

Research Article

Analogues of Conditional Wiener Integrals with Drift and Initial Distribution on a Function Space

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Let $C[0, T]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0, T]$, and define a stochastic process $Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by $Z(x, t) = \int_0^t h(u) dx(u) + x(0) + a(t)$, for $x \in C[0, T]$ and $t \in [0, T]$, where $h \in L_2[0, T]$ with $h \neq 0$ a.e. and a is a continuous function on $[0, T]$. Let $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ be given by $Z_n(x) = (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n))$ and $Z_{n+1}(x) = (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1}))$, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ is a partition of $[0, T]$. In this paper we derive two simple formulas for generalized conditional Wiener integrals of functions on $C[0, T]$ with the conditioning functions Z_n and Z_{n+1} which contain drift and initial distribution. As applications of these simple formulas we evaluate generalized conditional Wiener integrals of the function $\exp\{\int_0^T Z(x, t) dm_L(t)\}$ including the time integral on $C[0, T]$.

1. Introduction

A time integral is simply the Riemann integral of a function of the continuous random variable $X(x, t) = x(t)$ with respect to the parameter t for $x \in C_0[0, T]$ which is the Wiener space, the space of continuous real-valued functions x on $[0, T]$ with $x(0) = 0$ [1]. This means that the time integral of $X(x, t)$ is a random variable Y on $C_0[0, T]$ satisfying

$$Y(x) = \int_0^T F(t, X(x, t)) dm_L(t), \quad (1)$$

where $F(t, X(x, t))$ is Riemann integrable on $[0, T]$ and m_L is the Lebesgue measure on \mathbb{R} . A study of the Feynman integral provides ready examples of the utility of the time integral. The Feynman-Kac formula represents an important step in the process of providing a rigorous definition of the Feynman integral. A detailed explanation of this formula can be found in [2]. The Feynman-Kac functional is given by

$$\exp\left\{-\int_0^T V(t, X(x, t)) dm_L(t)\right\} \quad (2)$$

including the time integral, where $X(x, t)$ is a standard Brownian motion process on $C_0[0, T] \times [0, T]$ and V is

a complex-valued potential. Calculations involving the conditional expectations of the functional are important in the study of the Feynman integral. More examples of functionals involving time integrals are found in [2–4]. In [5] Park and Skoug introduced a generalized Brownian motion process $X : C_0[0, T] \times [0, T] \rightarrow \mathbb{R}$ defined by

$$X(x, t) = \int_0^t h(u) dx(u) \quad \text{for } x \in C_0[0, T], t \in [0, T], \quad (3)$$

where h is of bounded variation with $h \neq 0$ a.e. on $[0, T]$ and the integral denotes the Paley-Wiener-Zygmund stochastic integral of h according to x [6], and then they generalized various theories related to the conditional Wiener integrals on $C_0[0, T]$.

On the other hand let $C[0, T]$ denote the space of continuous real-valued functions on the interval $[0, T]$. Im et al. [6–8] introduced a probability measure w_φ on $C[0, T]$, where φ is a probability measure on the Borel class of \mathbb{R} . We note that w_φ is exactly the Wiener measure on $C_0[0, T]$ if $\varphi = \delta_0$, the Dirac measure concentrated at 0. Let a be

a continuous function on $[0, T]$. Define stochastic processes $X, Y, Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} X(x, t) &= \int_0^t h(u) dx(u), & Y(x, t) &= X(x, t) + a(t), \\ Z(x, t) &= Y(x, t) + x(0) \end{aligned} \tag{4}$$

for $x \in C[0, T]$ and for $t \in [0, T]$. Let $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ be given by

$$\begin{aligned} Z_n(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \\ Z_{n+1}(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1})), \end{aligned} \tag{5}$$

where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ is a partition of $[0, T]$. The author [9–11] derived various properties of X and then extended his works on X to Y . In fact he [9] established that X is a generalized Brownian process and derived a simple formula for a conditional expectation which evaluates conditional expectations in terms of ordinary expectations. Moreover he [10] extended the results on X to those on Y with drift using only a translation theorem [6, Theorem 3.1]. In [11] he investigated the distribution of Z and proved that Z is a generalized Brownian motion process if the initial distribution is degenerated. He also established a generalized Wiener integration theorem which extends Lemma 2.1 of [6]. Furthermore he derived a generalized Paley-Wiener theorem which generalizes Theorem 3.5 of [6]. As applications of the theorems he evaluated generalized Wiener integrals of various functions including

$$\begin{aligned} &Z(x, s_1) Z(x, s_2) \cdots Z(x, s_m), \\ &\exp \left\{ \int_0^T Z(x, t) dm_L(t) \right\} \end{aligned} \tag{6}$$

for $x \in C[0, T]$, where $0 < s_1 < s_2 < \dots < s_m \leq T$ and m is a positive integer.

In this paper, using the results in [11], we derive two simple formulas for generalized conditional Wiener integrals of functions on $C[0, T]$ with the conditioning functions Z_n and Z_{n+1} which contain drift and initial distribution. As applications of these simple formulas we evaluate several generalized conditional Wiener integrals of the functions given by (6). We note that if $h \equiv 1$ and $a \equiv 0$ on $[0, T]$, then X as above is exactly the stochastic process given by $X(x, t) = x(t)$ so that these works generalize the results of [5, 9, 10, 12–15] in which the works are the first results among them.

2. Simple Formulas for Generalized Conditional Wiener Integrals

In this section we derive two simple formulas for generalized conditional Wiener integrals on the space $(C[0, T], w_\varphi)$ which is introduced in the previous section.

