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Research Article

Positive Solutions and Mann Iterative Algorithms for a Nonlinear Three-Dimensional Difference System

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The existence of uncountably many positive solutions and Mann iterative approximations for a nonlinear three-dimensional difference system are proved by using the Banach fixed point theorem. Four illustrative examples are also provided.

1. Introduction

In recent years, the oscillation, nonoscillation, asymptotic behavior, existence and multiplicity of solutions, bounded solutions, unbounded solutions, positive solutions, and nonoscillatory solutions for some two- and three-dimensional difference systems have been studied by many authors and a significant number of important results have been found; see [1–12] and the references therein.

In order to solve the problem that the Picard iteration fails to converge under some conditions, Mann [13] introduced a modified iteration, which is now called Mann iterative scheme. It is well known that the Mann iterative schemes are often used in the fields of nonlinear differential equations, nonlinear equations, nonlinear mappings, optimization, variational inequalities, nonlinear analysis, and so forth.

Note that the difference systems in [1–12] are as follows:

$$\Delta x_n = b_n g(y_n), \quad n \ge n_0,$$
$$\Delta y_n = -a_n f(x_n), \quad n \ge n_0,$$
$$\Delta x_n = a_n f(y_n), \quad n \ge n_0,$$
$$\Delta y_n = b_n g(x_n), \quad n \ge n_0,$$
$$\Delta x_n = b_n g(y_n), \quad n \ge n_0,$$
$$\Delta y_{n-1} = -a_n f(x_n), \quad n \ge n_0,$$
$$\Delta x_n = b_n g(y_n), \quad n \ge n_0,$$
$$\Delta y_n = -a_n f(x_n) + r_n, \quad n \ge n_0, \tag{1}$$

where $n_0 \in \mathbb{N}_0$, $f, g \in C(\mathbb{R}, \mathbb{R})$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{b_n\}_{n\in\mathbb{N}_{n_0}}$ are nonnegative sequences, and $\{r_n\}_{n\in\mathbb{N}_{n_0}}$ is a real sequence with $\sum_{i=n_0}^{\infty} |r_i| < \infty$:

$$\begin{aligned} \Delta x_n &= b_n g(y_n), \quad n \ge n_0, \\ \Delta y_{n-1} &= -a_n f(x_n) + r_n, \quad n \ge n_0, \end{aligned} \tag{2}$$

where $n_0 \in \mathbb{N}_0$, $f, g \in C(\mathbb{R}, \mathbb{R})$, $\{a_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{r_n\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $\sum_{i=n_0}^{\infty} |r_i| < \infty$, and $\{b_n\}_{n\in\mathbb{N}_{n_0}}$ is a nonnegative sequence:

$$\begin{aligned} &\Delta\left(x_n + p_{1n}x_{n-\tau_1}\right) + f_1\left(n, x_{a_{1n}}, x_{a_{hn}}, y_{b_{1n}}, y_{b_{kn}}\right) = q_{1n}, \\ &\qquad n \ge n_0, \\ &\Delta\left(y_n + p_{2n}y_{n-\tau_2}\right) + f_2\left(n, x_{c_{1n}}, x_{c_{hn}}, y_{d_{1n}}, y_{d_{kn}}\right) = q_{2n}, \\ &\qquad n \ge n_0, \end{aligned} \tag{3}$$

where $h, \tau_i, k, n_0 \in \mathbb{N}$, $\{p_{in}\}_{n\in\mathbb{N}_{n_0}}, \{q_{in}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $f_i \in C(\mathbb{N}_{n_0} \times \mathbb{R}^{h+k}, \mathbb{R})$, $\{h_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{f_{ln}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{N}$ for $i \in \Lambda_2$, and $\{a_{ln}\}_{n\in\mathbb{N}_{n_0}}$, $\{b_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{c_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{d_{ln}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{Z}$ with

$$\lim_{n\to\infty} a_{ln} = \lim_{n\to\infty} b_{ln} = \lim_{n\to\infty} c_{ln} = \lim_{n\to\infty} d_{ln} = +\infty, \quad l \in \Lambda_k,$$

$$\begin{aligned} \Delta x_n &= a_n f(y_{n-l}), \quad n \ge n_0, \\ \Delta y_n &= b_n g(z_{n-m}), \quad n \ge n_0, \\ \Delta z_n &= \delta c_n h(x_{n-k}), \quad n \ge n_0. \end{aligned} \tag{4}$$

However, to the best of our knowledge, there exists no result in the literature dealing with the following nonlinear three-dimensional difference system:

$$\begin{aligned} &\Delta^2 \left(x_{1n} + b_{1n} x_{1(n-\tau_1)}\right) \\ &\quad + f_1 \left(n, x_{1r_{11n}}, \ldots, x_{1r_{1kn}}, x_{2s_{11n}}, \ldots, x_{2s_{1kn}}, x_{3t_{11n}}, \ldots, x_{3t_{1kn}}\right) = c_{1n}, \quad n \in \mathbb{N}_{n_0}, \\ &\Delta^2 \left(x_{2n} + b_{2n} x_{2(n-\tau_2)}\right) \\ &\quad + f_2 \left(n, x_{1r_{21n}}, \ldots, x_{1r_{2kn}}, x_{2s_{21n}}, \ldots, x_{2s_{2kn}}, x_{3t_{21n}}, \ldots, x_{3t_{2kn}}\right) = c_{2n}, \quad n \in \mathbb{N}_{n_0}, \\ &\Delta^2 \left(x_{3n} + b_{3n} x_{3(n-\tau_3)}\right) \\ &\quad + f_3 \left(n, x_{1r_{31n}}, \ldots, x_{1r_{3kn}}, x_{2s_{31n}}, \ldots, x_{2s_{3kn}}, x_{3t_{31n}}, \ldots, x_{3t_{3kn}}\right) = c_{3n}, \quad n \in \mathbb{N}_{n_0}, \end{aligned} \tag{5}$$

which is abbreviated as, for convenience,

$$\begin{aligned} &\Delta^2 \left(x_{an} + b_{an} x_{a(n-\tau_a)}\right) \\ &\quad + f_a \left(n, x_{1r_{a1n}}, \ldots, x_{1r_{akn}}, x_{2s_{a1n}}, \ldots, x_{2s_{akn}}, x_{3t_{a1n}}, \ldots, x_{3t_{akn}}\right) = c_{an}, \quad (n,a) \in \mathbb{N}_{n_0} \times \Lambda_3, \end{aligned} \tag{6}$$

where $n_0 \in \mathbb{N}_0$, $\tau_a, k \in \mathbb{N}$, $\{b_{an}\}_{n\in\mathbb{N}_{n_0}}, \{c_{an}\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $f_a \in C(\mathbb{N}_{n_0} \times \mathbb{R}^{3k}, \mathbb{R})$, and $\{r_{aln}, s_{aln}, t_{aln} : n \in \mathbb{N}_{n_0}, l \in \Lambda_k\} \subset \mathbb{N}$ with

$$\lim_{n\to\infty} r_{aln} = \lim_{n\to\infty} s_{aln} = \lim_{n\to\infty} t_{aln} = +\infty, \quad (l,a) \in \Lambda_k \times \Lambda_3. \tag{7}$$

The main purpose of this paper is to study solvability and convergence of the Mann iterative schemes for the system (6). Sufficient conditions for the existence of uncountably many positive solutions of the system (6) and convergence of the Mann iterative schemes relative to these positive solutions are provided by utilizing the Banach fixed point theorem. Four illustrative examples are given.

2. Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, and $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ stand for the sets of all integers, positive integers, and nonnegative integers, respectively, as

$$\begin{aligned} \Lambda_{n_0} &= \{1, 2, \ldots, n_0\}, \\ \mathbb{N}_{n_0} &= \{n : n \in \mathbb{N}_0 \text{ with } n \ge n_0\}, \quad n_0 \in \mathbb{N}_0, \\ \beta &= \min \left\{n_0 - \tau_a, \inf \left\{r_{aln}, s_{aln}, t_{aln} : n \in \mathbb{N}_{n_0}, l \in \Lambda_k\right\} : a \in \Lambda_3\right\} \in \mathbb{N}, \\ P_{an} &= \max \left\{r_{aln}, s_{aln}, t_{aln} : l \in \Lambda_k\right\}, \quad (n,a) \in \mathbb{N}_{n_0} \times \Lambda_3. \end{aligned} \tag{8}$$

Let l_β^∞ denote the Banach space of all sequences in $\mathbb{N}_\beta$ with norm

$$\|x\| = \sup_{n\in\mathbb{N}_\beta} \left|\frac{x_n}{n}\right| < +\infty \quad \text{for each } x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty \tag{9}$$

and put

$$\begin{aligned} \hat{d} &= \{nd\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty \quad \text{for each } d \in \mathbb{R}, \\ \Omega\left(\hat{d}, D\right) &= \left\{x = \{x_n\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty : \left\|x - \hat{d}\right\| \le D\right\} \\ &\quad \text{for each } \left(D, \hat{d}\right) \in \left(\mathbb{R}^+ \setminus \{0\}\right) \times l_\beta^\infty. \end{aligned} \tag{10}$$

It is easy to see that for each $(D_1, D_2, D_3, \hat{d}_1, \hat{d}_2, \hat{d}_3) \in (\mathbb{R}^+ \setminus \{0\})^3 \times (l_\beta^\infty)^3$, $\Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$ is a nonempty closed convex subset of the Banach space $(l_\beta^\infty)^3$ with norm $\|(x,y,z)\|_1 = \max\{\|x\|, \|y\|, \|z\|\}$ for each $(x,y,z) \in (l_\beta^\infty)^3$.

By a solution of the system (6), we mean a three-dimensional sequence $\{(x_{1n}, x_{2n}, x_{3n})\}_{n\in\mathbb{N}_\beta} \in (l_\beta^\infty)^3$ with a positive integer $T \ge 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ such that (6) holds for all $n \ge T$.

Let K be a nonempty convex subset of a Banach space X and let $S : K \to K$ be a mapping. For any given $x_0 \in K$, the sequence $\{x_m\}_{m\in\mathbb{N}_0}$ defined by

$$x_{m+1} = (1 - \alpha_m) x_m + \alpha_m S x_m, \quad m \in \mathbb{N}_0, \tag{11}$$

where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is a sequence in $[0,1]$ with certain condition, is called the Mann iterative scheme.

Lemma 1. *Let $\{p_t\}_{t\in\mathbb{N}}$ be a nonnegative sequence and $n, \tau \in \mathbb{N}$.*

(i) *If $\sum_{t=n}^{\infty} t p_t < +\infty$, then*

$$\sum_{s=n}^{\infty}\sum_{t=s}^{\infty} p_t = \sum_{t=n}^{\infty}(t-n+1)\,p_t \le \sum_{t=n}^{\infty} t p_t. \quad (12)$$

(ii) *If $\sum_{t=n+\tau}^{\infty} t^2 p_t < +\infty$, then*

$$\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty} p_t \le \sum_{t=n+\tau}^{\infty} t^2 p_t. \quad (13)$$

Proof. It is easy to see that

$$\begin{aligned}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty} p_t &= \sum_{t=n}^{\infty} p_t + \sum_{t=n+1}^{\infty} p_t + \sum_{t=n+2}^{\infty} p_t + \cdots \\ &= \sum_{t=n}^{\infty}(1+t-n)\,p_t \le \sum_{t=n}^{\infty} t p_t,\end{aligned} \quad (14)$$

which yields (12). Note that

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty} p_t &= \sum_{i=1}^{\infty}\left(\sum_{t=n+i\tau}^{\infty} p_t + \sum_{t=n+1+i\tau}^{\infty} p_t + \sum_{t=n+2+i\tau}^{\infty} p_t + \cdots\right) \\ &= \sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}(1+t-n-i\tau)\,p_t \le \sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty} t p_t \\ &= \sum_{t=n+\tau}^{\infty} t p_t + \sum_{t=n+2\tau}^{\infty} t p_t + \sum_{t=n+3\tau}^{\infty} t p_t + \cdots \\ &= \sum_{t=n+\tau}^{\infty}\left(1+\left[\frac{t-n-\tau}{\tau}\right]\right) t p_t \\ &\le \sum_{t=n+\tau}^{\infty}\left(1+\frac{t-n-\tau}{\tau}\right) t p_t \le \sum_{t=n+\tau}^{\infty}\frac{t-n}{\tau} t p_t \\ &\le \sum_{t=n+\tau}^{\infty} t^2 p_t,\end{aligned} \quad (15)$$

where $[(t-n-\tau)/\tau]$ denotes the largest integral number not exceeding $(t-n-\tau)/\tau$. Hence (13) holds. This completes the proof. □

3. Uncountably Many Positive Solutions and Mann Iterative Schemes

Our main results are as follows.

Theorem 2. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $d_a, D_a \in \mathbb{R}^+ \setminus \{0\}$ and nonnegative sequences $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$ and $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$d_a > D_a, \quad b_{an} = -1, \quad (n,a) \in \mathbb{N}_{n_1} \times \Lambda_3; \quad (16)$$

$$\begin{aligned}&\left|f_a(n,u_1,\ldots,u_k,v_1,\ldots,v_k,w_1,\ldots,w_k)\right. \\ &\quad \left.- f_a(n,\overline{u}_1,\ldots,\overline{u}_k,\overline{v}_1,\ldots,\overline{v}_k,\overline{w}_1,\ldots,\overline{w}_k)\right| \\ &\quad\le U_{an}\max\left\{|u_l-\overline{u}_l|,|v_l-\overline{v}_l|,|w_l-\overline{w}_l| : l\in\Lambda_k\right\}, \\ &(n,u_l,\overline{u}_l,v_l,\overline{v}_l,w_l,\overline{w}_l) \in \mathbb{N}_{n_0}\times\Pi_{w=1}^{3}[d_w-D_w,d_w+D_w]^2, \\ &\qquad (l,a)\in\Lambda_k\times\Lambda_3;\end{aligned} \quad (17)$$

$$\begin{aligned}&\left|f_a(n,u_1,\ldots,u_k,v_1,\ldots,v_k,w_1,\ldots,w_k)\right| \le F_{an}, \\ &(n,u_l,v_l,w_l)\in\mathbb{N}_{n_0}\times\Pi_{w=1}^{3}[d_w-D_w,d_w+D_w], \\ &\qquad (l,a)\in\Lambda_k\times\Lambda_3;\end{aligned} \quad (18)$$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},|c_{aj}|,U_{aj}P_{aj}\right\} = 0, \\ &\qquad a\in\Lambda_3.\end{aligned} \quad (19)$$

Then one has the following.

(a) For any $(L_1,L_2,L_3) \in \Pi_{w=1}^{3}(d_w-D_w,d_w+D_w)$, there exist $\theta \in (0,1)$ and $T \ge 1+n_0+n_1+\tau_1+\tau_2+\tau_3+\beta$ such that, for each $(x_1^0,x_2^0,x_3^0) = \{(x_{1n}^0,x_{2n}^0,x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$, the Mann iterative sequence $\{(x_1^m,x_2^m,x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m,x_{2n}^m,x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$ generated by the scheme

$$x_{an}^{m+1} = \begin{cases}(1-\alpha_m)x_{an}^m + \alpha_m\left\{nL_a + \sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}}^m,\ldots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\ldots,x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\ldots,x_{3t_{akj}}^m\right)-c_{aj}\right]\right\}, \\ \qquad n\ge T, \quad (m,a)\in\mathbb{N}_0\times\Lambda_3, \\ (1-\alpha_m)x_{an}^m + \alpha_m\left\{nL_a + \frac{n}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}}^m,\ldots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\ldots,x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\ldots,x_{3t_{akj}}^m\right)-c_{aj}\right]\right\}, \\ \qquad \beta\le n<T, \quad (m,a)\in\mathbb{N}_0\times\Lambda_3\end{cases} \quad (20)$$

converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ *of the system* (6) *with* $\lim_{n\to\infty}(z_{an}/n) = L_a$ *for each* $a \in \Lambda_3$ *and has the following error estimate:*

$$\begin{aligned}&\left\|\left(x_1^{m+1}, x_2^{m+1}, x_3^{m+1}\right) - (z_1, z_2, z_3)\right\|_1\\ &\quad\le e^{-(1-\theta)\sum_{n=0}^{m}\alpha_n}\left\|\left(x_1^0, x_2^0, x_3^0\right) - (z_1, z_2, z_3)\right\|_1, \\ &\quad m \in \mathbb{N}_0,\end{aligned} \tag{21}$$

where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *such that*

$$\sum_{m=0}^{\infty}\alpha_m = +\infty. \tag{22}$$

(b) The system (6) *has uncountably many positive solutions in* $\Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$.

