2 - t_1)^{p}\mathscr{L}(p) + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{p/2}(t_2 - t_1)^{p/2}\mathscr{R}(p) \\
&\quad\le 2^{p-1}(t_2 - t_1)^{p/2} \times \left\{(t_2 - t_1)^{p/2}\mathscr{L}(p) + \left[\frac{p(p-1)}{2}\right]^{p/2}\mathscr{R}(p)\right\} \\
&\quad\le 2^{p-1}(t_2 - t_1)^{p/2}\left\{\mathscr{L}(p) + \left[\frac{p(p-1)}{2}\right]^{p/2}\mathscr{R}(p)\right\}.
\end{aligned} \tag{45}$$

Thus, it follows from Lemma 5 that almost every sample path of $y_1(t)$ is locally but uniformly Hölder-continuous with exponent $\vartheta$ for $\vartheta \in (0, (p-2)/2p)$ and therefore almost every sample path of $y_1(t)$ is uniformly continuous on $t \ge 0$.

By a similar procedure as above, $y_2(t)$ can be proven. Thus, $(y_1(t), y_2(t))$ is uniformly continuous on $t \ge 0$. The proof of Lemma 9 is complete. □

**Theorem 10.** *Let* $(S_1)$, $(S_2)$, *and* $(S_3)$ *hold; then the unique positive solution of system* (3) *is globally attractive for initial data* $(y_1(0), y_2(0)) \in R_+^2$.

*Proof.* It follows from $(S_1)$ that, for initial data $(y_1(0), y_2(0)) \in R_+^2$, system (3) has a unique solution $(y_1(t), y_2(t)) \in R_+^2$ (see Lemma 7). Assume that $(y_1^*(t), y_2^*(t))$ is another solution of system (3) with initial values $y_1^*(0), y_2^*(0) > 0$.

Define a Lyapunov function $V(t)$ as

$$V(t) = \left|\ln y_1(t) - \ln y_1^*(t)\right| + \left|\ln y_2(t) - \ln y_2^*(t)\right|. \quad (46)$$

Using Itô's formula, a calculation of the right differential $D^+V(t)$ along (3) shows that

$$\begin{aligned}
D^+V(t) &= \operatorname{sgn}(y_1(t) - y_1^*(t))\, d(\ln y_1(t) - \ln y_1^*(t)) \\
&\quad + \operatorname{sgn}(y_2(t) - y_2^*(t))\, d(\ln y_2(t) - \ln y_2^*(t)) \\
&= \operatorname{sgn}(y_1(t) - y_1^*(t)) \\
&\quad \times \left\{\left[\frac{dy_1(t)}{y_1(t)} - \frac{(dy_1(t))^2}{2y_1^2(t)}\right] - \left[\frac{dy_1^*(t)}{y_1^*(t)} - \frac{(dy_1^*(t))^2}{2(y_1^*(t))^2}\right]\right\} \\
&\quad + \operatorname{sgn}(y_2(t) - y_2^*(t)) \\
&\quad \times \left\{\left[\frac{dy_2(t)}{y_2(t)} - \frac{(dy_2(t))^2}{2y_2^2(t)}\right] - \left[\frac{dy_2^*(t)}{y_2^*(t)} - \frac{(dy_2^*(t))^2}{2(y_2^*(t))^2}\right]\right\} \\
&= \operatorname{sgn}(y_1(t) - y_1^*(t)) \\
&\quad \times \left\{\left[\left(a_1 - b_1 y_1(t) - c_1 y_2(t) - d_1 y_1^2(t) - \frac{\sigma_1^2}{2}\right)dt \right.\right. \\
&\qquad \left. + \sigma_1 dw_1\right] \\
&\qquad - \left[\left(a_1 - b_1 y_1^*(t) - c_1 y_2^*(t) - d_1 y_1^*(t)^2 - \frac{\sigma_1^2}{2}\right)dt \right. \\
&\qquad \left.\left. + \sigma_1 dw_1\right]\right\} \\
&\quad + \operatorname{sgn}(y_2(t) - y_2^*(t)) \\
&\quad \times \left\{\left[\left(a_2 - b_2 y_2(t) - c_2 y_1(t) - d_2 y_2^2(t) - \frac{\sigma_2^2}{2}\right)dt \right.\right. \\
&\qquad \left. + \sigma_2 dw_2\right] \\
&\qquad - \left[\left(a_2 - b_2 y_2^*(t) - c_2 y_1^*(t) - d_2 y_2^*(t)^2 - \frac{\sigma_2^2}{2}\right)dt \right. \\
&\qquad \left.\left. + \sigma_2 dw_2\right]\right\} \\
&= \operatorname{sgn}(y_1(t) - y_1^*(t)) \\
&\quad \times \{-b_1(y_1(t) - y_1^*(t)) - c_1(y_2(t) - y_2^*(t)) \\
&\qquad - d_1(y_1(t) + y_1^*(t))(y_1(t) - y_1^*(t))\}\, dt \\
&\quad + \operatorname{sgn}(y_2(t) - y_2^*(t)) \\
&\quad \times \{-b_2(y_2(t) - y_2^*(t)) - c_2(y_1(t) - y_1^*(t)) \\
&\qquad - d_2(y_2(t) + y_2^*(t))(y_2(t) - y_2^*(t))\}\, dt \\
&\le \{-b_1 - d_1(y_1(t) + y_1^*(t)) + c_2\}\left|y_1(t) - y_1^*(t)\right| dt \\
&\quad + \{-b_2 - d_2(y_2(t) + y_2^*(t)) + c_1\}\left|y_2(t) - y_2^*(t)\right| dt \\
&\le -(b_1 - c_2)\left|y_1(t) - y_1^*(t)\right| dt \\
&\quad - (b_2 - c_1)\left|y_2(t) - y_2^*(t)\right| dt.
\end{aligned} \quad (47)$$