Let m_L denote the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of \mathbb{R} and let $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denote the dot product on \mathbb{R} . Let $(C[0, T], \mathcal{B}(C[0, T]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on $\mathcal{B}(\mathbb{R})$, where $\mathcal{B}(C[0, T])$ denotes the Borel class of $C[0, T]$ [6–8]. Let $F : C[0, T] \rightarrow \mathbb{C}$ be integrable and X_τ be a random vector on $C[0, T]$ assuming that the value space of X_τ is a normed space equipped with the Borel σ -algebra. Then we have the conditional expectation $E[F | X_\tau]$ of F given X_τ from a well-known probability theory. Furthermore there exists a P_{X_τ} -integrable complex valued function ψ on the value space of X_τ such that $E[F | X_\tau](x) = (\psi \circ X_\tau)(x)$ for w_φ a.e. $x \in C[0, T]$, where P_{X_τ} is the probability distribution of X_τ . The function ψ is called the conditional w_φ -integral of F given X_τ and it is also denoted by $E[F | X_\tau]$.

For v in $L_2[0, T]$ and x in $C[0, T]$ let (v, x) denote the Paley-Wiener-Zygmund integral of v according to x [6] and let $\langle \cdot, \cdot \rangle_2$ denote the inner product over $L_2[0, T]$. Let $h \in L_2[0, T]$ be of bounded variation with $h \neq 0$ a.e. on $[0, T]$ and let $a \in C[0, T]$. Define $X_0(x) = x(0)$ for $x \in C[0, T]$ and define stochastic processes $X, Y, Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} X(x, t) &= (\chi_{[0,t]} h, x), & Y(x, t) &= (\chi_{[0,t]} h, x) + a(t), \\ Z(x, t) &= (\chi_{[0,t]} h, x) + X_0(x) + a(t) \end{aligned} \tag{7}$$

for $x \in C[0, T]$ and for $t \in [0, T]$. Let $b(t) = \int_0^t (h(u))^2 du$ and let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ be a partition of $[0, T]$. For $j = 1, \dots, n + 1$ let

$$\alpha_j^b(t) = \frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})}, \quad \beta_j^b(t) = \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} \tag{8}$$

for $t \in [t_{j-1}, t_j]$ and

$$\sigma_j^b(s, t) = \alpha_j^b(t) \beta_j^b(s) (b(t_j) - b(t_{j-1})) \tag{9}$$

for $s, t \in [t_{j-1}, t_j]$. Define random vectors $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ by

$$\begin{aligned} Z_n(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \\ Z_{n+1}(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1})) \end{aligned} \tag{10}$$

for $x \in C[0, T]$. For any function f on $[0, T]$ define the polygonal function $P_{b, n+1}(f)$ of f by

$$\begin{aligned} &P_{b, n+1}(f)(t) \\ &= \sum_{j=1}^{n+1} \chi_{(t_{j-1}, t_j]}(t) (\alpha_j^b(t) f(t_{j-1}) \\ &\quad + \beta_j^b(t) f(t_j)) + \chi_{\{t_0\}}(t) f(t_0) \end{aligned} \tag{11}$$

for $t \in [0, T]$, where $\chi_{(t_{j-1}, t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ define

the polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by (11), where $f(t_j)$ is replaced by ξ_j , respectively, for $j = 0, 1, \dots, n+1$. If $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n}(\vec{\xi}_n)$ is interpreted as $\chi_{[0,t_n]} P_{b,n+1}(\vec{\xi}_{n+1})$ on $[0, T]$. For $x \in C[0, T]$ and for $t \in [0, T]$ let

$$\begin{aligned} Z_{b,n+1}(x, t) &= Z(x, t) - P_{b,n+1}(Z(x, \cdot))(t), \\ A_t &= a(t) - P_{b,n+1}(a)(t) \end{aligned} \tag{12}$$

and for $t \in [t_{j-1}, t_j]$ let

$$h_{j,t}(u) = \left[\alpha_j^b(t) \chi_{[t_{j-1},t]}(u) - \beta_j^b(t) \chi_{[t,t_j]}(u) \right] h(u) \tag{13}$$

for $u \in [0, T]$. It is not difficult to show that

$$Z_{b,n+1}(x, t) = (h_{j,t}, x) + A_t \quad \text{for } t \in [t_{j-1}, t_j], \tag{14}$$

$$\begin{aligned} P_{b,n+1}(Y(x, \cdot)) &= P_{b,n+1}(X(x, \cdot)) + P_{b,n+1}(a), \\ P_{b,n+1}(Z(x, \cdot)) &= P_{b,n+1}(Y(x, \cdot)) + X_0(x) \end{aligned} \tag{15}$$

for $x \in C[0, T]$.

Now we have the following theorem [11].

Theorem 1. For $t \in [0, T]$, X_0 and $X(\cdot, t)$ are independent so that X_0 and $Y(\cdot, t)$ are also independent.

Theorem 2. Let $t_{j-1} < t < t_j$ for some $j \in \{1, \dots, n+1\}$. Then $Z_{b,n+1}(\cdot, t)$ is normally distributed with mean A_t and variance $\sigma_j^b(t, t)$.

Proof. By (14) and Theorem 3.4 of [6], $Z_{b,n+1}(\cdot, t)$ is normally distributed with mean A_t and variance $\|h_{j,t}\|_2^2$. A simple calculation with an aid of (13) shows that

$$\begin{aligned} \|h_{j,t}\|_2^2 &= \left((b(t_j) - b(t))^2 (b(t) - b(t_{j-1})) \right. \\ &\quad \left. + (b(t) - b(t_{j-1}))^2 (b(t_j) - b(t)) \right) \\ &\quad \times \left((b(t_j) - b(t_{j-1}))^2 \right)^{-1} \\ &= \sigma_j^b(t, t) \end{aligned} \tag{16}$$

which completes the proof. □

Remark 3. (1) We can prove Theorem 2 using the Fourier transform of $Z_{b,n+1}(\cdot, t)$, Lemma 2.4 of [11], and Corollaries 2.11, 2.12 of [11], but the proof is tedious.