Proof. Firstly, we prove that (a) holds. Put $(L_1, L_2, L_3) \in \Pi_{w=1}^3(d_w - D_w, d_w + D_w)$. It follows from (19) that there exist $\theta \in (0,1)$ and $T \ge 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ satisfying

$$\theta = \max\{\theta_a : a \in \Lambda_3\}, \tag{23}$$

where

$$\theta_a = \frac{1}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}U_{aj}P_{aj}, \quad a \in \Lambda_3; \tag{24}$$

$$\frac{1}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right) < D_a - \left|L_a - d_a\right|, \quad a \in \Lambda_3. \tag{25}$$

Define mappings $A_{L_1}, A_{L_2}, A_{L_3} : \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w) \to l_\beta^\infty$ and $S_{L_1,L_2,L_3} : \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w) \to (l_\beta^\infty)^3$ by

$$A_{L_a}(x_{1n}, x_{2n}, x_{3n}) = \begin{cases} nL_a + \displaystyle\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}}, \dots, x_{1r_{akj}}, x_{2s_{a1j}}, \dots, x_{2s_{akj}}, x_{3t_{a1j}}, \dots, x_{3t_{akj}}\right) - c_{aj}\right], & n \ge T, \quad a \in \Lambda_3, \\ \dfrac{n}{T}A_{L_a}(x_{1T}, x_{2T}, x_{3T}), \quad \beta \le n < T,\ a \in \Lambda_3, \end{cases} \tag{26}$$

$$S_{L_1,L_2,L_3}(x_1, x_2, x_3) = \left(A_{L_1}(x_1, x_2, x_3), A_{L_2}(x_1, x_2, x_3), A_{L_3}(x_1, x_2, x_3)\right), \tag{27}$$

for each $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$.

Now we assert that

$$A_{L_a}\left(\Pi_{w=1}^3\Omega\left(\widehat{d}_w, D_w\right)\right) \subseteq \Omega\left(\widehat{d}_a, D_a\right), \quad a \in \Lambda_3, \tag{28}$$

$$S_{L_1,L_2,L_3}\left(\Pi_{w=1}^3\Omega\left(\widehat{d}_w, D_w\right)\right) \subseteq \Pi_{w=1}^3\Omega\left(\widehat{d}_w, D_w\right). \tag{29}$$

Using (18), (25), and (26), we get that, for any $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ and $a \in \Lambda_3$,

$$\begin{aligned}&\left\|A_{L_a}(x_1, x_2, x_3) - \widehat{d}_a\right\| \\ &= \sup_{n\in\mathbb{N}_\beta}\left|\frac{A_{L_a}(x_{1n}, x_{2n}, x_{3n}) - nd_a}{n}\right| \\ &= \max\left\{\sup_{n\ge T}\left|\frac{A_{L_a}(x_{1n}, x_{2n}, x_{3n})}{n} - d_a\right|, \right.\\ &\qquad \left.\sup_{\beta\le n<T}\left|\frac{n}{T}\cdot\frac{A_{L_a}(x_{1T}, x_{2T}, x_{3T})}{n} - d_a\right|\right\} \\ &= \sup_{n\ge T}\left|L_a - d_a + \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}}, \dots, x_{1r_{akj}}, x_{2s_{a1j}}, \dots, x_{2s_{akj}}, x_{3t_{a1j}}, \dots, x_{3t_{akj}}\right) - c_{aj}\right]\right| \\ &\le \sup_{n\ge T}\left(\left|L_a - d_a\right| + \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right)\right) \\ &\le \left|L_a - d_a\right| + \frac{1}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right) \\ &\le \left|L_a - d_a\right| + D_a - \left|L_a - d_a\right| = D_a,\end{aligned} \tag{30}$$

which implies (28). Thus (29) follows from (27) and (28).

Next we claim that

$$\begin{aligned}&\left\|A_{L_a}(x_1, x_2, x_3) - A_{L_a}(\overline{x}_1, \overline{x}_2, \overline{x}_3)\right\| \\ &\quad\le \theta_a\max\left\{\left\|x_w - \overline{x}_w\right\| : w \in \Lambda_3\right\}, \quad a \in \Lambda_3,\end{aligned} \tag{31}$$

$$\begin{aligned}&\left\|S_{L_1,L_2,L_3}(x_1, x_2, x_3) - S_{L_1,L_2,L_3}(\overline{x}_1, \overline{x}_2, \overline{x}_3)\right\|_1 \\ &\quad\le \theta\left\|(x_1, x_2, x_3) - (\overline{x}_1, \overline{x}_2, \overline{x}_3)\right\|_1,\end{aligned} \tag{32}$$

for all $(x_1,x_2,x_3), (\overline{x}_1,\overline{x}_2,\overline{x}_3) \in \Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$. In view of (17), (23), (26), and (27), we deduce that, for any $(x_1,x_2,x_3) = \{(x_{1n},x_{2n},x_{3n})\}_{n\in\mathbb{N}_\beta}$, $(\overline{x}_1,\overline{x}_2,\overline{x}_3) = \{(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$, and $a \in \Lambda_3$,

$$\begin{aligned}
&\left\|A_{L_a}(x_1,x_2,x_3) - A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\| \\
&\quad = \sup_{n\in\mathbb{N}_\beta}\left|\frac{A_{L_a}(x_{1n},x_{2n},x_{3n}) - A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n}\right| \\
&\quad = \max\left\{\sup_{n\geq T}\left|\frac{A_{L_a}(x_{1n},x_{2n},x_{3n}) - A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n}\right|, \sup_{\beta\leq n<T}\left|\frac{n}{T}\cdot\frac{A_{L_a}(x_{1T},x_{2T},x_{3T}) - A_{L_a}(\overline{x}_{1T},\overline{x}_{2T},\overline{x}_{3T})}{n}\right|\right\} \\
&\quad = \sup_{n\geq T}\frac{1}{n}\left|\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}},\ldots,x_{1r_{akj}}, x_{2s_{a1j}},\ldots,x_{2s_{akj}}, x_{3t_{a1j}},\ldots,x_{3t_{akj}}\right)\right.\right. \\
&\qquad\qquad \left.\left. -f_a\left(j, \overline{x}_{1r_{a1j}},\ldots,\overline{x}_{1r_{akj}}, \overline{x}_{2s_{a1j}},\ldots,\overline{x}_{2s_{akj}}, \overline{x}_{3t_{a1j}},\ldots,\overline{x}_{3t_{akj}}\right)\right]\right| \\
&\quad \leq \sup_{n\geq T}\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}\max\left\{\left|x_{1r_{alj}} - \overline{x}_{1r_{alj}}\right|, \left|x_{2s_{alj}} - \overline{x}_{2s_{alj}}\right|, \left|x_{3t_{alj}} - \overline{x}_{3t_{alj}}\right| : l \in \Lambda_k\right\}\right) \\
&\quad \leq \frac{1}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}P_{aj}\max\left\{\frac{\left|x_{1r_{alj}} - \overline{x}_{1r_{alj}}\right|}{r_{alj}}, \frac{\left|x_{2s_{alj}} - \overline{x}_{2s_{alj}}\right|}{s_{alj}}, \frac{\left|x_{3t_{alj}} - \overline{x}_{3t_{alj}}\right|}{t_{alj}} : l \in \Lambda_k\right\}\right) \\
&\quad \leq \theta_a\max\left\{\left\|x_w - \overline{x}_w\right\| : w \in \Lambda_3\right\} \\
&\quad = \theta_a\left\|(x_1,x_2,x_3) - (\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1, \\
&\left\|S_{L_1,L_2,L_3}(x_1,x_2,x_3) - S_{L_1,L_2,L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1 \\
&\quad = \left\|\left(A_{L_1}(x_1,x_2,x_3), A_{L_2}(x_1,x_2,x_3), A_{L_3}(x_1,x_2,x_3)\right) - \left(A_{L_1}(\overline{x}_1,\overline{x}_2,\overline{x}_3), A_{L_2}(\overline{x}_1,\overline{x}_2,\overline{x}_3), A_{L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right)\right\|_1 \\
&\quad = \max\left\{\left\|A_{L_a}(x_1,x_2,x_3) - A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\| : a \in \Lambda_3\right\} \\
&\quad \leq \max\left\{\theta_a\left\|(x_1,x_2,x_3) - (\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1 : a \in \Lambda_3\right\} \\
&\quad = \theta\left\|(x_1,x_2,x_3) - (\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1,
\end{aligned} \tag{33}$$

which yield (31) and (32). Clearly, (29) and (32) ensure that S_{L_1,L_2,L_3} is a contraction mapping in $\Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$ and it has a unique fixed point $(z_1,z_2,z_3) = \{(z_{1n},z_{2n},z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$, that is,

$$\begin{aligned}
z_{an} &= nL_a + \sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, z_{1r_{a1j}},\ldots,z_{1r_{akj}}, z_{2s_{a1j}},\ldots, z_{2s_{akj}}, z_{3t_{a1j}},\ldots,z_{3t_{akj}}\right) - c_{aj}\right], \\
&\qquad (n,a) \in \mathbb{N}_T \times \Lambda_3, \\
z_{a(n-\tau_a)} &= (n-\tau_a)L_a + \sum_{q=1}^{\infty}\sum_{i=n+(q-1)\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, z_{1r_{a1j}},\ldots,z_{1r_{akj}}, z_{2s_{a1j}},\ldots, z_{2s_{akj}}, z_{3t_{a1j}},\ldots,z_{3t_{akj}}\right) - c_{aj}\right], \\
&\qquad (n,a) \in \mathbb{N}_{T+\tau_a} \times \Lambda_3,
\end{aligned} \tag{34}$$

which guarantee that

$$\begin{aligned}
&\Delta^2\left(z_{an} - z_{a(n-\tau_a)}\right) \\
&\quad = -f_a\left(n, z_{1r_{a1n}},\ldots,z_{1r_{akn}}, z_{2s_{a1n}},\ldots,z_{2s_{akn}}, z_{3t_{a1n}},\ldots,z_{3t_{akn}}\right) \\
&\qquad + c_{an}, \quad (n,a) \in \mathbb{N}_{T+\tau_1+\tau_2+\tau_3} \times \Lambda_3,
\end{aligned} \tag{35}$$

which together with (16) means that $(z_1,z_2,z_3) = \{(z_{1n},z_{2n},z_{3n})\}_{n\in\mathbb{N}_\beta}$ is a positive solution of the system (6) in $\Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$. Observe that (18) and (19) give that, for each $a \in \Lambda_3$,

$$\begin{aligned}
&\left|\frac{z_{an}}{n} - L_a\right| \\
&\quad = \left|\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j, z_{1r_{a1j}},\ldots,z_{1r_{akj}}, z_{2s_{a1j}},\ldots, z_{2s_{akj}}, z_{3t_{a1j}},\ldots,z_{3t_{akj}}\right) - c_{aj}\right]\right| \\
&\quad \leq \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{36}$$

which implies that

$$\lim_{n\to\infty}\frac{z_{an}}{n}=L_a,\quad a\in\Lambda_3. \tag{37}$$

By means of (20), (26), and (31), we infer that

$$\begin{aligned}
&\frac{\left|x_{an}^{m+1}-z_{an}\right|}{n}\\
&=\Bigg|\left(1-\alpha_m\right)\frac{x_{an}^m-z_{an}}{n}+\frac{\alpha_m}{n}\\
&\quad\times\Bigg(nL_a+\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,x_{1r_{a1j}}^m,\dots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\dots,\\
&\qquad x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\dots,x_{3t_{akj}}^m\Big)\\
&\qquad -c_{aj}\Big]-z_{an}\Bigg)\Bigg|\\
&\le\left(1-\alpha_m\right)\left\|x_a^m-z_a\right\|\\
&\quad+\alpha_m\cdot\frac{1}{n}\left|A_{L_a}\left(x_{1n}^m,x_{2n}^m,x_{3n}^m\right)-A_{L_a}\left(z_{1n},z_{2n},z_{3n}\right)\right|\\
&\le\left(1-\alpha_m\right)\left\|x_a^m-z_a\right\|\\
&\quad+\alpha_m\left\|A_{L_a}\left(x_1^m,x_2^m,x_3^m\right)-A_{L_a}\left(z_1,z_2,z_3\right)\right\|\\
&\le\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|,\quad n\ge T,\ (m,a)\in\mathbb{N}_0\times\Lambda_3,\\
&\frac{\left|x_{an}^{m+1}-z_{an}\right|}{n}\\
&=\Bigg|\left(1-\alpha_m\right)\frac{x_{an}^m-z_{an}}{n}+\frac{\alpha_m}{n}\\
&\quad\times\Bigg(nL_a+\frac{n}{T}\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,x_{1r_{a1j}}^m,\dots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\dots,\\
&\qquad x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\dots,x_{3t_{akj}}^m\Big)\\
&\qquad -c_{aj}\Big]-z_{an}\Bigg)\Bigg|\\
&\le\left(1-\alpha_m\right)\left\|x_a^m-z_a\right\|+\frac{\alpha_m}{n}\cdot\frac{n}{T}\\
&\quad\times\Bigg|TL_a+\sum_{q=1}^{\infty}\sum_{i=T+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,x_{1r_{a1j}}^m,\dots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\dots,\\
&\qquad x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\dots,x_{3t_{akj}}^m\Big)\\
&\qquad -c_{aj}\Big]-A_{L_a}\left(z_{1T},z_{2T},z_{3T}\right)\Bigg|\\
&\le\left(1-\alpha_m\right)\left\|x_a^m-z_a\right\|\\
&\quad+\alpha_m\cdot\frac{1}{T}\left|A_{L_a}\left(x_{1T}^m,x_{2T}^m,x_{3T}^m\right)-A_{L_a}\left(z_{1T},z_{2T},z_{3T}\right)\right|\\
&\le\left(1-\alpha_m\right)\left\|x_a^m-z_a\right\|\\
&\quad+\alpha_m\left\|A_{L_a}\left(x_1^m,x_2^m,x_3^m\right)-A_{L_a}\left(z_1,z_2,z_3\right)\right\|\\
&\le\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|,\\
&\qquad\beta\le n<T,\quad (m,a)\in\mathbb{N}_0\times\Lambda_3,
\end{aligned}\tag{38}$$

which imply that

$$\begin{aligned}
&\left\|\left(x_1^{m+1},x_2^{m+1},x_3^{m+1}\right)-\left(z_1,z_2,z_3\right)\right\|_1\\
&\quad=\max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|:a\in\Lambda_3\right\}\\
&\quad=\max\left\{\sup_{n\ge T}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|,\sup_{\beta\le n<T}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|:a\in\Lambda_3\right\}\\
&\quad\le\max\left\{\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|:a\in\Lambda_3\right\}\\
&\quad\le\left(1-\left(1-\theta\right)\alpha_m\right)\left\|\left(x_1^m,x_2^m,x_3^m\right)-\left(z_1,z_2,z_3\right)\right\|_1\\
&\quad\le e^{-(1-\theta)\alpha_m}\left\|\left(x_1^m,x_2^m,x_3^m\right)-\left(z_1,z_2,z_3\right)\right\|_1\\
&\quad\le e^{-(1-\theta)\sum_{n=0}^m\alpha_n}\left\|\left(x_1^0,x_2^0,x_3^0\right)-\left(z_1,z_2,z_3\right)\right\|_1,\quad m\in\mathbb{N}_0;
\end{aligned}\tag{39}$$

that is, (21) holds. It follows from (21) and (22) that $\lim_{m\to\infty}\|(x_1^m,x_2^m,x_3^m)-(z_1,z_2,z_3)\|_1=0$.