Integrating both sides yields that

$$\begin{aligned}
V(t) - V(0) \le &-(b_1 - c_2)\int_0^t \left|y_1(s) - y_1^*(s)\right| ds \\
&- (b_2 - c_1)\int_0^t \left|y_2(s) - y_2^*(s)\right| ds;
\end{aligned} \quad (48)$$

that is,

$$\begin{aligned}
V(t) &+ (b_1 - c_2)\int_0^t \left|y_1(t) - y_1^*(t)\right| ds \\
&+ (b_2 - c_1)\int_0^t \left|y_2(t) - y_2^*(t)\right| ds \le V(0) < \infty.
\end{aligned} \quad (49)$$

In view of $(S_3)$ and $V(t) \ge 0$, then it follows from (49) that

$$\begin{aligned}
&\left|y_1(t) - y_1^*(t)\right| \in L^1[0,\infty), \\
&\left|y_2(t) - y_2^*(t)\right| \in L^1[0,\infty).
\end{aligned} \quad (50)$$

So recalling Lemmas 9 and 6, we can show that

$$\begin{aligned}
&\lim_{t\to+\infty}\left|y_1(t) - y_1^*(t)\right| = 0, \\
&\lim_{t\to+\infty}\left|y_2(t) - y_2^*(t)\right| = 0 \text{ a.s.}
\end{aligned} \quad (51)$$

This completes the proof of Theorem 10. □

## 5. Numerical Simulations

In this paper, we derived the sufficient conditions for the existence, uniqueness, stochastically ultimate boundness, and global attraction of positive solutions of system (3). In order to illustrate the above theoretical results, we will perform