(2) We can also prove Theorem 2 using Theorem 2.9 of [9] and Theorem 2.13 of [11].

Theorem 4. The process $\{Z_{b,n+1}(\cdot, t) : 0 \leq t \leq T\}$ and Z_{n+1} are stochastically independent.

Proof. Note that

$$\begin{aligned} Z_{b,n+1}(x, t) &= X(x, t) - P_{b,n+1}(X(x, \cdot))(t) + A_t, \\ Z_{n+1}(x) &= (X(x, t_0), \dots, X(x, t_{n+1})) \\ &\quad + (a(t_0) + X_0(x), \dots, a(t_{n+1}) + X_0(x)) \end{aligned} \tag{17}$$

for $x \in C[0, T]$. $X(x, t) - P_{b,n+1}(X(x, \cdot))(t)$ and $X(x, t_j)$ are independent by Theorem 2.10 of [9] for $j = 1, \dots, n+1$, and $X(\cdot, t) - P_{b,n+1}(X(\cdot, \cdot))(t)$ and X_0 are also independent by Theorem 1 which completes the proof of the theorem. □

Since $Z_{b,n+1}(x, t) = Y(x, t) - P_{b,n+1}(Y(x, \cdot))(t)$ for $x \in C[0, T]$ and $t \in [0, T]$, we have the following theorem by Theorem 16 of [10].

Theorem 5. The processes $\{Z_{b,n+1}(\cdot, t) : t_{j-1} \leq t \leq t_j\}$, where $j = 1, \dots, n+1$, are stochastically independent.

For a function $F : C[0, t] \rightarrow \mathbb{C}$ let $F_Z(x) = F(Z(x, \cdot))$ for $x \in C[0, T]$. Applying the same method as used in the proof of Theorem 2 in [15] with Problem 4 of [16, page 216], we have the following theorem from Theorem 4.

Theorem 6. Let F be a complex valued function on $C[0, T]$ and let F_Z be integrable over $C[0, T]$. Then for a Borel subset B of \mathbb{R}^{n+2}

$$\begin{aligned} &\int_{Z_{n+1}^{-1}(B)} F_Z(x) dw_\varphi(x) \\ &= \int_B E \left[F \left(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}) \right) \right] dP_{Z_{n+1}}(\vec{\xi}_{n+1}), \end{aligned} \tag{18}$$

where $P_{Z_{n+1}}$ is the probability distribution of Z_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$, so that for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) = E \left[F \left(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}) \right) \right], \tag{19}$$

where the expectation is taken over the variable x .

Using similar method as used in the proof of Theorem 18 of [10], we can prove the following theorem.

Theorem 7. Let F be a complex valued function on $C[0, T]$ and let F_Z be integrable over $C[0, T]$. Let P_{Z_n} be the probability distribution of Z_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E[F_Z | Z_n](\vec{\xi}_n) \\ &= \left[\frac{1}{2\pi(b(T) - b(t_n))} \right]^{1/2} \\ &\quad \times \int_{\mathbb{R}} E \left[F \left(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}) \right) \right] \\ &\quad \times \exp \left\{ - \frac{(\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2}{2(b(T) - b(t_n))} \right\} dm_L(\xi_{n+1}), \end{aligned} \tag{20}$$

where $\vec{\xi}_{n+1} = (\xi_0, \dots, \xi_n, \xi_{n+1})$.

Remark 8. (1) The conditioning functions Z_{n+1} and Z_n describe the positions of paths at the times $t_0 = 0, t_1, \dots, t_n, t_{n+1} = T$ (the present time). $Z_{n+1}(x)$ depends on the present position $Z(x, T)$ of the path $Z(x, \cdot)$ for $x \in C[0, T]$ while Z_n does not. Moreover if $a(0) = 0, h = \sqrt{b'}$, and $\varphi = \delta_0$ which is the Dirac measure concentrated at 0, then we can obtain the space $C_{a,b}[0, T]$ in [12] by Theorem 2.13 of [11]. Furthermore if Z is replaced by the generalized Brownian motion process $x(t)$ on $C_{a,b}[0, T] \times [0, T]$ and $\varphi = \delta_0$, then we can also obtain Theorem 3.4 of [12] by Theorem 6. If $a \equiv 0$ and $\varphi = \delta_0$, then we can obtain Theorem 3 of [5] by Theorem 6. If $n = 0$ and $\varphi = \delta_0$, then we can obtain Remark 2.2 of [17] by Theorem 6. If $h = 1$ a.e. and $a \equiv 0$, then we can obtain Theorem 2.9 of [13] and Theorem 2.5 of [14] by Theorems 6 and 7, respectively. Finally if $h = 1$ a.e., $\varphi = \delta_0$, and $a \equiv 0$, then we can obtain Theorem 2 of [15] by Theorem 6 which is among the first result expressing the conditional Wiener integrals of functions on $C_0[0, T]$ as ordinary Wiener integrals.

(2) Theorems 6 and 7 are not generalizations of Theorem 2.12 of [9] and Theorems 17 and 18 of [10]. In Theorems 6 and 7 the conditioning functions depend on the initial distribution φ while the conditioning functions in [9, 10] do not.

3. A Multivariate Normal Distribution

In this section we investigate the joint distribution of $\{Z_{b,n+1}(\cdot, s_l) : l = 1, \dots, m\}$, where $t_{j-1} < s_1 < \dots < s_m < t_j$ for $j = 1, \dots, n + 1$. In fact we prove that the random vector as above has a multivariate normal distribution which plays a key role in the next sections.

Now we begin this section with the introduction of the following lemma.