Secondly, we show that (b) holds. Let (L_{11},L_{21},L_{31}), $(L_{12},L_{22},L_{32})\in\Pi_{w=1}^3(d_w-D_w,d_w+D_w)$ with $\max\{|L_{a1}-L_{a2}|:a\in\Lambda_3\}>0$. As in the proof of (a), we infer similarly that, for each $l\in\Lambda_2$, there exist constants $\theta^l,\theta_1^l,\theta_2^l,\theta_3^l\in(0,1)$, $T_l\ge 1+n_0+n_1+\tau_1+\tau_2+\tau_3+\beta$ and mappings $A_{L_{1l}},A_{L_{2l}},A_{L_{3l}}:\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)\to l_\beta^\infty$ and $S_{L_{1l},L_{2l},L_{3l}}:\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)\to(l_\beta^\infty)^3$ satisfying (23)~(32), where θ, θ_1, θ_2, θ_3, T, L_1, L_2, L_3, A_{L_1}, A_{L_2}, A_{L_3}, and S_{L_1,L_2,L_3} are replaced by $\theta^l,\theta_1^l,\theta_2^l,\theta_3^l,T_l,L_{1l},L_{2l},L_{3l},A_{L_{1l}},A_{L_{2l}},A_{L_{3l}}$, and $S_{L_{1l},L_{2l},L_{3l}}$, respectively, and the contraction mapping $S_{L_{1l},L_{2l},L_{3l}}$ has a unique fixed point $(z_1^l,z_2^l,z_3^l)=\{(z_{1n}^l,z_{2n}^l,z_{3n}^l)\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$, which is a positive solution of the system (6); that is,

$$\begin{aligned}
z_{an}^l&=nL_{al}\\
&\quad+\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,z_{1r_{a1j}}^l,\dots,z_{1r_{akj}}^l,z_{2s_{a1j}}^l,\dots,\\
&\qquad z_{2s_{akj}}^l,z_{3t_{a1j}}^l,\dots,z_{3t_{akj}}^l\Big)-c_{aj}\Big],\\
&\qquad (n,l,a)\in\mathbb{N}_{T_1}\times\Lambda_2\times\Lambda_3.
\end{aligned}\tag{40}$$

On account of (17), (23), and (40), we conclude that

$$\begin{aligned}
\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| &= \Bigg| L_{a1} - L_{a2} + \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\left(j, z^1_{1r_{a1j}},\ldots,z^1_{1r_{akj}}, z^1_{2s_{a1j}},\ldots,z^1_{2s_{akj}}, z^1_{3t_{a1j}},\ldots,z^1_{3t_{akj}}\right) \\
&\qquad - f_a\left(j, z^2_{1r_{a1j}},\ldots,z^2_{1r_{akj}}, z^2_{2s_{a1j}},\ldots,z^2_{2s_{akj}}, z^2_{3t_{a1j}},\ldots,z^2_{3t_{akj}}\right)\Big]\Bigg| \\
&\geq \left|L_{a1} - L_{a2}\right| - \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}\max\left\{\left|z^1_{1r_{alj}} - z^2_{1r_{alj}}\right|, \left|z^1_{2s_{alj}} - z^2_{2s_{alj}}\right|, \left|z^1_{3t_{alj}} - z^2_{3t_{alj}}\right| : l \in \Lambda_k\right\}\right) \\
&\geq \left|L_{a1} - L_{a2}\right| - \frac{1}{\max\{T_1, T_2\}} \\
&\quad \times \sum_{q=1}^{\infty}\sum_{i=\max\{T_1,T_2\}+q\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}P_{aj}\max\left\{\frac{\left|z^1_{1r_{alj}} - z^2_{1r_{alj}}\right|}{r_{alj}}, \frac{\left|z^1_{2s_{alj}} - z^2_{2s_{alj}}\right|}{s_{alj}}, \frac{\left|z^1_{3t_{alj}} - z^2_{3t_{alj}}\right|}{t_{alj}} : l \in \Lambda_k\right\}\right) \\
&\geq \left|L_{a1} - L_{a2}\right| - \left(\frac{1}{\max\{T_1, T_2\}}\sum_{q=1}^{\infty}\sum_{i=\max\{T_1,T_2\}+q\tau_a}^{\infty}\sum_{j=i}^{\infty}U_{aj}P_{aj}\right)\max\left\{\left\|z_1^1 - z_1^2\right\|, \left\|z_2^1 - z_2^2\right\|, \left\|z_3^1 - z_3^2\right\|\right\} \\
&\geq \left|L_{a1} - L_{a2}\right| - \max\left\{\theta^1, \theta^2\right\}\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1, \quad (n,a) \in \mathbb{N}_{\max\{T_1,T_2\}} \times \Lambda_3,
\end{aligned} \tag{41}$$

which means that

$$\begin{aligned}
&\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1 \\
&\quad = \max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\
&\quad \geq \max\left\{\sup_{n\in\mathbb{N}_{\max\{T_1,T_2\}}}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\
&\quad \geq \max\left\{\left|L_{a1} - L_{a2}\right| - \max\left\{\theta^1, \theta^2\right\}\right. \\
&\qquad\quad \left. \times \left\|(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)\right\|_1 : a \in \Lambda_3\right\} \\
&\quad = \max\left\{\left|L_{a1} - L_{a2}\right| : a \in \Lambda_3\right\} \\
&\qquad - \max\left\{\theta^1, \theta^2\right\}\left\|(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)\right\|_1,
\end{aligned} \tag{42}$$

which gives that

$$\begin{aligned}
\left\|(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)\right\|_1 &\geq \frac{\max\left\{\left|L_{a1} - L_{a2}\right| : a \in \Lambda_3\right\}}{1 + \max\left\{\theta^1, \theta^2\right\}} \\
&> 0;
\end{aligned} \tag{43}$$

that is, $(z_1^1, z_2^1, z_3^1) \neq (z_1^2, z_2^2, z_3^2)$. Hence the system (6) possesses uncountably many positive solutions in $\Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$. This completes the proof. □

Theorem 3. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, d_a, $D_a \in \mathbb{R}^+ \setminus \{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$, and $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ satisfying (17), (18), and (22) as follows:*

$$d_a > D_a, \quad b_{an} = 1, \quad (n,a) \in \mathbb{N}_{n_1} \times \Lambda_3, \tag{44}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=n+\tau_a}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj}, \left|c_{aj}\right|, U_{aj}P_{aj}\right\} = 0, \quad a \in \Lambda_3. \tag{45}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in \Pi_{w=1}^3(d_w - D_w, d_w + D_w)$, there exist $\theta \in (0,1)$ and $T \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ such that, for each $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$, the Mann iterative sequence $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ generated by the scheme

$$x_{an}^{m+1} = \begin{cases} (1-\alpha_m) x_{an}^m + \alpha_m \left\{ nL_a - \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, x_{1r_{a1j}}^m, \ldots, x_{1r_{akj}}^m, x_{2s_{a1j}}^m, \ldots, x_{2s_{akj}}^m, x_{3t_{a1j}}^m, \ldots, x_{3t_{akj}}^m\right) - c_{aj} \right] \right\}, \\ \qquad n \geq T, \quad (m,a) \in \mathbb{N}_0 \times \Lambda_3, \\ (1-\alpha_m) x_{an}^m + \alpha_m \left\{ nL_a - \frac{n}{T} \sum_{q=1}^{\infty} \sum_{i=T+(2q-1)\tau_a}^{T+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, x_{1r_{a1j}}^m, \ldots, x_{1r_{akj}}^m, x_{2s_{a1j}}^m, \ldots, x_{2s_{akj}}^m, x_{3t_{a1j}}^m, \ldots, x_{3t_{akj}}^m\right) - c_{aj} \right] \right\}, \\ \qquad \beta \leq n < T, \quad (m,a) \in \mathbb{N}_0 \times \Lambda_3 \end{cases} \tag{46}$$

converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n \in \mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ *of the system* (6) *with* $\lim_{n\to\infty}(z_{an}/n) = L_a$ *for each* $a \in \Lambda_3$ *and has the error estimate* (21).

(b) The system (6) *has uncountably many positive solutions in* $\Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$.

Proof. Firstly, we prove that (a) holds. Set $(L_1, L_2, L_3) \in \Pi_{w=1}^3 (d_w - D_w, d_w + D_w)$. It follows from (45) that there exist $\theta \in (0,1)$ and $T \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ satisfying

$$\theta = \max\{\theta_a : a \in \Lambda_3\}, \tag{47}$$

where

$$\theta_a = \frac{1}{T} \sum_{i=T+\tau_a}^{\infty} \sum_{j=i}^{\infty} U_{aj} P_{aj}, \quad a \in \Lambda_3, \tag{48}$$

$$\frac{1}{T} \sum_{i=T+\tau_a}^{\infty} \sum_{j=i}^{\infty} \left(F_{aj} + |c_{aj}|\right) < D_a - |L_a - d_a|, \quad a \in \Lambda_3. \tag{49}$$

Define mappings $A_{L_1}, A_{L_2}, A_{L_3} : \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w) \to l_\beta^\infty$ and $S_{L_1,L_2,L_3} : \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w) \to (l_\beta^\infty)^3$ by (27) and

$$A_{L_a}(x_{1n}, x_{2n}, x_{3n}) = \begin{cases} nL_a - \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, x_{1r_{a1j}}, \ldots, x_{1r_{akj}}, x_{2s_{a1j}}, \ldots, x_{2s_{akj}}, x_{3t_{a1j}}, \ldots, x_{3t_{akj}}\right) - c_{aj} \right], \\ \qquad n \geq T, \quad a \in \Lambda_3, \\ \frac{n}{T} A_{L_a}(x_{1T}, x_{2T}, x_{3T}), \quad \beta \leq n < T, \ a \in \Lambda_3, \end{cases} \tag{50}$$

for each $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n \in \mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$. It follows from (18), (49), and (50) that, for any $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n \in \mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ and $a \in \Lambda_3$,

$$\begin{aligned} &\left\| A_{L_a}(x_1, x_2, x_3) - \widehat{d}_a \right\| \\ &= \sup_{n \in \mathbb{N}_\beta} \left| \frac{A_{L_a}(x_{1n}, x_{2n}, x_{3n}) - nd_a}{n} \right| \\ &= \max\left\{ \sup_{n \geq T} \left| \frac{A_{L_a}(x_{1n}, x_{2n}, x_{3n})}{n} - d_a \right|, \right. \\ &\qquad \left. \sup_{\beta \leq n < T} \left| \frac{n}{T} \frac{A_{L_a}(x_{1T}, x_{2T}, x_{3T})}{n} - d_a \right| \right\} \\ &= \sup_{n \geq T} \left| L_a - d_a - \frac{1}{n} \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, x_{1r_{a1j}}, \ldots, x_{1r_{akj}}, x_{2s_{a1j}}, \ldots, x_{2s_{akj}}, x_{3t_{a1j}}, \ldots, x_{3t_{akj}}\right) - c_{aj} \right] \right| \\ &\leq \sup_{n \geq T} \left(|L_a - d_a| + \frac{1}{n} \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left(F_{aj} + |c_{aj}|\right) \right) \\ &\leq |L_a - d_a| + \frac{1}{T} \sum_{i=T+\tau_a}^{\infty} \sum_{j=i}^{\infty} \left(F_{aj} + |c_{aj}|\right) \\ &\leq |L_a - d_a| + D_a - |L_a - d_a| = D_a, \end{aligned} \tag{51}$$

which means (28). It is easy to see that (27) and (28) yield (29). By virtue of (17), (47), and (50), we know that for any $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n\in\mathbb{N}_\beta}, (\overline{x}_1, \overline{x}_2, \overline{x}_3) = \{(\overline{x}_{1n}, \overline{x}_{2n}, \overline{x}_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$, and $a \in \Lambda_3$,

$$\begin{aligned}
&\left\| A_{L_a}(x_1,x_2,x_3) - A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3) \right\| \\
&\quad = \sup_{n\in\mathbb{N}_\beta} \left| \frac{A_{L_a}(x_{1n},x_{2n},x_{3n}) - A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n} \right| \\
&\quad = \max\left\{ \sup_{n\ge T} \left| \frac{A_{L_a}(x_{1n},x_{2n},x_{3n}) - A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n} \right|, \sup_{\beta\le n<T} \left| \frac{n}{T}\cdot\frac{A_{L_a}(x_{1T},x_{2T},x_{3T}) - A_{L_a}(\overline{x}_{1T},\overline{x}_{2T},\overline{x}_{3T})}{n} \right| \right\} \\
&\quad = \sup_{n\ge T}\frac{1}{n} \left| \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, x_{1r_{a1j}},\dots,x_{1r_{akj}},x_{2s_{a1j}},\dots,x_{2s_{akj}},x_{3t_{a1j}},\dots,x_{3t_{akj}}\right) \right.\right. \\
&\qquad\qquad \left.\left. - f_a\left(j, \overline{x}_{1r_{a1j}},\dots,\overline{x}_{1r_{akj}},\overline{x}_{2s_{a1j}},\dots,\overline{x}_{2s_{akj}},\overline{x}_{3t_{a1j}},\dots,\overline{x}_{3t_{akj}}\right) \right] \right| \\
&\quad \le \sup_{n\ge T}\frac{1}{n} \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left(U_{aj} \max\left\{ \left|x_{1r_{alj}} - \overline{x}_{1r_{alj}}\right|, \left|x_{2s_{alj}} - \overline{x}_{2s_{alj}}\right|, \left|x_{3t_{alj}} - \overline{x}_{3t_{alj}}\right| : l \in \Lambda_k \right\} \right) \\
&\quad \le \frac{1}{T} \sum_{i=T+\tau_a}^{\infty} \sum_{j=i}^{\infty} \left(U_{aj} P_{aj} \max\left\{ \frac{\left|x_{1r_{alj}} - \overline{x}_{1r_{alj}}\right|}{r_{alj}}, \frac{\left|x_{2s_{alj}} - \overline{x}_{2s_{alj}}\right|}{s_{alj}}, \frac{\left|x_{3t_{alj}} - \overline{x}_{3t_{alj}}\right|}{t_{alj}} : l \in \Lambda_k \right\} \right) \\
&\quad \le \theta_a \max\left\{ \|x_w - \overline{x}_w\| : w \in \Lambda_3 \right\} \\
&\quad = \theta_a \left\| (x_1,x_2,x_3) - (\overline{x}_1,\overline{x}_2,\overline{x}_3) \right\|_1, \\
&\left\| S_{L_1,L_2,L_3}(x_1,x_2,x_3) - S_{L_1,L_2,L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3) \right\|_1 \\
&\quad = \left\| \left(A_{L_1}(x_1,x_2,x_3), A_{L_2}(x_1,x_2,x_3), A_{L_3}(x_1,x_2,x_3) \right) - \left(A_{L_1}(\overline{x}_1,\overline{x}_2,\overline{x}_3), A_{L_2}(\overline{x}_1,\overline{x}_2,\overline{x}_3), A_{L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3) \right) \right\|_1 \\
&\quad = \max\left\{ \left\| A_{L_a}(x_1,x_2,x_3) - A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3) \right\| : a \in \Lambda_3 \right\} \\
&\quad \le \theta \left\| (x_1,x_2,x_3) - (\overline{x}_1,\overline{x}_2,\overline{x}_3) \right\|_1,
\end{aligned} \tag{52}$$

which yield (31) and (32). Thus (29) and (32) guarantee that S_{L_1,L_2,L_3} is a contraction mapping in $\Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ and it has a unique fixed point $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$, that is,

$$\begin{aligned}
z_{an} &= nL_a - \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, z_{1r_{a1j}},\dots,z_{1r_{akj}},z_{2s_{a1j}},\dots,\right.\right. \\
&\qquad \left.\left. z_{2s_{akj}},z_{3t_{a1j}},\dots,z_{3t_{akj}}\right) - c_{aj} \right], \\
&\qquad (n,a) \in \mathbb{N}_T \times \Lambda_3, \\
z_{a(n-\tau_a)} &= (n-\tau_a) L_a - \sum_{q=1}^{\infty} \sum_{i=n+2(q-1)\tau_a}^{n+(2q-1)\tau_a-1} \sum_{j=i}^{\infty} \left[f_a\left(j, z_{1r_{a1j}},\dots,z_{1r_{akj}},z_{2s_{a1j}},\dots,\right.\right. \\
&\qquad \left.\left. z_{2s_{akj}},z_{3t_{a1j}},\dots,z_{3t_{akj}}\right) - c_{aj} \right], \\
&\qquad (n,a) \in \mathbb{N}_{T+\tau_a} \times \Lambda_3,
\end{aligned} \tag{53}$$