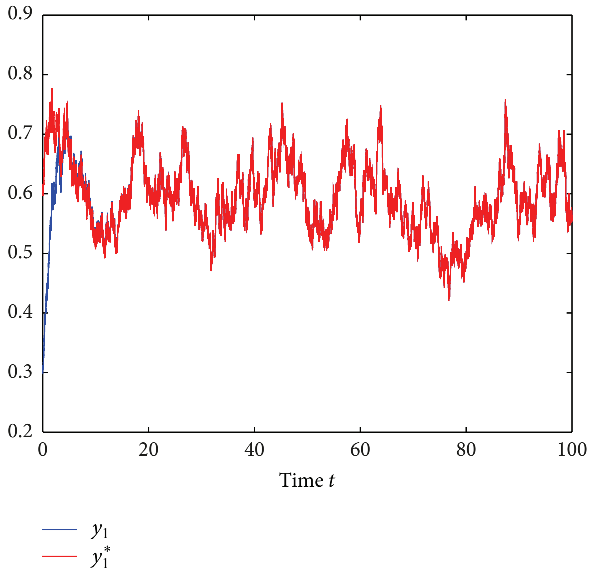


FIGURE 1: The sample path of $(y_1(t); y_1^*(t))$ in the same coordinate system.

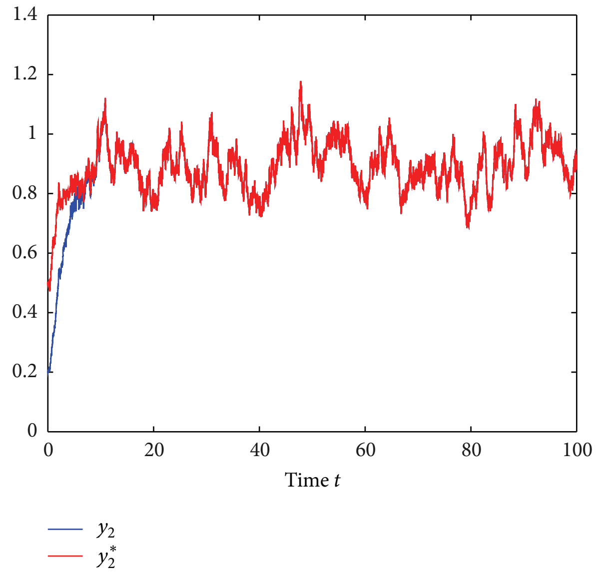


FIGURE 2: The sample path of $(y_2(t); y_2^*(t))$ in the same coordinate system.

a specific example. Motivated by the Milsten method mentioned in Higham [25], we can obtain the following discrete version of system (3):

$$\begin{aligned} y_1(k+1) &= y_1(k) + y_1(k)\left[a_1 - b_1 y_1(k) - c_1 y_2(k) - d_1 y_1^2(k)\right]\Delta t \\ &\quad + \sigma_1 y_1(k)\sqrt{\Delta t}\,\xi_1(k) \\ &\quad + 0.5\sigma_1^2 y_1(k)\left(\xi_1^2(k) - 1\right)\Delta t, \\ y_2(k+1) &= y_2(k) + y_2(k)\left[a_2 - b_2 y_2(k) - c_2 y_1(k) - d_2 y_2^2(k)\right]\Delta t \\ &\quad + \sigma_2 y_2(k)\sqrt{\Delta t}\,\xi_2(k) \\ &\quad + 0.5\sigma_2^2 y_2(k)\left(\xi_2^2(k) - 1\right)\Delta t, \end{aligned} \tag{52}$$

where $\xi_1(k)$ and $\xi_2(k)$ are Gaussian random variables that follow $N(0,1)$. Let $a_1 = 0.6$, $b_1 = 0.5$, $c_1 = 0.2$, $d_1 = 0.3$, and $\sigma_1 = 0.1$; $a_2 = 0.7$, $b_2 = 0.4$, $c_2 = 0.3$, $d_2 = 0.2$, and $\sigma_2 = 0.1$; $\Delta t = 0.001$; and the initial value $(y_1(0), y_2(0)) = (0.3, 0.2)$, $(y_1^*(0), y_2^*(0)) = (0.6, 0.5)$. After a calculation, we can see that the assumptions of Theorems 8 and 10 hold. Figures 1 and 2 show that the positive solution of system (52) is stochastically ultimately bounded and globally attractive.

It follows from Theorem 8 that a preliminary result on the stochastically ultimate boundness of system (3) is obtained. We would like to mention here that an interesting but challenging problem associated with the investigation of system (3) should be the stochastic permanence; we leave this for future work.

## Conflict of Interests

The authors declare that they have no conflict of interests.

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