Lemma 9. *Let $t_{j-1} = s_0 < s_1 < \dots < s_m < t_j$ and h_{j,s_l} be given by (13) for $l = 1, \dots, m$. Then $\{h_{j,s_l} : l = 1, \dots, m\}$ is a linearly independent set in $L_2[0, T]$.*

Proof. For $c_1, \dots, c_m \in \mathbb{R}$ let

$$\begin{aligned} \sum_{l=1}^m c_l h_{j,s_l}(u) &= \sum_{l=1}^m c_l \left[\alpha_j^b(s_l) \chi_{[t_{j-1}, s_l]}(u) \right. \\ &\quad \left. - \beta_j^b(s_l) \chi_{[s_l, t_j]}(u) \right] h(u) \end{aligned} \tag{21}$$

= 0

for a.e. $u \in [0, T]$. For $k = 1, \dots, m$ take $u_k \in (s_{k-1}, s_k)$ which satisfies the above equation and $h(u_k) \neq 0$. Replacing u by u_k we have the following linear equation system with unknowns c_1, \dots, c_m :

$$\begin{aligned} \alpha_j^b(s_1) c_1 + \alpha_j^b(s_2) c_2 + \dots + \alpha_j^b(s_{m-2}) c_{m-2} \\ + \alpha_j^b(s_{m-1}) c_{m-1} + \alpha_j^b(s_m) c_m = 0, \end{aligned}$$

$$\begin{aligned} -\beta_j^b(s_1) c_1 + \alpha_j^b(s_2) c_2 + \dots + \alpha_j^b(s_{m-2}) c_{m-2} \\ + \alpha_j^b(s_{m-1}) c_{m-1} + \alpha_j^b(s_m) c_m = 0, \\ \vdots \\ -\beta_j^b(s_1) c_1 - \beta_j^b(s_2) c_2 - \dots - \beta_j^b(s_{m-2}) c_{m-2} \\ + \alpha_j^b(s_{m-1}) c_{m-1} + \alpha_j^b(s_m) c_m = 0, \\ -\beta_j^b(s_1) c_1 - \beta_j^b(s_2) c_2 - \dots - \beta_j^b(s_{m-2}) c_{m-2} \\ - \beta_j^b(s_{m-1}) c_{m-1} + \alpha_j^b(s_m) c_m = 0. \end{aligned} \tag{22}$$

The determinant of coefficient matrix of the system is given by

$$\begin{aligned} &\begin{vmatrix} \alpha_j^b(s_1) & \alpha_j^b(s_2) & \dots & \alpha_j^b(s_{m-2}) & \alpha_j^b(s_{m-1}) & \alpha_j^b(s_m) \\ -\beta_j^b(s_1) & \alpha_j^b(s_2) & \dots & \alpha_j^b(s_{m-2}) & \alpha_j^b(s_{m-1}) & \alpha_j^b(s_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\beta_j^b(s_1) & -\beta_j^b(s_2) & \dots & -\beta_j^b(s_{m-2}) & \alpha_j^b(s_{m-1}) & \alpha_j^b(s_m) \\ -\beta_j^b(s_1) & -\beta_j^b(s_2) & \dots & -\beta_j^b(s_{m-2}) & -\beta_j^b(s_{m-1}) & \alpha_j^b(s_m) \end{vmatrix} \\ &= \begin{vmatrix} \alpha_j^b(s_1) & \alpha_j^b(s_2) & \dots & \alpha_j^b(s_{m-2}) & \alpha_j^b(s_{m-1}) & \alpha_j^b(s_m) \\ -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & -1 & 0 & 0 \\ -1 & -1 & \dots & -1 & -1 & 0 \end{vmatrix} \\ &= (-1)^{1+m} \alpha_j^b(s_m) (-1)^{m-1} = \alpha_j^b(s_m) \neq 0. \end{aligned} \tag{23}$$

Now we have $c_1 = c_2 = \dots = c_m = 0$ which completes the proof. \square

A random variable X_t on $C[0, t]$ is said to be degenerated if there exists a constant c satisfying $w_\varphi(X_t = c) = 1$ [18].

Lemma 10. *Let the assumptions and notations be as given in Lemma 9. If $c_1, \dots, c_m \in \mathbb{R}$ and $\sum_{l=1}^m c_l Z_{b,n+1}(\cdot, s_l)$ is degenerated, then $c_l = 0$ for all $l = 1, \dots, m$.*

Proof. By (14) we can take a real constant c satisfying

$$c = \sum_{l=1}^m c_l Z_{b,n+1}(x, s_l) = \left(\sum_{l=1}^m c_l h_{j,s_l}, x \right) + \sum_{l=1}^m c_l A_{s_l} \tag{24}$$

for a.e. $x \in C[0, T]$. Then we have by Theorem 3.4 of [6] that

$$\begin{aligned} 0 &= \int_{C[0,T]} \left[\left(\sum_{l=1}^m c_l h_{j,s_l}, x \right) + \sum_{l=1}^m c_l A_{s_l} - c \right]^2 dw_\varphi(x) \\ &= \left\| \sum_{l=1}^m c_l h_{j,s_l} \right\|_2^2 + \left[\sum_{l=1}^m c_l A_{s_l} - c \right]^2 \end{aligned} \tag{25}$$

so that $\sum_{l=1}^m c_l h_{j,s_l} = 0$ in $L_2[0, T]$. Now $c_1 = c_2 = \dots = c_m = 0$ by Lemma 9 which completes the proof. \square

Lemma 11. *Let the assumptions and notations be as given in Lemma 9. Then for $\vec{s} = (s_1, \dots, s_m)$ the covariance matrix $\Sigma_m(\vec{s}) = [\sigma(s_l, s_k)]_{m \times m}$ of the random variables $Z_{b,n+1}(\cdot, s_l)$, $l = 1, \dots, m$, exists and is positive definite. Moreover $\sigma(s_l, s_k)$ is given by*

$$\sigma(s_l, s_k) = \langle h_{j,s_l}, h_{j,s_k} \rangle_2 = \sigma_j^b(\min\{s_l, s_k\}, \max\{s_l, s_k\}) \tag{26}$$

and the determinant $|\Sigma_m(\vec{s})|$ is positive so that $\Sigma_m(\vec{s})$ is nonsingular and the inverse matrix $\Sigma_m^{-1}(\vec{s})$ of $\Sigma_m(\vec{s})$ is also positive definite.