which mean that

$$\begin{aligned}
\Delta\left(z_{an} + z_{a(n-\tau_a)}\right) &= 2L_a + \sum_{j=n}^{\infty} \left[f_a\left(j, z_{1r_{a1j}},\dots,z_{1r_{akj}},z_{2s_{a1j}},\dots,z_{2s_{akj}},\right.\right. \\
&\qquad \left.\left. z_{3t_{a1j}},\dots,z_{3t_{akj}}\right) - c_{aj} \right], \\
&\qquad (n,a) \in \mathbb{N}_{T+\tau_1+\tau_2+\tau_3} \times \Lambda_3,
\end{aligned} \tag{54}$$

which implies that

$$\begin{aligned}
\Delta^2\left(z_{an} + z_{a(n-\tau_a)}\right) &= -f_a\left(n, z_{1r_{a1n}},\dots,z_{1r_{akn}},z_{2s_{a1n}},\dots,z_{2s_{akn}},z_{3t_{a1n}},\dots,z_{3t_{akn}}\right) \\
&\quad + c_{an}, \quad (n,a) \in \mathbb{N}_{T+\tau_1+\tau_2+\tau_3} \times \Lambda_3,
\end{aligned} \tag{55}$$

which together with (44) ensures that $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta}$ is a positive solution of the system (6) in $\Pi_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$. Obviously, (18) and (45) yield that

$$\begin{aligned}
\left|\frac{z_{an}}{n} - L_a\right|
&= \frac{1}{n}\left|\sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left[f_a\left(j, z_{1r_{a1j}}, \ldots, z_{1r_{akj}}, z_{2s_{a1j}}, \ldots, z_{2s_{akj}}, z_{3t_{a1j}}, \ldots, z_{3t_{akj}}\right) - c_{aj}\right]\right| \\
&\le \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right) \\
&\le \frac{1}{n}\sum_{i=n+\tau_a}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + \left|c_{aj}\right|\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{56}$$

which means that

$$\lim_{n\to\infty}\frac{z_{an}}{n} = L_a, \quad a \in \Lambda_3. \tag{57}$$

In light of (31), (46), and (50), we get that

$$\begin{aligned}
\frac{\left|x_{an}^{m+1} - z_{an}\right|}{n}
&= \left|(1-\alpha_m)\frac{x_{an}^{m} - z_{an}}{n} + \frac{\alpha_m}{n}\left(nL_a + \sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}}^{m}, \ldots, x_{1r_{akj}}^{m}, x_{2s_{a1j}}^{m}, \ldots, x_{2s_{akj}}^{m}, x_{3t_{a1j}}^{m}, \ldots, x_{3t_{akj}}^{m}\right) - c_{aj}\right] - z_{an}\right)\right| \\
&\le (1-\alpha_m)\left\|x_a^m - z_a\right\| + \alpha_m \cdot \frac{1}{n}\left|A_{L_a}\left(x_{1n}^m, x_{2n}^m, x_{3n}^m\right) - A_{L_a}\left(z_{1n}, z_{2n}, z_{3n}\right)\right| \\
&\le (1-\alpha_m)\left\|x_a^m - z_a\right\| + \alpha_m\left\|A_{L_a}\left(x_1^m, x_2^m, x_3^m\right) - A_{L_a}\left(z_1, z_2, z_3\right)\right\| \\
&\le \left(1 - (1-\theta_a)\alpha_m\right)\left\|x_a^m - z_a\right\|, \quad n \ge T, \quad (m, a) \in \mathbb{N}_0 \times \Lambda_3,
\end{aligned}$$

$$\begin{aligned}
\frac{\left|x_{an}^{m+1} - z_{an}\right|}{n}
&= \left|(1-\alpha_m)\frac{x_{an}^{m} - z_{an}}{n} + \frac{\alpha_m}{n}\left(nL_a + \frac{n}{T}\sum_{q=1}^{\infty}\sum_{i=T+(2q-1)\tau_a}^{T+2q\tau_a-1}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}}^{m}, \ldots, x_{1r_{akj}}^{m}, x_{2s_{a1j}}^{m}, \ldots, x_{2s_{akj}}^{m}, x_{3t_{a1j}}^{m}, \ldots, x_{3t_{akj}}^{m}\right) - c_{aj}\right] - z_{an}\right)\right| \\
&\le (1-\alpha_m)\left\|x_a^m - z_a\right\| + \frac{\alpha_m}{n}\cdot\frac{n}{T} \times \left|TL_a + \sum_{q=1}^{\infty}\sum_{i=T+(2q-1)\tau_a}^{T+2q\tau_a-1}\sum_{j=i}^{\infty}\left[f_a\left(j, x_{1r_{a1j}}^{m}, \ldots, x_{1r_{akj}}^{m}, x_{2s_{a1j}}^{m}, \ldots, x_{2s_{akj}}^{m}, x_{3t_{a1j}}^{m}, \ldots, x_{3t_{akj}}^{m}\right) - c_{aj}\right] - A_{L_a}\left(z_{1T}, z_{2T}, z_{3T}\right)\right| \\
&\le (1-\alpha_m)\left\|x_a^m - z_a\right\| + \alpha_m \cdot \frac{1}{T}\left|A_{L_a}\left(x_{1T}^m, x_{2T}^m, x_{3T}^m\right) - A_{L_a}\left(z_{1T}, z_{2T}, z_{3T}\right)\right| \\
&\le (1-\alpha_m)\left\|x_a^m - z_a\right\| + \alpha_m\left\|A_{L_a}\left(x_1^m, x_2^m, x_3^m\right) - A_{L_a}\left(z_1, z_2, z_3\right)\right\| \\
&\le \left(1 - (1-\theta_a)\alpha_m\right)\left\|x_a^m - z_a\right\|, \quad \beta \le n < T, \quad (m, a) \in \mathbb{N}_0 \times \Lambda_3,
\end{aligned} \tag{58}$$

which yield that

$$\begin{aligned}
&\left\|\left(x_1^{m+1}, x_2^{m+1}, x_3^{m+1}\right) - \left(z_1, z_2, z_3\right)\right\|_1 \\
&= \max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{x_{an}^{m+1} - z_{an}}{n}\right| : a \in \Lambda_3\right\} \\
&= \max\left\{\sup_{n\ge T}\left|\frac{x_{an}^{m+1} - z_{an}}{n}\right|, \sup_{\beta\le n<T}\left|\frac{x_{an}^{m+1} - z_{an}}{n}\right| : a \in \Lambda_3\right\} \\
&\le \max\left\{\left(1 - (1-\theta_a)\alpha_m\right)\left\|x_a^m - z_a\right\| : a \in \Lambda_3\right\} \\
&\le \left(1 - (1-\theta)\alpha_m\right)\left\|\left(x_1^m, x_2^m, x_3^m\right) - \left(z_1, z_2, z_3\right)\right\|_1 \\
&\le e^{-(1-\theta)\alpha_m}\left\|\left(x_1^m, x_2^m, x_3^m\right) - \left(z_1, z_2, z_3\right)\right\|_1 \\
&\le e^{-(1-\theta)\sum_{n=0}^{m}\alpha_n}\left\|\left(x_1^0, x_2^0, x_3^0\right) - \left(z_1, z_2, z_3\right)\right\|_1, \quad m \in \mathbb{N}_0;
\end{aligned} \tag{59}$$

that is, (21) holds. It follows from (21) and (22) that $\lim_{m\to\infty}\|(x_1^m, x_2^m, x_3^m) - (z_1, z_2, z_3)\|_1 = 0$.

Secondly, we show that (b) holds. Put (L_{11}, L_{21}, L_{31}), $(L_{12}, L_{22}, L_{32}) \in \Pi_{w=1}^3(d_w - D_w, d_w + D_w)$ with $\max\{|L_{a1} - L_{a2}| : a \in \Lambda_3\} > 0$. Similar to the proof of (a), we conclude that, for each $l \in \Lambda_2$, there exist constants θ^l, θ_1^l, θ_2^l, $\theta_3^l \in (0,1)$, $T_l \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ and mappings $A_{L_{1l}}, A_{L_{2l}}, A_{L_{3l}} : \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w) \to l_\beta^\infty$ and $S_{L_{1l},L_{2l},L_{3l}} : \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w) \to (l_\beta^\infty)^3$ satisfying (27)~(32) and (47)~(50), where θ, θ_1, θ_2, θ_3, T, L_1, L_2, L_3, A_{L_1}, A_{L_2}, A_{L_3}, and S_{L_1,L_2,L_3} are replaced by θ^l, θ_1^l, θ_2^l, θ_3^l, T_l, L_{1l}, L_{2l}, L_{3l}, $A_{L_{1l}}$, $A_{L_{2l}}$, $A_{L_{3l}}$, and $S_{L_{1l},L_{2l},L_{3l}}$, respectively, and the contraction mapping $S_{L_{1l},L_{2l},L_{3l}}$ has a unique fixed point $(z_1^l, z_2^l, z_3^l) = \{(z_{1n}^l, z_{2n}^l, z_{3n}^l)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$, which is a positive solution of the system (6); that is,

$$z_{an}^l = nL_{al} - \sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left[f_a\left(j, z_{1r_{a1j}}^l, \ldots, z_{1r_{akj}}^l, z_{2s_{a1j}}^l, \ldots, z_{2s_{akj}}^l, z_{3t_{a1j}}^l, \ldots, z_{3t_{akj}}^l\right) - c_{aj}\right],$$
$$(n,l,a) \in \mathbb{N}_{T_l} \times \Lambda_2 \times \Lambda_3. \tag{60}$$

Using (17), (47), and (60), we infer that, for each $n \geq \max\{T_1, T_2\}$ and $a \in \Lambda_3$,

$$\begin{aligned}
\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| &= \Bigg|L_{a1} - L_{a2} - \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\Big[f_a\left(j, z_{1r_{a1j}}^1, \ldots, z_{1r_{akj}}^1, z_{2s_{a1j}}^1, \ldots, z_{2s_{akj}}^1, z_{3t_{a1j}}^1, \ldots, z_{3t_{akj}}^1\right) \\
&\qquad - f_a\left(j, z_{1r_{a1j}}^2, \ldots, z_{1r_{akj}}^2, z_{2s_{a1j}}^2, \ldots, z_{2s_{akj}}^2, z_{3t_{a1j}}^2, \ldots, z_{3t_{akj}}^2\right)\Big]\Bigg| \\
&\geq |L_{a1} - L_{a2}| - \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left(U_{aj}\max\left\{\left|z_{1r_{alj}}^1 - z_{1r_{alj}}^2\right|, \left|z_{2s_{alj}}^1 - z_{2s_{alj}}^2\right|, \left|z_{3t_{alj}}^1 - z_{3t_{alj}}^2\right| : l \in \Lambda_k\right\}\right) \\
&\geq |L_{a1} - L_{a2}| - \frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+(2q-1)\tau_a}^{n+2q\tau_a-1}\sum_{j=i}^{\infty}\left(U_{aj}P_{aj}\max\left\{\frac{\left|z_{1r_{alj}}^1 - z_{1r_{alj}}^2\right|}{r_{alj}}, \frac{\left|z_{2s_{alj}}^1 - z_{2s_{alj}}^2\right|}{s_{alj}}, \frac{\left|z_{3t_{alj}}^1 - z_{3t_{alj}}^2\right|}{t_{alj}} : l \in \Lambda_k\right\}\right) \\
&\geq |L_{a1} - L_{a2}| - \left(\frac{1}{\max\{T_1, T_2\}}\sum_{i=\max\{T_1,T_2\}+\tau_a}^{\infty}\sum_{j=i}^{\infty}U_{aj}P_{aj}\right)\max\left\{\left\|z_1^1 - z_1^2\right\|, \left\|z_2^1 - z_2^2\right\|, \left\|z_3^1 - z_3^2\right\|\right\} \\
&\geq |L_{a1} - L_{a2}| - \max\{\theta^1, \theta^2\}\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1,
\end{aligned} \tag{61}$$

which implies that

$$\begin{aligned}
&\|(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)\|_1 \\
&\quad = \max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\
&\quad \geq \max\left\{\sup_{n\in\mathbb{N}_{\max\{T_1,T_2\}}}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\
&\quad \geq \max\Big\{|L_{a1} - L_{a2}| - \max\{\theta^1, \theta^2\} \\
&\qquad \times \left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1 : \\
&\qquad a \in \Lambda_3\Big\} \\
&\quad = \max\{|L_{a1} - L_{a2}| : a \in \Lambda_3\} \\
&\qquad - \max\{\theta^1, \theta^2\}\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1,
\end{aligned} \tag{62}$$

which gives that

$$\|(z_1^1, z_2^1, z_3^1) - (z_1^2, z_2^2, z_3^2)\|_1 \geq \frac{\max\{|L_{a1} - L_{a2}| : a \in \Lambda_3\}}{1 + \max\{\theta^1, \theta^2\}} > 0; \tag{63}$$

that is, $(z_1^1, z_2^1, z_3^1) \neq (z_1^2, z_2^2, z_3^2)$. Hence the system (6) possesses uncountably many positive solutions in $\Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$. This completes the proof. □

Theorem 4. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, d_a, $D_a, b_a \in \mathbb{R}^+\backslash\{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$,*

$\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (17), (18), *and* (22) *as follows:*

$$1<\frac{d_a}{D_a}<\frac{2-b_a}{b_a},\quad 0\le b_{an}\le b_a<1,\quad (n,a)\in\mathbb{N}_{n_1}\times\Lambda_3,\tag{64}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},\left|c_{aj}\right|,U_{aj}P_{aj}\right\}=0,\quad a\in\Lambda_3.\tag{65}$$

Then one has the following.

(a) For any $(L_1,L_2,L_3)\in\Pi_{w=1}^3(b_w(d_w+D_w)+d_w-D_w,d_w+D_w)$*, there exist* $\theta\in(0,1)$ *and* $T\ge 1+n_0+n_1+\tau_1+\tau_2+\tau_3+\beta$ *such that, for each* $(x_1^0,x_2^0,x_3^0)=\{(x_{1n}^0,x_{2n}^0,x_{3n}^0)\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$*, the Mann iterative sequence* $\{(x_1^m,x_2^m,x_3^m)\}_{m\in\mathbb{N}_0}=\{\{(x_{1n}^m,x_{2n}^m,x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}\subset\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$ *generated by the scheme*

$$x_{an}^{m+1}=\begin{cases}(1-\alpha_m)x_{an}^m+\alpha_m\left\{nL_a-b_{an}x_{a(n-\tau_a)}^m-\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}}^m,\ldots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\ldots,x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\ldots,x_{3t_{akj}}^m\right)-c_{aj}\right]\right\}, & n\ge T,\quad (m,a)\in\mathbb{N}_0\times\Lambda_3,\\ (1-\alpha_m)x_{an}^m+\alpha_m\left\{nL_a-\frac{n}{T}b_{aT}x_{a(T-\tau_a)}^m-\frac{n}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}}^m,\ldots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\ldots,x_{2s_{akj}}^m,x_{3t_{a1j}}^m,\ldots,x_{3t_{akj}}^m\right)-c_{aj}\right]\right\}, & \beta\le n<T,\quad (m,a)\in\mathbb{N}_0\times\Lambda_3\end{cases}\tag{66}$$

converges to a positive solution $(z_1,z_2,z_3)=\{(z_{1n},z_{2n},z_{3n})\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$ *of the system* (6) *with* $\lim_{n\to\infty}(z_{an}+b_{an}z_{a(n-\tau_a)})/n=L_a$ *for each* $a\in\Lambda_3$ *and has the error estimate* (21).

(b) The system (6) *has uncountably many positive solutions in* $\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$.

Proof. Firstly, we show that (a) holds. Let $(L_1,L_2,L_3)\in\Pi_{w=1}^3(b_w(d_w+D_w)+d_w-D_w,d_w+D_w)$. Observe that (65) implies that there exist $\theta\in(0,1)$ and $T\ge 1+n_0+n_1+\tau_1+\tau_2+\tau_3+\beta$ satisfying

$$\theta=\max\left\{\theta_a:a\in\Lambda_3\right\},\tag{67}$$

where

$$\theta_a=\left|b_a\right|+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}U_{aj}P_{aj},\quad a\in\Lambda_3,\tag{68}$$

$$\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj}+\left|c_{aj}\right|\right)<\min\left\{d_a+D_a-L_a,L_a-b_a\left(d_a+D_a\right)-\left(d_a-D_a\right)\right\},\quad a\in\Lambda_3.\tag{69}$$

Define mappings $A_{L_1},A_{L_2},A_{L_3}:\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)\to l_\beta^\infty$ and $S_{L_1,L_2,L_3}:\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)\to(l_\beta^\infty)^3$ by (27) and

$$A_{L_a}\left(x_{1n},x_{2n},x_{3n}\right)=\begin{cases}nL_a-b_{an}x_{a(n-\tau_a)}-\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}},\ldots,x_{1r_{akj}},x_{2s_{a1j}},\ldots,x_{2s_{akj}},x_{3t_{a1j}},\ldots,x_{3t_{akj}}\right)-c_{aj}\right], & n\ge T,\quad a\in\Lambda_3,\\ \frac{n}{T}A_{L_a}\left(x_{1T},x_{2T},x_{3T}\right), & \beta\le n<T,\ a\in\Lambda_3,\end{cases}\tag{70}$$

for each $(x_1,x_2,x_3)=\{(x_{1n},x_{2n},x_{3n})\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$. It follows from (18), (69), and (70) that, for any $(x_1,x_2,x_3)=\{(x_{1n},x_{2n},x_{3n})\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^3\Omega(\widehat{d}_w,D_w)$,

$$\begin{aligned}\frac{1}{n}A_{L_a}\left(x_{1n},x_{2n},x_{3n}\right)&=L_a-\frac{b_{an}x_{a(n-\tau_a)}}{n}\\&\quad-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}},\ldots,x_{1r_{akj}},x_{2s_{a1j}},\ldots,x_{2s_{akj}},x_{3t_{a1j}},\ldots,x_{3t_{akj}}\right)-c_{aj}\right]\\&\le L_a+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj}+\left|c_{aj}\right|\right)\\&<L_a+\min\left\{d_a+D_a-L_a,L_a-b_a\left(d_a+D_a\right)-\left(d_a-D_a\right)\right\}\\&\le d_a+D_a,\quad (n,a)\in\mathbb{N}_T\times\Lambda_3,\end{aligned}$$

$$\begin{aligned}\frac{1}{n}A_{L_a}\left(x_{1n},x_{2n},x_{3n}\right)&=\frac{1}{n}\cdot\frac{n}{T}A_{L_a}\left(x_{1T},x_{2T},x_{3T}\right)\\&=\frac{1}{T}A_{L_a}\left(x_{1T},x_{2T},x_{3T}\right)<d_a+D_a,\\&\qquad\beta\le n<T,\quad a\in\Lambda_3,\end{aligned}$$

$$\begin{aligned}
&\frac{1}{n}A_{L_a}(x_{1n},x_{2n},x_{3n})\\
&\quad= L_a-\frac{b_{an}x_{a(n-\tau_a)}}{n}\\
&\qquad-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,x_{1r_{a1j}},\dots,x_{1r_{akj}},x_{2s_{a1j}},\dots,x_{2s_{akj}},\\
&\qquad\qquad x_{3t_{a1j}},\dots,x_{3t_{akj}}\Big)-c_{aj}\Big]\\
&\quad\ge L_a-b_a(d_a+D_a)-\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj}+|c_{aj}|\right)\\
&\quad> L_a-b_a(d_a+D_a)\\
&\qquad-\min\{d_a+D_a-L_a,L_a-b_a(d_a+D_a)-(d_a-D_a)\}\\
&\quad\ge d_a-D_a,\quad (n,a)\in\mathbb{N}_T\times\Lambda_3,\\
&\frac{1}{n}A_{L_a}(x_{1n},x_{2n},x_{3n})=\frac{1}{n}\cdot\frac{n}{T}A_{L_a}(x_{1T},x_{2T},x_{3T})\\
&\qquad=\frac{1}{T}A_{L_a}(x_{1T},x_{2T},x_{3T})>d_a-D_a,\\
&\qquad\beta\le n<T,\quad a\in\Lambda_3,
\end{aligned}\tag{71}$$

which mean (28). Consequently, (29) follows from (27) and (28). By virtue of (17), (67), and (70), we infer that, for any $(x_1,x_2,x_3)=\{(x_{1n},x_{2n},x_{3n})\}_{n\in\mathbb{N}_\beta}$, $(\overline{x}_1,\overline{x}_2,\overline{x}_3)=\{(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$, and $a\in\Lambda_3$,

$$\begin{aligned}
&\left\|A_{L_a}(x_1,x_2,x_3)-A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|\\
&=\sup_{n\in\mathbb{N}_\beta}\left|\frac{A_{L_a}(x_{1n},x_{2n},x_{3n})-A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n}\right|\\
&=\max\left\{\sup_{n\ge T}\left|\frac{A_{L_a}(x_{1n},x_{2n},x_{3n})-A_{L_a}(\overline{x}_{1n},\overline{x}_{2n},\overline{x}_{3n})}{n}\right|,\right.\\
&\qquad\left.\sup_{\beta\le n<T}\left|\frac{n}{T}\cdot\frac{A_{L_a}(x_{1T},x_{2T},x_{3T})-A_{L_a}(\overline{x}_{1T},\overline{x}_{2T},\overline{x}_{3T})}{n}\right|\right\}\\
&=\sup_{n\in\mathbb{N}_\beta}\left(\frac{1}{n}\left|b_{an}\left(x_{a(n-\tau_a)}-\overline{x}_{a(n-\tau_a)}\right)\right.\right.\\
&\qquad+\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,x_{1r_{a1j}},\dots,x_{1r_{akj}},x_{2s_{a1j}},\dots,x_{2s_{akj}},\\
&\qquad\qquad x_{3t_{a1j}},\dots,x_{3t_{akj}}\Big)\\
&\qquad-f_a\Big(j,\overline{x}_{1r_{a1j}},\dots,\overline{x}_{1r_{akj}},\overline{x}_{2s_{a1j}},\dots,\\
&\qquad\qquad\left.\left.\overline{x}_{2s_{akj}},\overline{x}_{3t_{a1j}},\dots,\overline{x}_{3t_{akj}}\Big)\Big]\right|\right)\\
&\le b_a\|x_a-\overline{x}_a\|\\
&\quad+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\Big(U_{aj}\max\Big\{\left|x_{1r_{alj}}-\overline{x}_{1r_{alj}}\right|,\left|x_{2s_{alj}}-\overline{x}_{2s_{alj}}\right|,\\
&\qquad\left|x_{3t_{alj}}-\overline{x}_{3t_{alj}}\right|:l\in\Lambda_k\Big\}\Big)\\
&\le b_a\|x_a-\overline{x}_a\|\\
&\quad+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}P_{aj}\max\left\{\frac{\left|x_{1r_{alj}}-\overline{x}_{1r_{alj}}\right|}{r_{alj}},\frac{\left|x_{2s_{alj}}-\overline{x}_{2s_{alj}}\right|}{s_{alj}},\right.\right.\\
&\qquad\left.\left.\frac{\left|x_{3t_{alj}}-\overline{x}_{3t_{alj}}\right|}{t_{alj}}:l\in\Lambda_k\right\}\right)\\
&\le\theta_a\max\{\|x_w-\overline{x}_w\|:w\in\Lambda_3\}\\
&=\theta_a\left\|(x_1,x_2,x_3)-(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1,\\
&\left\|S_{L_1,L_2,L_3}(x_1,x_2,x_3)-S_{L_1,L_2,L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1\\
&=\Big\|\left(A_{L_1}(x_1,x_2,x_3),A_{L_2}(x_1,x_2,x_3),A_{L_3}(x_1,x_2,x_3)\right)\\
&\qquad-\left(A_{L_1}(\overline{x}_1,\overline{x}_2,\overline{x}_3),A_{L_2}(\overline{x}_1,\overline{x}_2,\overline{x}_3),A_{L_3}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right)\Big\|_1\\
&=\max\left\{\left\|A_{L_a}(x_1,x_2,x_3)-A_{L_a}(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|:a\in\Lambda_3\right\}\\
&\le\theta\left\|(x_1,x_2,x_3)-(\overline{x}_1,\overline{x}_2,\overline{x}_3)\right\|_1,
\end{aligned}\tag{72}$$

which yield (31) and (32). Thus (29) and (32) ensure that S_{L_1,L_2,L_3} is a contraction mapping in $\Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$ and it has a unique fixed point $(z_1,z_2,z_3)=\{(z_{1n},z_{2n},z_{3n})\}_{n\in\mathbb{N}_\beta}\in\Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$, that is,

$$\begin{aligned}
z_{an}&=nL_a-b_{an}z_{a(n-\tau_a)}\\
&\quad-\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j,z_{1r_{a1j}},\dots,z_{1r_{akj}},z_{2s_{a1j}},\dots,z_{2s_{akj}},\\
&\qquad z_{3t_{a1j}},\dots,z_{3t_{akj}}\Big)-c_{aj}\Big],\\
&\qquad(n,a)\in\mathbb{N}_T\times\Lambda_3,
\end{aligned}\tag{73}$$

which yields that

$$\begin{aligned}
&\Delta\left(z_{an}+b_{an}z_{a(n-\tau_a)}\right)\\
&\quad=L_a+\sum_{j=n}^{\infty}\Big[f_a\Big(j,z_{1r_{a1j}},\dots,z_{1r_{akj}},z_{2s_{a1j}},\dots,z_{2s_{akj}},\\
&\qquad z_{3t_{a1j}},\dots,z_{3t_{akj}}\Big)-c_{aj}\Big],\\
&\qquad(n,a)\in\mathbb{N}_{T+\tau_1+\tau_2+\tau_3}\times\Lambda_3,
\end{aligned}\tag{74}$$

which implies that

$$\begin{aligned}&\Delta^2\left(z_{an}+b_{an}z_{a(n-\tau_a)}\right)\\&=-f_a\left(n,z_{1r_{a1n}},\ldots,z_{1r_{akn}},z_{2s_{a1n}},\ldots,z_{2s_{akn}},z_{3t_{a1n}},\ldots,z_{3t_{akn}}\right)\\&\quad+c_{an},\quad (n,a)\in\mathbb{N}_{T+\tau_1+\tau_2+\tau_3}\times\Lambda_3,\end{aligned}\tag{75}$$

which together with (64) means that $(z_1,z_2,z_3) = \{(z_{1n},z_{2n},z_{3n})\}_{n\in\mathbb{N}_\beta}$ is a positive solution of the system (6) in $\Pi_{w=1}^{3}\Omega(\widehat{d}_w,D_w)$. Note that (18) and (65) give that, for each $a\in\Lambda_3$,

$$\begin{aligned}&\left|\frac{z_{an}+b_{an}z_{a(n-\tau_a)}}{n}-L_a\right|\\&=\frac{1}{n}\left|\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,z_{1r_{a1j}},\ldots,z_{1r_{akj}},z_{2s_{a1j}},\ldots,z_{2s_{akj}},\right.\right.\right.\\&\qquad\left.\left.\left.z_{3t_{a1j}},\ldots,z_{3t_{akj}}\right)-c_{aj}\right]\right|\\&\le\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj}+\left|c_{aj}\right|\right)\longrightarrow 0\quad\text{as } n\longrightarrow\infty,\end{aligned}\tag{76}$$

which means that

$$\lim_{n\to\infty}\frac{z_{an}+b_{an}z_{a(n-\tau_a)}}{n}=L_a,\quad a\in\Lambda_3.\tag{77}$$

Making use of (31), (66), and (70), we get that

$$\begin{aligned}&\frac{\left|x_{an}^{m+1}-z_{an}\right|}{n}\\&=\left|(1-\alpha_m)\frac{x_{an}^{m}-z_{an}}{n}\right.\\&\quad+\frac{\alpha_m}{n}\left(nL_a-b_{an}x_{a(n-\tau_a)}\right.\\&\qquad-\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}},\ldots,x_{1r_{akj}},x_{2s_{a1j}},\ldots,\right.\right.\\&\qquad\left.x_{2s_{akj}},x_{3t_{a1j}},\ldots,x_{3t_{akj}}\right)\\&\qquad\left.\left.\left.-c_{aj}\right]-z_{an}\right)\right|\\&\le(1-\alpha_m)\left\|x_a^m-z_a\right\|\\&\quad+\alpha_m\cdot\frac{1}{n}\left|A_{L_a}\left(x_{1n}^m,x_{2n}^m,x_{3n}^m\right)-A_{L_a}\left(z_{1n},z_{2n},z_{3n}\right)\right|\\&\le(1-\alpha_m)\left\|x_a^m-z_a\right\|\\&\quad+\alpha_m\left\|A_{L_a}\left(x_1^m,x_2^m,x_3^m\right)-A_{L_a}\left(z_1,z_2,z_3\right)\right\|\\&\le\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|,\quad n\ge T,\ (m,a)\in\mathbb{N}_0\times\Lambda_3,\\&\frac{\left|x_{an}^{m+1}-z_{an}\right|}{n}\\&=\left|(1-\alpha_m)\frac{x_{an}^{m}-z_{an}}{n}\right.\\&\quad+\frac{\alpha_m}{n}\left(nL_a-\frac{n}{T}b_{aT}x_{a(T-\tau_a)}\right.\\&\qquad-\frac{n}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}},\ldots,x_{1r_{akj}},x_{2s_{a1j}},\ldots,\right.\right.\\&\qquad\left.x_{2s_{akj}},x_{3t_{a1j}},\ldots,x_{3t_{akj}}\right)\\&\qquad\left.\left.\left.-c_{aj}\right]-z_{an}\right)\right|\\&\le(1-\alpha_m)\left\|x_a^m-z_a\right\|\\&\quad+\frac{\alpha_m}{n}\cdot\frac{n}{T}\left|TL_a-b_{aT}x_{a(T-\tau_a)}\right.\\&\qquad-\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left[f_a\left(j,x_{1r_{a1j}}^m,\ldots,x_{1r_{akj}}^m,x_{2s_{a1j}}^m,\ldots,\right.\right.\\&\qquad\left.\left.x_{2s_{akj}}^m,\ x_{3t_{a1j}}^m,\ldots,x_{3t_{akj}}^m\right)-c_{aj}\right]\\&\qquad\left.-\ A_{L_a}\left(z_{1T},z_{2T},z_{3T}\right)\right|\\&\le(1-\alpha_m)\left\|x_a^m-z_a\right\|\\&\quad+\alpha_m\cdot\frac{1}{T}\left|A_{L_a}\left(x_{1T}^m,x_{2T}^m,x_{3T}^m\right)-A_{L_a}\left(z_{1T},z_{2T},z_{3T}\right)\right|\\&\le(1-\alpha_m)\left\|x_a^m-z_a\right\|\\&\quad+\alpha_m\left\|A_{L_a}\left(x_1^m,x_2^m,x_3^m\right)-A_{L_a}\left(z_1,z_2,z_3\right)\right\|\\&\le\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|,\\&\qquad\beta\le n<T,\quad (m,a)\in\mathbb{N}_0\times\Lambda_3,\end{aligned}\tag{78}$$

which yield that

$$\begin{aligned}&\left\|\left(x_1^{m+1},x_2^{m+1},x_3^{m+1}\right)-\left(z_1,z_2,z_3\right)\right\|_1\\&=\max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|:a\in\Lambda_3\right\}\\&=\max\left\{\sup_{n\ge T}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|,\ \sup_{\beta\le n<T}\left|\frac{x_{an}^{m+1}-z_{an}}{n}\right|:a\in\Lambda_3\right\}\\&\le\max\left\{\left(1-\left(1-\theta_a\right)\alpha_m\right)\left\|x_a^m-z_a\right\|:a\in\Lambda_3\right\}\\&\le\left(1-(1-\theta)\alpha_m\right)\left\|\left(x_1^m,x_2^m,x_3^m\right)-\left(z_1,z_2,z_3\right)\right\|_1\end{aligned}$$

$$\leq e^{-(1-\theta)\alpha_m} \|(x_1^m, x_2^m, x_3^m) - (z_1, z_2, z_3)\|_1$$
$$\leq e^{-(1-\theta)\sum_{n=0}^{m}\alpha_n} \left\|\left(x_1^0, x_2^0, x_3^0\right) - (z_1, z_2, z_3)\right\|_1, \quad m \in \mathbb{N}_0; \tag{79}$$

that is, (21) holds. It follows from (21) and (22) that $\lim_{m\to\infty}\|(x_1^m, x_2^m, x_3^m) - (z_1, z_2, z_3)\|_1 = 0$.