Proof. By Theorem 3.4 of [6]

$$\begin{aligned} & \|h_{j,s_l}\|_2^2 + 2 \int_{C[0,T]} (h_{j,s_l}, x)(h_{j,s_k}, x) dw_\varphi(x) + \|h_{j,s_k}\|_2^2 \\ &= \int_{C[0,T]} (h_{j,s_l} + h_{j,s_k}, x)^2 dw_\varphi(x) \\ &= \|h_{j,s_l}\|_2^2 + 2 \langle h_{j,s_l}, h_{j,s_k} \rangle_2 + \|h_{j,s_k}\|_2^2 \end{aligned} \tag{27}$$

so that the covariance $\sigma(s_l, s_k)$ of $Z_{b,n+1}(\cdot, s_l)$ and $Z_{b,n+1}(\cdot, s_k)$ is given by $\sigma(s_l, s_k) = \langle h_{j,s_l}, h_{j,s_k} \rangle_2$. Now we have for $l \leq k$

$$\begin{aligned} \sigma(s_l, s_k) &= \int_0^T \left[\alpha_j^b(s_l) \chi_{[t_{j-1}, s_l]}(u) - \beta_j^b(s_l) \chi_{[s_l, t_j]}(u) \right] \\ &\quad \times \left[\alpha_j^b(s_k) \chi_{[t_{j-1}, s_k]}(u) - \beta_j^b(s_k) \chi_{[s_k, t_j]}(u) \right] \\ &\quad \times (h(u))^2 dm_L(u) \\ &= \alpha_j^b(s_l) \alpha_j^b(s_k) (b(s_l) - b(t_{j-1})) \\ &\quad - \beta_j^b(s_l) \alpha_j^b(s_k) (b(s_k) - b(s_l)) \\ &\quad + \beta_j^b(s_l) \beta_j^b(s_k) (b(t_j) - b(s_k)) \\ &= \frac{(b(t_j) - b(s_k))(b(s_l) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})} \\ &= \sigma_j^b(s_l, s_k) \end{aligned} \tag{28}$$

which proves (26). We have for $\vec{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$

$$\begin{aligned} & \langle \Sigma_m(\vec{s}) \vec{c}, \vec{c} \rangle_{\mathbb{R}} \\ &= \sum_{l=1}^m \sum_{k=1}^m \sigma(s_l, s_k) c_l c_k \\ &= E \left[\left(\sum_{l=1}^m c_l (Z_{b,n+1}(\cdot, s_l) - A_{s_l}) \right)^2 \right] \geq 0. \end{aligned} \tag{29}$$

Moreover if $\langle \Sigma_m(\vec{s}) \vec{c}, \vec{c} \rangle_{\mathbb{R}} = 0$, then

$$\sum_{l=1}^m c_l Z_{b,n+1}(x, s_l) = \sum_{l=1}^m c_l A_{s_l} \tag{30}$$

for a.e. $x \in C[0, T]$; that is, $\sum_{l=1}^m c_l Z_{b,n+1}(\cdot, s_l)$ is degenerated. Thus the covariance matrix $\Sigma_m(\vec{s})$ is positive definite by Lemma 10. Since $\Sigma_m(\vec{s})$ is symmetric and positive definite, the eigenvalues $\lambda_1, \dots, \lambda_m$ of $\Sigma_m(\vec{s})$ are real and positive. Since $|\Sigma_m(\vec{s})| = \lambda_1 \dots \lambda_m$, we have $|\Sigma_m(\vec{s})| > 0$ so that $\Sigma_m(\vec{s})$ is invertible. Since

$$\begin{aligned} & \langle \Sigma_m^{-1}(\vec{s}) \vec{c}, \vec{c} \rangle_{\mathbb{R}} = \langle \vec{c}, \Sigma_m^{-1}(\vec{s}) \vec{c} \rangle_{\mathbb{R}} \\ &= \langle \Sigma_m(\vec{s}) \Sigma_m^{-1}(\vec{s}) \vec{c}, \Sigma_m^{-1}(\vec{s}) \vec{c} \rangle_{\mathbb{R}} \geq 0 \end{aligned} \tag{31}$$

and $\langle \Sigma_m^{-1}(\vec{s}) \vec{c}, \vec{c} \rangle_{\mathbb{R}} = 0$ implies $\Sigma_m^{-1}(\vec{s}) \vec{c} = \vec{0}$; that is, $\vec{c} = \vec{0}$, $\Sigma_m^{-1}(\vec{s})$ is positive definite. \square

Remark 12. Using the same process as used in the proof of Theorem 3.4 in [9] we can prove (26).

For simplicity let

$$\begin{aligned} \Psi_m(\vec{s}, \vec{u}) &= \left[\frac{1}{(2\pi)^m |\Sigma_m(\vec{s})|} \right]^{1/2} \\ &\quad \times \exp \left\{ -\frac{1}{2} \langle \Sigma_m^{-1}(\vec{s}) (\vec{u} - A(\vec{s})), \vec{u} - A(\vec{s}) \rangle_{\mathbb{R}} \right\} \end{aligned} \tag{32}$$

for $\vec{u} \in \mathbb{R}^m$ and $\vec{s} = (s_1, \dots, s_m) \in \mathbb{R}^m$, where $t_{j-1} < s_1 < \dots < s_m < t_j$ and $A(\vec{s}) = (A_{s_1}, \dots, A_{s_m})$. By Lemmas 9, 10, 11, and Theorem 4 of [1] we have the following theorem which is our main result in this section.