Secondly, we show that (b) holds. Let (L_{11}, L_{21}, L_{31}), $(L_{12}, L_{22}, L_{32}) \in \prod_{w=1}^{3}(b_w(d_w + D_w) + d_w - D_w, d_w + D_w)$ with $\max\{|L_{a1} - L_{a2}| : a \in \Lambda_3\} > 0$. Similarly we infer that, for each $l \in \Lambda_2$, there exist constants $\theta^l, \theta_1^l, \theta_2^l, \theta_3^l \in (0,1)$, $T_l \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ and mappings $A_{L_{1l}}, A_{L_{2l}}, A_{L_{3l}} : \prod_{w=1}^{3}\Omega(\widehat{d}_w, D_w) \to l_\beta^\infty$ and $S_{L_{1l},L_{2l},L_{3l}} : \prod_{w=1}^{3}\Omega(\widehat{d}_w, D_w) \to (l_\beta^\infty)^3$ satisfying (27)~(32) and (67)~(70), where θ, θ_1, θ_2, θ_3, T, L_1, L_2, L_3, A_{L_1}, A_{L_2}, A_{L_3}, and S_{L_1,L_2,L_3} are replaced by θ^l, θ_1^l, θ_2^l, θ_3^l, T_l, L_{1l}, L_{2l}, L_{3l}, $A_{L_{1l}}$, $A_{L_{2l}}$, $A_{L_{3l}}$, and $S_{L_{1l},L_{2l},L_{3l}}$, respectively, and the contraction mapping $S_{L_{1l},L_{2l},L_{3l}}$ has a unique fixed point $(z_1^l, z_2^l, z_3^l) = \{(z_{1n}^l, z_{2n}^l, z_{3n}^l)\}_{n\in\mathbb{N}_\beta} \in \prod_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$, which is a positive solution of the system (6); that is,

$$\begin{aligned} z_{an}^l = nL_{al} - b_{an} z_{a(n-\tau_a)}^l & - \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j, z_{1r_{a1j}}^l, \ldots, z_{1r_{akj}}^l, z_{2s_{a1j}}^l, \ldots, z_{2s_{akj}}^l, \\ & z_{3t_{a1j}}^l, \ldots, z_{3t_{akj}}^l\Big) - c_{aj}\Big], \\ & (n, l, a) \in \mathbb{N}_{T_l} \times \Lambda_2 \times \Lambda_3. \end{aligned} \tag{80}$$

Using (17), (67), (70), and (80), we know that, for each $(n, a) \in \mathbb{N}_{\max\{T_1,T_2\}} \times \Lambda_3$,

$$\begin{aligned} \left|\frac{z_{an}^1 - z_{an}^2}{n}\right| & = \Bigg| L_{a1} - L_{a2} - \frac{1}{n} b_{an}\left(z_{a(n-\tau_a)}^1 - z_{a(n-\tau_a)}^2\right) \\ & \quad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j, z_{1r_{a1j}}^1, \ldots, z_{1r_{akj}}^1, z_{2s_{a1j}}^1, \ldots, z_{2s_{akj}}^1, \\ & \qquad z_{3t_{a1j}}^1, \ldots, z_{3t_{akj}}^1\Big) \\ & \qquad - f_a\Big(j, z_{1r_{a1j}}^2, \ldots, z_{1r_{akj}}^2, z_{2s_{a1j}}^2, \ldots, z_{2s_{akj}}^2, \\ & \qquad z_{3t_{a1j}}^2, \ldots, z_{3t_{akj}}^2\Big)\Big]\Bigg| \\ & \geq |L_{a1} - L_{a2}| - b_a\left\|z_a^1 - z_a^2\right\| \\ & \quad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big(U_{aj}\max\Big\{\left|z_{1r_{alj}}^1 - z_{1r_{alj}}^2\right|, \left|z_{2s_{alj}}^1 - z_{2s_{alj}}^2\right|, \\ & \qquad \left|z_{3t_{alj}}^1 - z_{3t_{alj}}^2\right| : l \in \Lambda_k\Big\}\Big) \\ & \geq |L_{a1} - L_{a2}| - b_a\left\|z_a^1 - z_a^2\right\| \\ & \quad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left(U_{aj}P_{aj}\max\left\{\frac{\left|z_{1r_{alj}}^1 - z_{1r_{alj}}^2\right|}{r_{alj}}, \frac{\left|z_{2s_{alj}}^1 - z_{2s_{alj}}^2\right|}{s_{alj}}, \right.\right. \\ & \qquad \left.\left. \frac{\left|z_{3t_{alj}}^1 - z_{3t_{alj}}^2\right|}{t_{alj}} : l \in \Lambda_k\right\}\right) \\ & \geq |L_{a1} - L_{a2}| - \left(b_a + \frac{1}{\max\{T_1, T_2\}}\sum_{i=\max\{T_1,T_2\}}^{\infty}\sum_{j=i}^{\infty}U_{aj}P_{aj}\right) \\ & \quad \times \max\left\{\left\|z_1^1 - z_1^2\right\|, \left\|z_2^1 - z_2^2\right\|, \left\|z_3^1 - z_3^2\right\|\right\} \\ & \geq |L_{a1} - L_{a2}| - \max\left\{\theta^1, \theta^2\right\}\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1, \end{aligned} \tag{81}$$

which implies that

$$\begin{aligned} & \left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1 \\ & = \max\left\{\sup_{n\in\mathbb{N}_\beta}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\ & \geq \max\left\{\sup_{n\in\mathbb{N}_{\max\{T_1,T_2\}}}\left|\frac{z_{an}^1 - z_{an}^2}{n}\right| : a \in \Lambda_3\right\} \\ & \geq \max\Big\{|L_{a1} - L_{a2}| - \max\left\{\theta^1, \theta^2\right\} \\ & \qquad \times \left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1 : a \in \Lambda_3\Big\} \\ & = \max\left\{|L_{a1} - L_{a2}| : a \in \Lambda_3\right\} \\ & \quad - \max\left\{\theta^1, \theta^2\right\}\left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1, \end{aligned} \tag{82}$$

which gives that

$$\begin{aligned} \left\|\left(z_1^1, z_2^1, z_3^1\right) - \left(z_1^2, z_2^2, z_3^2\right)\right\|_1 & \geq \frac{\max\left\{|L_{a1} - L_{a2}| : a \in \Lambda_3\right\}}{1 + \max\left\{\theta^1, \theta^2\right\}} \\ & > 0; \end{aligned} \tag{83}$$

that is, $(z_1^1, z_2^1, z_3^1) \neq (z_1^2, z_2^2, z_3^2)$. Hence the system (6) possesses uncountably many positive solutions in $\prod_{w=1}^{3}\Omega(\widehat{d}_w, D_w)$. This completes the proof. □

Theorem 5. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $d_a, D_a \in \mathbb{R}^+ \setminus \{0\}$, $b_a \in \mathbb{R}_- \setminus \{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$, and $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (17), (18), (22), (65), *and*

$$1 < \frac{d_a}{D_a} < \frac{b_a+2}{-b_a}, \quad -1 < b_a \le b_{an} \le 0, \quad (n,a) \in \mathbb{N}_{n_1} \times \Lambda_3. \tag{84}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in \Pi_{w=1}^3 (d_w - D_w, (1+b_w)(d_w + D_w))$, there exist $\theta \in (0,1)$ and $T \ge 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ such that, for each $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$, the Mann iterative sequence $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_{n_0}}\}_{m\in\mathbb{N}_0} \in \Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$ generated by (66) *converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$ of the system* (6) *with $\lim_{n\to\infty}(z_{an} + b_{an} z_{a(n-\tau_a)})/n = L_a$ for each $a \in \Lambda_3$ and has the error estimate* (21).

(b) The system (6) *has uncountably many positive solutions in $\Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$.*

Proof. Let $(L_1, L_2, L_3) \in \Pi_{w=1}^3 (d_w - D_w, (1+b_w)(d_w + D_w))$. It follows from (65) that there exist $\theta \in (0,1)$ and $T \ge 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ satisfying (67) and

$$\begin{aligned} &\frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + |c_{aj}|\right) \\ &\quad < \min\left\{(1+b_a)(d_a+D_a) - L_a, L_a - (d_a - D_a)\right\}, \\ &\qquad a \in \Lambda_3. \end{aligned} \tag{85}$$

Let A_{L_1}, A_{L_2}, A_{L_3}, and S_{L_1,L_2,L_3} be defined by (27) and (70), respectively. It follows from (18), (70), and (85) that, for any $(x_1, x_2, x_3) = \{(x_{1n}, x_{2n}, x_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\hat{d}_w, D_w)$,

$$\begin{aligned} &\frac{1}{n}A_{L_a}(x_{1n}, x_{2n}, x_{3n}) \\ &\quad = L_a - \frac{1}{n}b_{an}x_{a(n-\tau_a)} \\ &\qquad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j, x_{1r_{a1j}}, \dots, x_{1r_{akj}}, x_{2s_{a1j}}, \dots, x_{2s_{akj}}, \\ &\qquad\qquad x_{3t_{a1j}}, \dots, x_{3t_{akj}}\Big) - c_{aj}\Big] \\ &\quad \ge L_a - \frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + |c_{aj}|\right) \\ &\quad > L_a - \min\left\{(1+b_a)(d_a+D_a) - L_a, L_a - (d_a - D_a)\right\} \\ &\quad \ge d_a - D_a, \quad (n,a) \in \mathbb{N}_T \times \Lambda_3, \\ &\frac{1}{n}A_{L_a}(x_{1n}, x_{2n}, x_{3n}) = \frac{1}{n}\cdot\frac{n}{T}A_{L_a}(x_{1T}, x_{2T}, x_{3T}) \\ &\qquad = \frac{1}{T}A_{L_a}(x_{1T}, x_{2T}, x_{3T}) > d_a - D_a, \\ &\qquad\qquad \beta \le n < T, \quad a \in \Lambda_3, \\ &\frac{1}{n}A_{L_a}(x_{1n}, x_{2n}, x_{3n}) \\ &\quad = L_a - \frac{1}{n}b_{an}x_{a(n-\tau_a)} \\ &\qquad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\Big[f_a\Big(j, x_{1r_{a1j}}, \dots, x_{1r_{akj}}, x_{2s_{a1j}}, \dots, x_{2s_{akj}}, \\ &\qquad\qquad x_{3t_{a1j}}, \dots, x_{3t_{akj}}\Big) - c_{aj}\Big] \\ &\quad \le L_a - b_a(d_a + D_a) + \frac{1}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left(F_{aj} + |c_{aj}|\right) \\ &\quad < L_a - b_a(d_a + D_a) \\ &\qquad + \min\left\{(1+b_a)(d_a+D_a) - L_a, L_a - (d_a - D_a)\right\} \\ &\quad \le d_a + D_a, \quad (n,a) \in \mathbb{N}_T \times \Lambda_3, \\ &\frac{1}{n}A_{L_a}(x_{1n}, x_{2n}, x_{3n}) = \frac{1}{n}\cdot\frac{n}{T}A_{L_a}(x_{1T}, x_{2T}, x_{3T}) \\ &\qquad = \frac{1}{T}A_{L_a}(x_{1T}, x_{2T}, x_{3T}) < d_a + D_a, \\ &\qquad\qquad \beta \le n < T, \quad a \in \Lambda_3, \end{aligned} \tag{86}$$

which lead to (28). The rest of the proof is similar to that of Theorem 4 and is omitted. This completes the proof. □

Similar to the proofs of Theorems 2~5, we have the following results and omit their proofs.

Theorem 6. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $d_a, D_a, b_3 \in \mathbb{R}^+ \setminus \{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$, and $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ for each $a \in \Lambda_3$ satisfying* (17), (18), *and* (22) *as follows:*

$$\begin{aligned} &d_1 > D_1, \quad d_2 > D_2, \quad 1 < \frac{d_3}{D_3} < \frac{2-b_3}{b_3}, \\ &b_{1n} = -1, \quad b_{2n} = 1, \quad 0 \le b_{3n} \le b_3 < 1, \quad n \in \mathbb{N}_{n_1} \end{aligned} \tag{87}$$

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_1}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{1j}, |c_{1j}|, U_{1j}P_{1j}\right\} = 0, \\ &\lim_{n\to\infty}\frac{1}{n}\sum_{i=n+\tau_2}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{2j}, |c_{2j}|, U_{2j}P_{2j}\right\} = 0, \\ &\lim_{n\to\infty}\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{3j}, |c_{3j}|, U_{3j}P_{3j}\right\} = 0. \end{aligned} \tag{88}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in (d_1 - D_1, d_1 + D_1) \times (d_2 - D_2, d_2 + D_2) \times (b_3(d_3 + D_3) + d_3 - D_3, d_3 + D_3)$, *there exist* $\theta \in (0,1)$ *and* $T \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ *such that, for each* $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$, *the Mann iterative sequence* $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ *generated by the scheme*

$$x_{1n}^{m+1} = \begin{cases} (1-\alpha_m) x_{1n}^m + \alpha_m \left\{ nL_1 + \sum_{q=1}^{\infty} \sum_{i=n+q\tau_1}^{\infty} \sum_{j=i}^{\infty} \left[f_1\left(j, x_{1r_{11j}}^m, \ldots, x_{1r_{1kj}}^m, x_{2s_{11j}}^m, \ldots, x_{2s_{1kj}}^m, x_{3t_{11j}}^m, \ldots, x_{3t_{1kj}}^m\right) - c_{1j} \right] \right\}, \\ \qquad n \geq T, \quad m \in \mathbb{N}_0, \\ (1-\alpha_m) x_{1n}^m + \alpha_m \left\{ nL_1 + \frac{n}{T} \sum_{q=1}^{\infty} \sum_{i=T+q\tau_1}^{\infty} \sum_{j=i}^{\infty} \left[f_1\left(j, x_{1r_{11j}}^m, \ldots, x_{1r_{1kj}}^m, x_{2s_{11j}}^m, \ldots, x_{2s_{1kj}}^m, x_{3t_{11j}}^m, \ldots, x_{3t_{1kj}}^m\right) - c_{1j} \right] \right\}, \\ \qquad \beta \leq n < T, \quad m \in \mathbb{N}_0, \end{cases} \tag{89}$$

$$x_{2n}^{m+1} = \begin{cases} (1-\alpha_m) x_{2n}^m + \alpha_m \left\{ nL_2 - \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_2}^{n+2q\tau_2-1} \sum_{j=i}^{\infty} \left[f_2\left(j, x_{1r_{21j}}^m, \ldots, x_{1r_{2kj}}^m, x_{2s_{21j}}^m, \ldots, x_{2s_{2kj}}^m, x_{3t_{21j}}^m, \ldots, x_{3t_{2kj}}^m\right) - c_{2j} \right] \right\}, \\ \qquad n \geq T, \quad m \in \mathbb{N}_0, \\ (1-\alpha_m) x_{2n}^m + \alpha_m \left\{ nL_2 - \frac{n}{T} \sum_{q=1}^{\infty} \sum_{i=T+(2q-1)\tau_2}^{T+2q\tau_2-1} \sum_{j=i}^{\infty} \left[f_2\left(j, x_{1r_{21j}}^m, \ldots, x_{1r_{2kj}}^m, x_{2s_{21j}}^m, \ldots, x_{2s_{2kj}}^m, x_{3t_{21j}}^m, \ldots, x_{3t_{2kj}}^m\right)\right] - c_{2j} \right\}, \\ \qquad \beta \leq n < T, \quad m \in \mathbb{N}_0, \end{cases} \tag{90}$$

$$x_{3n}^{m+1} = \begin{cases} (1-\alpha_m) x_{3n}^m + \alpha_m \left\{ nL_3 - b_{3n} x_{3(n-\tau_3)}^m - \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} \left[f_3\left(j, x_{1r_{31j}}^m, \ldots, x_{1r_{3kj}}^m, x_{2s_{31j}}^m, \ldots, x_{2s_{3kj}}^m, x_{3t_{31j}}^m, \ldots, x_{3t_{3kj}}^m\right) - c_{3j} \right] \right\}, \\ \qquad n \geq T, \quad m \in \mathbb{N}_0, \\ (1-\alpha_m) x_{3n}^m + \alpha_m \left\{ nL_3 - \frac{n}{T} b_{3T} x_{3(T-\tau_3)}^m - \frac{n}{T} \sum_{i=T}^{\infty} \sum_{j=i}^{\infty} \left[f_3\left(j, x_{1r_{31j}}^m, \ldots, x_{1r_{3kj}}^m, x_{2s_{31j}}^m, \ldots, x_{2s_{3kj}}^m, x_{3t_{31j}}^m, \ldots, x_{3t_{3kj}}^m\right) - c_{3j} \right] \right\}, \\ \qquad \beta \leq n < T, \quad m \in \mathbb{N}_0 \end{cases} \tag{91}$$

converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ *of the system* (6) *with*

$$\lim_{n\to\infty} \frac{z_{1n}}{n} = L_1, \qquad \lim_{n\to\infty} \frac{z_{2n}}{n} = L_2, \qquad \lim_{n\to\infty} \frac{z_{3n} + b_{3n} z_{3(n-\tau_3)}}{n} = L_3 \tag{92}$$

and has the error estimate (21).