Theorem 13. *Let the assumptions and notations be as given in Lemma 11. Then for every Borel measurable function $f : \mathbb{R}^m \rightarrow \mathbb{C}$*

$$\begin{aligned} & \int_{C[0,T]} f(Z_{b,n+1}(x, s_1), \dots, Z_{b,n+1}(x, s_m)) dw_\varphi(x) \\ &= \int_{\mathbb{R}^m} f(\vec{u}) \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}) \\ &= \left[\frac{1}{(2\pi)^m} \right]^{1/2} \int_{\mathbb{R}^m} f \left(\sqrt{\Sigma_m(\vec{s})} \vec{u} + A(\vec{s}) \right) \\ &\quad \times \exp \left\{ -\frac{1}{2} \|\vec{u}\|_{\mathbb{R}}^2 \right\} d(m_L)^m(\vec{u}), \end{aligned} \tag{33}$$

where $\sqrt{\Sigma_m(\vec{s})}$ is the positive definite matrix satisfying $\sqrt{\Sigma_m(\vec{s})}^2 = \Sigma_m(\vec{s})$ and $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

4. Conditional Expectations of Functions on Time Integrals

In this section we evaluate generalized conditional Wiener integrals of the function $\exp\{\int_0^T Z(x, t)dm_L(t)\}$ including a time integral. To do this we have the following theorem by Theorems 6 and 13 and Theorem 3.3 of [11].

Theorem 14. *Let the assumptions and notations be as given in Theorem 13 and let $H_m(\vec{s}, x) = x(s_1) \cdots x(s_m)$ for $x \in C[0, T]$. Suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) = \int_{\mathbb{R}^m} (u_1 + P_{b,n+1}(\vec{\xi}_{n+1})(s_1))$$

$$\begin{aligned} |\Sigma_2(\vec{s})| &= \begin{vmatrix} \sigma_j^b(s_1, s_1) & \sigma_j^b(s_1, s_2) \\ \sigma_j^b(s_1, s_2) & \sigma_j^b(s_2, s_2) \end{vmatrix} = \frac{(b(t_j) - b(s_2))(b(s_2) - b(s_1))(b(s_1) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})}, \\ \Sigma_2^{-1}(\vec{s}) &= \begin{bmatrix} \frac{b(s_2) - b(t_{j-1})}{(b(s_2) - b(s_1))(b(s_1) - b(t_{j-1}))} & -\frac{b(s_1) - b(t_{j-1})}{(b(s_2) - b(s_1))(b(s_1) - b(t_{j-1}))} \\ -\frac{b(s_1) - b(t_{j-1})}{(b(s_2) - b(s_1))(b(s_1) - b(t_{j-1}))} & \frac{b(t_j) - b(s_1)}{(b(t_j) - b(s_2))(b(s_2) - b(s_1))} \end{bmatrix}. \end{aligned} \tag{36}$$

Moreover for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} E[(H_2(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) &= \int_{\mathbb{R}^2} (u_1 + P_{b,n+1}(\vec{\xi}_{n+1})(s_1))(u_2 + P_{b,n+1}(\vec{\xi}_{n+1})(s_2)) \\ &\quad \times \Psi_2(\vec{s}, \vec{u}) d(m_L)^2(\vec{u}) \\ &= \sigma_j^b(s_1, s_2) + (A_{s_1} + P_{b,n+1}(\vec{\xi}_{n+1})(s_1)) \\ &\quad \times (A_{s_2} + P_{b,n+1}(\vec{\xi}_{n+1})(s_2)) \end{aligned} \tag{37}$$

which is a generalization of (2) in Theorem 23 of [10].

By Theorem 7 we have the following theorem.

Theorem 16. *Let the assumptions and notations be as given in Theorem 14 and let $j \in \{1, \dots, n-1\}$. Then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$*

$$\begin{aligned} E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) &= \int_{\mathbb{R}^m} (u_1 + P_{b,n}(\vec{\xi}_n)(s_1)) \cdots (u_m + P_{b,n}(\vec{\xi}_n)(s_m)) \\ &\quad \times \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}). \end{aligned} \tag{38}$$

$$\begin{aligned} &\cdots (u_m + P_{b,n+1}(\vec{\xi}_{n+1})(s_m)) \\ &\times \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}), \end{aligned} \tag{34}$$

where $\vec{u} = (u_1, \dots, u_m)$.

Example 15. Let the assumptions and notations be as given in Theorem 14. If $m = 1$, then by Theorem 2 we have for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[(H_1(s_1, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) = A_{s_1} + P_{b,n+1}(\vec{\xi}_{n+1})(s_1). \tag{35}$$

If $m = 2$, then

Example 17. Let the assumptions and notations be as given in Theorem 16. If $m = 1$, then by Theorem 16 we have for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$E[(H_1(s_1, \cdot))_Z | Z_n](\vec{\xi}_n) = A_{s_1} + P_{b,n}(\vec{\xi}_n)(s_1). \tag{39}$$

If $m = 2$, then we have for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} E[(H_2(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) &= \sigma_j^b(s_1, s_2) + (A_{s_1} + P_{b,n}(\vec{\xi}_n)(s_1)) \\ &\quad \times (A_{s_2} + P_{b,n}(\vec{\xi}_n)(s_2)) \end{aligned} \tag{40}$$

which is a generalization of (2) in Theorem 24 of [10].

Theorem 18. *Let the assumptions and notations be as given in Theorem 14 and let $t_n < s_1 < \cdots < s_m < t_{n+1} = T$. For $\xi_n \in \mathbb{R}$ and $\vec{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ let*

$$\begin{aligned} P_{\vec{s}, \vec{u}, \xi_n}(z) &= \left(u_1 + \xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} \right. \\ &\quad \left. \times (a(T) - a(t_n)) + \frac{b(s_1) - b(t_n)}{\sqrt{b(T) - b(t_n)}} z \right) \end{aligned}$$

$$\begin{aligned} & \cdots \left(u_m + \xi_n + \frac{b(s_m) - b(t_n)}{b(T) - b(t_n)} \right. \\ & \quad \left. \times (a(T) - a(t_n)) + \frac{b(s_m) - b(t_n)}{\sqrt{b(T) - b(t_n)}} z \right) \\ & = \sum_{l=0}^m a_l(\vec{s}, \vec{u}, \xi_n) z^l \end{aligned} \tag{41}$$

for $z \in \mathbb{R}$, where $\vec{s} = (s_1, \dots, s_m)$. Then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) \\ & = \sum_{l=0}^{[m/2]} \frac{(2l)!}{2^l l!} \int_{\mathbb{R}^m} a_{2l}(\vec{s}, \vec{u}, \xi_n) \\ & \quad \times \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}), \end{aligned} \tag{42}$$

where $[m/2]$ denotes the greatest integer less than or equal to $m/2$.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. By Theorems 7 and 14 we have for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} & E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) \\ & = \left[\frac{1}{2\pi(b(T) - b(t_n))} \right]^{1/2} \\ & \quad \times \int_{\mathbb{R}^m} \Psi_m(\vec{s}, \vec{u}) \\ & \quad \times \int_{\mathbb{R}} (u_1 + P_{b,n+1}(\vec{\xi}_{n+1})(s_1)) \\ & \quad \cdots (u_m + P_{b,n+1}(\vec{\xi}_{n+1})(s_m)) \\ & \quad \times \exp \left\{ -\frac{(\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2}{2(b(T) - b(t_n))} \right\} dm_L(\xi_{n+1}) d(m_L)^m(\vec{u}) \\ & = \left[\frac{1}{2\pi(b(T) - b(t_n))} \right]^{1/2} \\ & \quad \times \int_{\mathbb{R}^m} \Psi_m(\vec{s}, \vec{u}) \\ & \quad \times \int_{\mathbb{R}} \left(u_1 + \xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} \right. \\ & \quad \quad \left. \times (u + a(T) - a(t_n)) \right) \\ & \quad \cdots \left(u_m + \xi_n + \frac{b(s_m) - b(t_n)}{b(T) - b(t_n)} \right. \\ & \quad \quad \left. \times (u + a(T) - a(t_n)) \right) \\ & \quad \times \exp \left\{ -\frac{u^2}{2(b(T) - b(t_n))} \right\} dm_L(u) d(m_L)^m(\vec{u}). \end{aligned} \tag{43}$$

Let $z = u/\sqrt{b(T) - b(t_n)}$. Then by the change of variable theorem

$$\begin{aligned} & E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) \\ & = \left(\frac{1}{2\pi} \right)^{1/2} \int_{\mathbb{R}^m} \Psi_m(\vec{s}, \vec{u}) \\ & \quad \times \int_{\mathbb{R}} P_{\vec{s}, \vec{u}, \xi_n}(z) \exp \left\{ -\frac{z^2}{2} \right\} dm_L(z) d(m_L)^m(\vec{u}) \\ & = \sum_{l=0}^{[m/2]} \left(\frac{1}{2\pi} \right)^{1/2} \int_{\mathbb{R}^m} \Psi_m(\vec{s}, \vec{u}) a_{2l}(\vec{s}, \vec{u}, \xi_n) \\ & \quad \times \int_{\mathbb{R}} z^{2l} \exp \left\{ -\frac{z^2}{2} \right\} dm_L(z) d(m_L)^m(\vec{u}) \\ & = \sum_{l=0}^{[m/2]} \frac{(2l)!}{2^l l!} \int_{\mathbb{R}^m} a_{2l}(\vec{s}, \vec{u}, \xi_n) \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}) \end{aligned} \tag{44}$$

which completes the proof. \square

Example 19. Let the assumptions and notations be as given in Theorem 18. If $m = 1$, then by Theorem 18 we have for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & E[(H_1(s_1, \cdot))_Z | Z_n](\vec{\xi}_n) \\ & = \left[\frac{1}{2\pi\sigma_{n+1}^b(s_1, s_1)} \right]^{1/2} \\ & \quad \times \int_{\mathbb{R}} \left(u_1 + \xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} \right. \\ & \quad \quad \left. \times (a(T) - a(t_n)) \right) \\ & \quad \times \exp \left\{ -\frac{(u_1 - A_{s_1})^2}{2\sigma_{n+1}^b(s_1, s_1)} \right\} dm_L(u_1) \\ & = a(s_1) - a(t_n) + \xi_n. \end{aligned} \tag{45}$$

If $m = 2$, then we have for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} & E[(H_2(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n) \\ & = \sum_{l=0}^1 \frac{(2l)!}{2^l l!} \int_{\mathbb{R}^2} a_{2l}(\vec{s}, \vec{u}, \xi_n) \Psi_2(\vec{s}, \vec{u}) d(m_L)^2(\vec{u}) \\ & = \int_{\mathbb{R}^2} \left(u_1 + \xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} (a(T) - a(t_n)) \right) \\ & \quad \times \left(u_2 + \xi_n + \frac{b(s_2) - b(t_n)}{b(T) - b(t_n)} \right. \\ & \quad \quad \left. \times (a(T) - a(t_n)) \right) \Psi_2(\vec{s}, \vec{u}) d(m_L)^2(\vec{u}) \\ & \quad + \frac{(b(s_1) - b(t_n))(b(s_2) - b(t_n))}{b(T) - b(t_n)} \end{aligned}$$