(b) The system (6) *has uncountably many positive solutions in* $\Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$.

Theorem 7. *Assume that there exist constants* $n_1 \in \mathbb{N}_{n_0}$, $d_a, D_a \in \mathbb{R}^+ \setminus \{0\}$, $b_3 \in \mathbb{R}_- \setminus \{0\}$ *and nonnegative sequences* $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ *for each* $a \in \Lambda_3$ *satisfying* (17), (18), (22), (88), *and*

$$d_1 > D_1, \quad d_2 > D_2, \quad 1 < \frac{d_3}{D_3} < \frac{b_3 + 2}{-b_3}, \qquad b_{1n} = -1, \quad b_{2n} = 1, \quad -1 < b_3 \leq b_{3n} \leq 0, \quad n \in \mathbb{N}_{n_1}. \tag{93}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in (d_1 - D_1, d_1 + D_1) \times (d_2 - D_2, d_2 + D_2) \times (d_3 - D_3, (1 + b_3)(d_3 + D_3))$, *there exist* $\theta \in (0,1)$ *and*

$T \geq 1+n_0+n_1+\tau_1+\tau_2+\tau_3+\beta$ such that, for each $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$, the Mann iterative sequence $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ generated by the scheme (89)~(91) converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ of the system (6) with

$$\lim_{n\to\infty}\frac{z_{1n}}{n} = L_1, \qquad \lim_{n\to\infty}\frac{z_{2n}}{n} = L_2,$$
$$\lim_{n\to\infty}\frac{z_{3n}+b_{3n}z_{3(n-\tau_3)}}{n} = L_3 \tag{94}$$

and has the error estimate (21).

(b) The system (6) has uncountably many positive solutions in $\Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$.

Theorem 8. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, $d_a, D_a, b_3 \in \mathbb{R}^+\setminus\{0\}$, $b_2 \in \mathbb{R}_-\setminus\{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$, and $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ for each $a \in \Lambda_3$ satisfying (17), (18), and (22) as follows:*

$$d_1 > D_1, \quad 1 < \frac{d_2}{D_2} < \frac{b_2+2}{-b_2}, \quad 1 < \frac{d_3}{D_3} < \frac{2-b_3}{b_3},$$
$$b_{1n} = -1, \quad -1 < b_2 \leq b_{2n} \leq 0,$$
$$0 \leq b_{3n} \leq b_3 < 1, \quad n \in \mathbb{N}_{n_1}, \tag{95}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\tau_1}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{1j}, \left|c_{1j}\right|, U_{1j}P_{1j}\right\} = 0,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{2j}, \left|c_{2j}\right|, U_{2j}P_{2j}, F_{3j}, \left|c_{3j}\right|, U_{3j}P_{3j}\right\} = 0. \tag{96}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in (d_1 - D_1, d_1 + D_1) \times (d_2 - D_2, (1+b_2)(d_2+D_2)) \times (b_3(d_3+D_3) + d_3 - D_3, d_3 + D_3)$, there exist $\theta \in (0,1)$ and $T \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ such that, for each $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$, the Mann iterative sequence $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ generated by the scheme (89), (91), and

$$x_{2n}^{m+1} = \begin{cases} (1-\alpha_m)x_{2n}^m + \alpha_m\left\{nL_2 - b_{2n}x_{2(n-\tau_2)}^m - \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\left[f_2\left(j, x_{1r_{21j}}^m, \ldots, x_{1r_{2kj}}^m, x_{2s_{21j}}^m, \ldots, x_{2s_{2kj}}^m, x_{3t_{21j}}^m, \ldots, x_{3t_{2kj}}^m\right) - c_{2j}\right]\right\}, \\ \qquad n \geq T, \quad m \in \mathbb{N}_0, \\ (1-\alpha_m)x_{2n}^m + \alpha_m\left\{nL_2 - \frac{n}{T}b_{2T}x_{2(T-\tau_2)}^m - \frac{n}{T}\sum_{i=T}^{\infty}\sum_{j=i}^{\infty}\left[f_2\left(j, x_{1r_{21j}}^m, \ldots, x_{1r_{2kj}}^m, x_{2s_{21j}}^m, \ldots, x_{2s_{2kj}}^m, x_{3t_{21j}}^m, \ldots, x_{3t_{2kj}}^m\right) - c_{2j}\right]\right\}, \\ \qquad \beta \leq n < T, \quad m \in \mathbb{N}_0 \end{cases} \tag{97}$$

converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$ of the system (6) with

$$\lim_{n\to\infty}\frac{z_{1n}}{n} = L_1, \qquad \lim_{n\to\infty}\frac{z_{2n}+b_{2n}z_{2(n-\tau_2)}}{n} = L_2,$$
$$\lim_{n\to\infty}\frac{z_{3n}+b_{3n}z_{3(n-\tau_3)}}{n} = L_3 \tag{98}$$

and has the error estimate (21).

(b) The system (6) has uncountably many positive solutions in $\Pi_{w=1}^3\Omega(\widehat{d}_w, D_w)$.

Theorem 9. *Assume that there exist constants $n_1 \in \mathbb{N}_{n_0}$, d_a, $D_a, b_3 \in \mathbb{R}^+\setminus\{0\}$, $b_2 \in \mathbb{R}_-\setminus\{0\}$ and nonnegative sequences $\{\alpha_m\}_{m\in\mathbb{N}_0} \subset [0,1]$, $\{F_{an}\}_{n\in\mathbb{N}_{n_0}}$, and $\{U_{an}\}_{n\in\mathbb{N}_{n_0}}$ for each $a \in \Lambda_3$ satisfying (17), (18), and (22) as follows:*

$$d_1 > D_1, \quad 1 < \frac{d_2}{D_2} < \frac{b_2+2}{-b_2}, \quad 1 < \frac{d_3}{D_3} < \frac{2-b_3}{b_3},$$
$$b_{1n} = 1, \quad -1 < b_2 \leq b_{2n} \leq 0,$$
$$0 \leq b_{3n} \leq b_3 < 1, \quad n \in \mathbb{N}_{n_1},$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=n+q\tau_1}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{1j}, \left|c_{1j}\right|, U_{1j}P_{1j}\right\} = 0,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{2j}, \left|c_{2j}\right|, U_{2j}P_{2j}, F_{3j}, \left|c_{3j}\right|, U_{3j}P_{3j}\right\} = 0. \tag{99}$$

Then one has the following.

(a) For any $(L_1, L_2, L_3) \in (d_1 - D_1, d_1 + D_1) \times (d_2 - D_2, (1 + b_2)(d_2 + D_2)) \times (b_3(d_3 + D_3) + d_3 - D_3, d_3 + D_3)$, there exist $\theta \in (0,1)$ and $T \geq 1 + n_0 + n_1 + \tau_1 + \tau_2 + \tau_3 + \beta$ such that, for each $(x_1^0, x_2^0, x_3^0) = \{(x_{1n}^0, x_{2n}^0, x_{3n}^0)\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$, the Mann iterative sequence $\{(x_1^m, x_2^m, x_3^m)\}_{m\in\mathbb{N}_0} = \{\{(x_{1n}^m, x_{2n}^m, x_{3n}^m)\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0} \subset \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ generated by the scheme (91), (96), *and*

$$x_{1n}^{m+1} = \begin{cases} (1-\alpha_m) x_{1n}^m + \alpha_m \left\{ nL_1 - \sum_{q=1}^{\infty} \sum_{i=n+(2q-1)\tau_1}^{n+2q\tau_1-1} \sum_{j=i}^{\infty} \left[f_1\left(j, x_{1r_{11j}}^m, \ldots, x_{1r_{1kj}}^m, x_{2s_{11j}}^m, \ldots, x_{2s_{1kj}}^m, x_{3t_{11j}}^m, \ldots, x_{3t_{1kj}}^m\right) - c_{1j} \right] \right\}, \\ \qquad n \geq T, \quad m \in \mathbb{N}_0, \\ (1-\alpha_m) x_{1n}^m + \alpha_m \left\{ nL_1 - \frac{n}{T} \sum_{q=1}^{\infty} \sum_{i=T+(2q-1)\tau_1}^{T+2q\tau_1-1} \sum_{j=i}^{\infty} \left[f_1\left(j, x_{1r_{11j}}^m, \ldots, x_{1r_{1kj}}^m, x_{2s_{11j}}^m, \ldots, x_{2s_{1kj}}^m, x_{3t_{11j}}^m, \ldots, x_{3t_{1kj}}^m\right) - c_{1j} \right] \right\}, \\ \qquad \beta \leq n < T, \quad m \in \mathbb{N}_0 \end{cases} \tag{100}$$

converges to a positive solution $(z_1, z_2, z_3) = \{(z_{1n}, z_{2n}, z_{3n})\}_{n\in\mathbb{N}_\beta} \in \Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$ of the system (6) *with*

$$\lim_{n\to\infty} \frac{z_{1n}}{n} = L_1, \qquad \lim_{n\to\infty} \frac{z_{2n} + b_{2n} z_{2(n-\tau_2)}}{n} = L_2, \qquad \lim_{n\to\infty} \frac{z_{3n} + b_{3n} z_{3(n-\tau_3)}}{n} = L_3 \tag{101}$$

and has the error estimate (21).

(b) The system (6) *has uncountably many positive solutions in $\Pi_{w=1}^3 \Omega(\widehat{d}_w, D_w)$.*

4. Examples

Now we suggest four examples to explain the main results in Section 3.

Example 1. Consider the nonlinear three-dimensional difference system

$$\begin{aligned} &\Delta^2\left(x_{1n} - x_{1(n-\tau_1)}\right) + \frac{4nx_{1(2n+6)}^3 - x_{2(n+1)}^2 + 2x_{3(n-2)}}{n^6+1} \\ &= \frac{n^2 - 3n + 1}{n^7 + n^4 + 1}, \quad n \geq 8, \\ &\Delta^2\left(x_{2n} - x_{2(n-\tau_2)}\right) + \frac{(n+3)\, x_{1(3n-10)}^2 + n^2 x_{2(n^2-1)} + x_{3(4n^2-1)}^3}{n^8 + \ln n} \\ &= \frac{2n + (-1)^n \ln n}{n^6 + 3n^2 + 5n + 1}, \quad n \geq 8, \\ &\Delta^2\left(x_{3n} - x_{3(n-\tau_3)}\right) + \frac{\sin^2\left(n^2 x_{2(n-2)} - n^3 x_{3(n-7)}^3\right)}{(n+2)^8 + n\left|x_{1(2n-1)}\right|} \\ &= \frac{n - \sin n}{n^8 + 3n^2 + 2n + 5}, \quad n \geq 8, \end{aligned} \tag{102}$$

where $n_0 = 8$ and $\tau_1, \tau_2, \tau_3 \in \mathbb{N}$ are fixed. Let $n_1 = 3$, $k = 1$, $d_1 = 6$, $D_1 = 2$, $d_2 = 10$, $D_2 = 6$, $d_3 = 14$, $D_3 = 10$, $\beta = \min\{8 - \tau_1, 8 - \tau_2, 8 - \tau_3, 1\} \in \mathbb{N}$, and

$$\begin{gathered} b_{1n} = b_{2n} = b_{3n} = -1, \qquad r_{11n} = 2n + 6, \qquad s_{11n} = n + 1, \\ t_{11n} = n - 2, \qquad r_{21n} = 3n - 10, \qquad s_{21n} = n^2 - 1, \\ t_{21n} = 4n^2 - 1, \qquad r_{31n} = 2n - 1, \qquad s_{31n} = n - 2, \\ t_{31n} = n - 7, \qquad c_{1n} = \frac{n^2 - 3n + 1}{n^7 + n^4 + 1}, \\ c_{2n} = \frac{2n + (-1)^n \ln n}{n^6 + 3n^2 + 5n + 1}, \qquad c_{3n} = \frac{n - \sin n}{n^8 + 3n^2 + 2n + 5}, \\ f_1(n, u, v, w) = \frac{4nu^3 - v^2 + 2w}{n^6 + 1}, \\ f_2(n, u, v, w) = \frac{(n+3)\, u^2 + n^2 v + w^3}{n^8 + \ln n}, \\ f_3(n, u, v, w) = \frac{\sin^2\left(n^2 v - n^3 w\right)}{(n+2)^8 + n|u|}, \\ U_{1n} = \frac{800}{n^5}, \qquad U_{2n} = \frac{40}{n^6}, \qquad U_{3n} = \frac{3}{n^5}, \\ F_{1n} = \frac{2500}{n^5}, \qquad F_{2n} = \frac{15000}{n^6}, \qquad F_{3n} = \frac{1}{n^8}, \\ P_{1n} = 2n + 6, \qquad P_{2n} = 4n^2 - 1, \qquad P_{3n} = 2n - 1, \\ (n, u, v, w) \in \mathbb{N}_{n_0} \times \mathbb{R}^3. \end{gathered} \tag{103}$$

It is clear that (16)~(18) are satisfied. Note that Lemma 1 implies that

$$\begin{aligned}
&\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},\left|c_{aj}\right|,U_{aj}P_{aj}:a\in\Lambda_3\right\}\\
&\le\frac{1}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\max\left\{\frac{2500}{j^5},\frac{15000}{j^6},\frac{1}{j^8},\frac{2}{j^5},\frac{3}{j^5},\right.\\
&\qquad\frac{1}{j^7},\frac{800}{j^5}\cdot 3j,\\
&\qquad\left.\frac{1300}{j^6}\cdot 4j^2,\frac{3}{j^5}\cdot 2j\right\}\\
&\le\frac{15000}{n}\sum_{q=1}^{\infty}\sum_{i=n+q\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\frac{1}{j^4}\\
&\le\frac{15000}{n}\sum_{j=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}j^2\cdot\frac{1}{j^4}\\
&\le\frac{15000}{n}\sum_{j=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\frac{1}{j^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned}\tag{104}$$

which yield that (19) holds. Thus Theorem 2 implies that the system (102) possesses uncountably many positive solutions in $\Pi_{w=1}^{3}\Omega(\hat{d}_w,D_w)$.

Example 2. Consider the nonlinear three-dimensional difference system

$$\begin{aligned}
&\Delta^2\left(x_{1n}+x_{1(n-\tau_1)}\right)+\frac{n^2x_{2(n+1)}+x_{3(n^2-3)}\ln n}{n^8+\left|x_{1(2n-1)}\right|}\\
&\quad=\frac{(-1)^n\sqrt{n-5}}{n^9+3n-1},\quad n\ge 40,\\
&\Delta^2\left(x_{2n}+x_{2(n-\tau_2)}\right)+\frac{x_{1(n^2-1)}+(-1)^nx_{3(2n+1)}}{n^5+x_{2(n-1)}^2}\\
&\quad=\frac{n^3-\sin(n-1)}{n^{12}+3n^2+11},\quad n\ge 40,\\
&\Delta^2\left(x_{3n}+x_{3(n-\tau_3)}\right)+\frac{n^2x_{1(n-30)}+nx_{2(2n-2)}+x_{3(3n+1)}^3}{n^7+\sin(n^3+1)}\\
&\quad=\frac{n\cos(n^3-2n)}{n^7+2n^3+1},\quad n\ge 40,
\end{aligned}\tag{105}$$

where $n_0=40$ and $\tau_1,\tau_2,\tau_3\in\mathbb{N}$ are fixed. Let $n_1=40$, $k=1$, $d_1=3$, $D_1=1$, $d_2=4$, $D_2=2$, $d_3=5$, $D_3=3$, $\beta=\min\{40-\tau_1,40-\tau_2,40-\tau_3,10\}\in\mathbb{N}$, and

$$\begin{aligned}
&b_{1n}=b_{2n}=b_{3n}=1,\qquad r_{11n}=2n-1,\qquad s_{11n}=n+1,\\
&t_{11n}=n^2-3,\qquad r_{21n}=n^2-1,\qquad s_{21n}=n-1,\\
&t_{21n}=2n+1,\qquad r_{31n}=n-30,\qquad s_{31n}=2n-2,\\
&t_{31n}=3n+1,\qquad c_{1n}=\frac{(-1)^n\sqrt{n-5}}{n^9+3n-1},\\
&c_{2n}=\frac{n^3-\sin(n-1)}{n^{12}+3n^2+11},\qquad c_{3n}=\frac{n\cos(n^3-2n)}{n^7+2n^3+1},\\
&f_1(n,u,v,w)=\frac{n^2v+w\ln n}{n^8+|u|},\\
&f_2(n,u,v,w)=\frac{u+(-1)^nw}{n^5+v^2},\\
&f_3(n,u,v,w)=\frac{n^2u+nv+w^3}{n^7+\sin(n^3+1)},\\
&U_{1n}=\frac{24}{n^6},\qquad U_{2n}=\frac{3}{n^5},\qquad U_{3n}=\frac{400}{n^5},\\
&F_{1n}=\frac{7}{n^6},\qquad F_{2n}=\frac{12}{n^5},\qquad F_{3n}=\frac{600}{n^5},\\
&P_{1n}=n^2-3,\qquad P_{2n}=n^2-1,\qquad P_{3n}=3n+1,\\
&(n,u,v,w)\in\mathbb{N}_{n_0}\times\mathbb{R}^3.
\end{aligned}\tag{106}$$

Obviously, (17) and (18) are satisfied. Note that Lemma 1 yields that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},\left|c_{aj}\right|,U_{aj}P_{aj}:a\in\Lambda_3\right\}\\
&\le\frac{1}{n}\sum_{i=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\max\left\{\frac{7}{j^6},\frac{12}{j^5},\frac{600}{j^5},\frac{1}{j^8},\frac{2}{j^9},\frac{1}{j^6},\right.\\
&\qquad\left.\frac{24}{j^6}\cdot j^2,\frac{3}{j^5}\cdot j^2,\frac{400}{j^5}\cdot 4j\right\}\\
&\le\frac{600}{n}\sum_{i=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\sum_{j=i}^{\infty}\frac{1}{j^3}\\
&\le\frac{600}{n}\sum_{j=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}j\cdot\frac{1}{j^3}\\
&=\frac{600}{n}\sum_{j=n+\min\{\tau_a:a\in\Lambda_3\}}^{\infty}\frac{1}{j^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned}\tag{107}$$

which implies that (45) holds. Thus Theorem 3 implies that the system (105) possesses uncountably many positive solutions in $\Pi_{w=1}^{3}\Omega(\hat{d}_w,D_w)$.

Example 3. Consider the nonlinear three-dimensional difference system

$$\Delta^2\left(x_{1n}+\frac{3n\cos^2 n}{5n+3}x_{1(n-\tau_1)}\right)+\left(\sqrt{n}x_{1(n^2-1)}x_{2(n+5)}+\sin\left(x_{1(n-2)}x_{2(n+7)}\right)+\cos\left(x_{3(2n+1)}x_{3(2n-1)}\right)\right)\left(n^{15}+\ln n\right)^{-1}=\frac{(-1)^n n^3}{n^7+1},\quad n\ge 5,$$
$$\Delta^2\left(x_{2n}+\frac{3n^3\sin^4 n}{6n^3+1}x_{2(n-\tau_2)}\right)+\left(x_{1(n+2)}x_{2(3n+1)}+nx_{1(2n+2)}x_{3(3n+5)}+\sqrt{n-2}x^2_{2(n^2-2)}x^2_{3(4n+1)}\right)\left(n^6+n^2+1\right)^{-1}=\frac{(-1)^n\left(n^2-1\right)}{n^8+\sqrt{n+1}},\quad n\ge 5,$$
$$\Delta^2\left(x_{3n}+\frac{4n^{21}}{5n^{21}+\ln n}x_{3(n-\tau_3)}\right)+\frac{x_{1(n+3)}x_{2(n+1)}+x_{2(n+4)}x_{3(n^2-2)}+x_{1(2n+3)}x_{3(n^2-3)}}{n^6+nx^2_{1(n+3)}+x^2_{3(n^2-2)}}=\frac{n\sin n}{n^9+1},\quad n\ge 5, \tag{108}$$

where $n_0 = 5$ and $\tau_1,\tau_2,\tau_3\in\mathbb{N}$ are fixed. Let $n_1 = 8$, $k = 2$, $d_1 = 2$, $D_1 = 1$, $d_2 = 4$, $D_2 = 2$, $d_3 = 5$, $D_3 = 4$, $\beta = \min\{5-\tau_1, 5-\tau_2, 5-\tau_3, 3\}\in\mathbb{N}$, $b_1 = 3/5$, $b_2 = 1/2$, $b_3 = 4/5$, and

$$b_{1n}=\frac{3n\cos^2 n}{5n+3},\qquad b_{2n}=\frac{3n^3\sin^4 n}{6n^3+1},\qquad b_{3n}=\frac{4n^{21}}{5n^{21}+\ln n},$$
$$c_{1n}=\frac{(-1)^n n^3}{n^7+1},\qquad c_{2n}=\frac{(-1)^n\left(n^2-1\right)}{n^8+\sqrt{n+1}},\qquad c_{3n}=\frac{n\sin n}{n^9+1},$$
$$r_{11n}=n^2-1,\qquad r_{12n}=n-2,$$
$$s_{11n}=n+5,\qquad s_{12n}=n+7,$$
$$t_{11n}=2n+1,\qquad t_{12n}=2n-1,$$
$$r_{21n}=n+2,\qquad r_{22n}=2n+2,$$
$$s_{21n}=3n+1,\qquad s_{22n}=n^2-2,$$
$$t_{21n}=3n+5,\qquad t_{22n}=4n+1,$$
$$r_{31n}=n+3,\qquad r_{32n}=2n+3,$$
$$s_{31n}=n+1,\qquad s_{32n}=n+4,$$
$$t_{31n}=n^2-2,\qquad t_{32n}=n^2-3,$$
$$f_1\left(n,u_1,u_2,v_1,v_2,w_1,w_2\right)=\frac{\sqrt{n}u_1v_1+\sin\left(u_2v_2\right)+\cos\left(w_1w_2\right)}{n^{15}+\ln n},$$
$$f_2\left(n,u_1,u_2,v_1,v_2,w_1,w_2\right)=\frac{u_1v_1+nu_2w_1+\sqrt{n-2}v_2^2w_2^2}{n^6+n^2+1},$$
$$f_3\left(n,u_1,u_2,v_1,v_2,w_1,w_2\right)=\frac{u_1v_1+v_2w_1+u_2w_2}{n^6+nu_1^2+w_1^2},$$
$$U_{1n}=\frac{40}{n^{14}},\qquad U_{2n}=\frac{1700}{n^5},\qquad U_{3n}=\frac{5000}{n^6},$$
$$F_{1n}=\frac{20}{n^{14}},\qquad F_{2n}=\frac{3000}{n^5},\qquad F_{3n}=\frac{100}{n^6},$$
$$P_{1n}=n^2-1,\qquad P_{2n}=n^2-2,\qquad P_{3n}=n^2-2,$$
$$\left(n,u_1,u_2,v_1,v_2,w_1,w_2\right)\in\mathbb{N}_{n_0}\times\mathbb{R}^6. \tag{109}$$

It is easy to see that (17), (18), and (64) are satisfied. Note that Lemma 1 yields that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},\left|c_{aj}\right|,U_{aj}P_{aj}: a\in\Lambda_3\right\}\\
&\le\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{\frac{20}{j^{14}},\frac{3000}{j^5},\frac{100}{j^6},\frac{1}{j^4},\frac{1}{j^6},\frac{2}{j^8},\frac{40}{j^{14}}\cdot j^2,\frac{1700}{j^5}\cdot j^2,\frac{5000}{j^6}\cdot j^2\right\}\\
&\le\frac{5000}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\frac{1}{j^3}\\
&\le\frac{5000}{n}\sum_{j=n}^{\infty}j\cdot\frac{1}{j^3}\\
&=\frac{5000}{n}\sum_{j=n}^{\infty}\frac{1}{j^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned} \tag{110}$$

which implies that (65) holds. Thus Theorem 4 implies that the system (108) possesses uncountably many positive solutions in $\Pi_{w=1}^{3}\Omega(\hat{d}_w, D_w)$.

Example 4. Consider the nonlinear three-dimensional difference system

$$\begin{aligned}
&\Delta^2\left(x_{1n}-\frac{4n^3+2}{5n^3+3}x_{1(n-\tau_1)}\right)\\
&\quad+\frac{n^2-x_{1(n-1)}x_{2(n^2-1)}+x_{2(n^2-1)}x_{3(n^3-2)}}{n^7+1}\\
&=\frac{(-1)^n n^2-1}{n^8+4n^3+5},\quad n\ge 4,\\
&\Delta^2\left(x_{2n}-\frac{3n\cos^6\left(n^2-1\right)}{5n+1}x_{2(n-\tau_2)}\right)\\
&\quad+\frac{n^{3/2}x_{1(n+3)}+x_{2(n^2-3)}x_{3(3n-1)}\ln n}{n^8+1}\\
&=\frac{\sin\left(n^2-1\right)}{n^5+1},\quad n\ge 4,\\
&\Delta^2\left(x_{3n}-\frac{2n^2\sin^2\left(n^2+3\right)}{4n^2+3}x_{3(n-\tau_3)}\right)+\frac{nx_{2(n-3)}+x_{3(4n-3)}}{n^6+x_{1(n^2-2)}^2}\\
&=\frac{(-1)^{n+1}\sqrt{n}}{n^6+n^4+n^2+1},\quad n\ge 4,
\end{aligned}\tag{111}$$

where $n_0=4$ and $\tau_1,\tau_2,\tau_3\in\mathbb{N}$ are fixed. Let $n_1=10$, $k=1$, $d_1=4$, $D_1=3$, $d_2=2$, $D_2=1$, $d_3=5$, $D_3=4$, $\beta=\min\{4-\tau_1,4-\tau_2,4-\tau_3,1\}\in\mathbb{N}$, $b_1=-4/5$, $b_2=-3/5$, $b_3=-1/2$, and

$$\begin{gathered}
b_{1n}=-\frac{4n^3+2}{5n^3+3},\qquad b_{2n}=-\frac{3n\cos^6\left(n^2-1\right)}{5n+1},\\
b_{3n}=-\frac{2n^2\sin^2\left(n^2+3\right)}{4n^2+3},\\
c_{1n}=\frac{(-1)^n n^2-1}{n^8+4n^3+5},\qquad c_{2n}=\frac{\sin\left(n^2-1\right)}{n^5+1},\\
c_{3n}=\frac{(-1)^{n+1}\sqrt{n}}{n^6+n^4+n^2+1},\\
r_{11n}=n-1,\qquad s_{11n}=n^2-1,\qquad t_{11n}=n^3-2,\\
r_{21n}=n+3,\qquad s_{21n}=n^2-3,\qquad t_{21n}=3n-1,\\
r_{31n}=n^2-2,\qquad s_{31n}=n-3,\qquad t_{31n}=4n-3,\\
f_1(n,u,v,w)=\frac{n^2-uv+vw}{n^7+1},\\
f_2(n,u,v,w)=\frac{n^{3/2}u^2+vw\ln n}{n^8+1},\\
f_3(n,u,v,w)=\frac{nv+w}{n^6+u^2},\\
U_{1n}=\frac{22}{n^7},\qquad U_{2n}=\frac{26}{n^6},\qquad U_{3n}=\frac{300}{n^5},\\
F_{1n}=\frac{47}{n^5},\qquad F_{2n}=\frac{10}{n^6},\qquad F_{3n}=\frac{6}{n^5},\\
P_{1n}=n^3-2,\qquad P_{2n}=n^2-3,\qquad P_{3n}=n^2-4,\\
(n,u,v,w)\in\mathbb{N}_{n_0}\times\mathbb{R}^3.
\end{gathered}\tag{112}$$

Clearly, (17), (18), and (84) hold. Observe that Lemma 1 means that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{F_{aj},\left|c_{aj}\right|,U_{aj}P_{aj}:a\in\Lambda_3\right\}\\
&\le\frac{1}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\max\left\{\frac{4}{j^5},\frac{10}{j^6},\frac{6}{j^5},\frac{2}{j^6},\frac{1}{j^5},\frac{1}{j^5},\frac{22}{j^7}\cdot j^3,\right.\\
&\qquad\left.\frac{26}{j^6}\cdot j^2,\frac{300}{j^5}\cdot j^2\right\}\\
&\le\frac{300}{n}\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\frac{1}{j^3}\\
&\le\frac{300}{n}\sum_{j=n}^{\infty}j\cdot\frac{1}{j^3}\\
&=\frac{300}{n}\sum_{j=n}^{\infty}\frac{1}{j^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty;
\end{aligned}\tag{113}$$

that is, (65) holds. Thus Theorem 5 implies that the system (111) possesses uncountably many positive solutions in $\Pi_{w=1}^{3}\Omega(\hat{d}_w,D_w)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] R. P. Agarwal, W.-T. Li, and P. Y. H. Pang, "Asymptotic behavior of nonlinear difference systems," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 307–316, 2003.

[2] J. R. Graef and E. Thandapani, "Oscillation of two-dimensional difference systems," *Computers and Mathematics with Applications*, vol. 38, no. 7, pp. 157–165, 1999.

[3] H.-F. Huo and W.-T. Li, "Oscillation of certain two-dimensional nonlinear difference systems," *Computers and Mathematics with Applications*, vol. 45, no. 6–9, pp. 1221–1226, 2003.

[4] H.-F. Huo and W.-T. Li, "Oscillation of the emden-fowler difference systems," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 2, pp. 478–485, 2001.

[5] J. Jiang and X. Tang, "Oscillation and asymptotic behavior of two-dimensional difference systems," *Computers and Mathematics with Applications*, vol. 54, no. 9-10, pp. 1240–1249, 2007.

[6] W.-T. Li, "Classification schemes for nonoscillatory solutions of two-dimensional nonlinear difference systems," *Computers and Mathematics with Applications*, vol. 42, no. 3–5, pp. 341–355, 2001.

[7] Z. Liu, S. M. Kang, and Y. C. Kwun, "Positive solutions and iterative approximations for a nonlinear two-dimensional difference system with multiple delays," *Abstract and Applied Analysis*, vol. 2012, Article ID 240378, 57 pages, 2012.

[8] Z. Liu, L. Zhao, S. M. Kang, and Y. C. Kwun, "Existence of bounded positive solutions for a system of difference equations," *Applied Mathematics and Computation*, vol. 218, no. 6, pp. 2889–2912, 2011.

[9] S. Matucci and P. Řehák, "Nonoscillatory solutions of a second-order nonlinear discrete system," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 833–845, 2007.

[10] E. Schmeidel, "Boundedness of solutions of nonlinear three-dimensional difference systems with delays," *Fasciculi Mathematici*, vol. 44, pp. 107–113, 2010.

[11] E. Schmeidel, "Oscillation of nonlinear three-dimensional difference systems with delays," *Mathematica Bohemica*, vol. 135, pp. 163–170, 2010.

[12] E. Thandapani and B. Ponnammal, "Oscillatory properties of solutions of three-dimensional difference systems," *Mathematical and Computer Modelling*, vol. 42, no. 5-6, pp. 641–650, 2005.

[13] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical*, vol. 4, no. 3, pp. 506–510, 1953.