$$\begin{aligned}
 &= \sigma_{n+1}^b(s_1, s_2) + A_{s_1} A_{s_2} + A_{s_1} \\
 &\quad \times \left(\xi_n + \frac{b(s_2) - b(t_n)}{b(T) - b(t_n)} (a(T) - a(t_n)) \right) \\
 &\quad + A_{s_2} \left(\xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} (a(T) - a(t_n)) \right) \\
 &\quad + \left(\xi_n + \frac{b(s_2) - b(t_n)}{b(T) - b(t_n)} (a(T) - a(t_n)) \right) \\
 &\quad \times \left(\xi_n + \frac{b(s_1) - b(t_n)}{b(T) - b(t_n)} (a(T) - a(t_n)) \right) \\
 &\quad + \frac{(b(s_1) - b(t_n))(b(s_2) - b(t_n))}{b(T) - b(t_n)} \\
 &= \xi_n^2 + \xi_n (A_{s_1} + A_{s_2} + (a(T) - a(t_n))) \\
 &\quad \times (\beta_{n+1}^b(s_1) + \beta_{n+1}^b(s_2)) \\
 &\quad + (A_{s_1} + \beta_{n+1}^b(s_1) (a(T) - a(t_n))) \\
 &\quad \times (A_{s_2} + \beta_{n+1}^b(s_2) (a(T) - a(t_n))) \\
 &\quad + \beta_{n+1}^b(s_1) \beta_{n+1}^b(s_2) (b(T) - b(t_n)) \\
 &\quad + \sigma_{n+1}^b(s_1, s_2)
 \end{aligned} \tag{46}$$

which is a generalization of (5) in Theorem 24 of [10].

Theorem 20. *Let the notations be as given in Theorem 14 and let $H(x) = \exp\{\int_0^T x(t) dm_L(t)\}$ for a.e. $x \in C[0, T]$. Suppose that $\int_{\mathbb{R}} \exp\{T|u|\} d\varphi(u) < \infty$ and $E[\exp\{\int_{t_{j-1}}^{t_j} X(\cdot, t) dm_L(t)\}] < \infty$ for $j = 1, \dots, n + 1$. Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ $E[\exp\{\int_{t_{j-1}}^{t_j} Z(\cdot, t) dm_L(t)\} | Z_{n+1}](\vec{\xi}_{n+1})$ exists and*

$$\begin{aligned}
 &E[H_Z | Z_{n+1}](\vec{\xi}_{n+1}) \\
 &= \prod_{j=1}^{n+1} \left[1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1}) d(m_L)^m(\vec{s}) \right],
 \end{aligned} \tag{47}$$

where $\Delta_{m,j} = \{(s_1, \dots, s_m) : t_{j-1} < s_1 < \dots < s_m < t_j\}$ and $E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}]$ is as given in Theorem 14. Moreover if $E[\exp\{\int_{t_{j-1}}^{t_j} Z(\cdot, t) dm_L(t)\} | Z_{n+1}](\vec{0})$ exists for $j = 1, \dots, n + 1$, then $E[H_Z | Z_{n+1}](\vec{\xi}_{n+1})$ can be represented by

$$\begin{aligned}
 &E[H_Z | Z_{n+1}](\vec{\xi}_{n+1}) \\
 &= \exp \left\{ \int_0^T P_{b,n+1}(\vec{\xi}_{n+1})(t) dm_L(t) \right\} \\
 &\quad \times \prod_{j=1}^{n+1} \left[1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} \int_{\mathbb{R}^m} u_1 \cdots u_m \Psi_m(\vec{s}, \vec{u}) d(m_L)^m(\vec{u}) d(m_L)^m(\vec{s}) \right].
 \end{aligned} \tag{48}$$

Proof. By Theorem 2.1 and Lemma 2.1 of [6] we have for $j = 1, \dots, n + 1$

$$\begin{aligned}
 &E \left[\exp \left\{ \left| \int_{t_{j-1}}^{t_j} Z(x, t) dm_L(t) \right| \right\} \right] \\
 &\leq \int_{\mathbb{R}} \exp\{T|u|\} d\varphi(u) E \left[\exp \left\{ \left| \int_{t_{j-1}}^{t_j} X(x, t) dm_L(t) \right| \right\} \right] \\
 &\quad \times \exp \left\{ \left| \int_{t_{j-1}}^{t_j} a(t) dm_L(t) \right| \right\} < \infty
 \end{aligned} \tag{49}$$

so that $E[\exp\{\int_{t_{j-1}}^{t_j} Z(\cdot, t) dm_L(t)\} | Z_{n+1}](\vec{\xi}_{n+1})$ exists and

$$\begin{aligned}
 &E \left[\exp \left\{ \int_0^T Z(x, t) dm_L(t) \right\} \right] \\
 &\leq E \left[\exp \left\{ \sum_{j=1}^{n+1} \left| \int_{t_{j-1}}^{t_j} Z(x, t) dm_L(t) \right| \right\} \right] < \infty
 \end{aligned} \tag{50}$$

which implies the existence of $E[H_Z | Z_{n+1}]$. By the monotone convergence theorem we have

$$\begin{aligned}
 &\sum_{m=0}^{\infty} \frac{1}{m!} T^m \int_{\mathbb{R}} |u|^m d\varphi(u) \\
 &= \int_{\mathbb{R}} \exp\{T|u|\} d\varphi(u) < \infty
 \end{aligned} \tag{51}$$

so that we have $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$ for each positive integer m . Furthermore for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ we have by Theorem 6

$$\begin{aligned}
 &\sum_{m=0}^{\infty} \frac{1}{m!} \int_{C[0,T]} \left| \int_{t_{j-1}}^{t_j} (Z_{b,n+1}(x, t) \right. \\
 &\quad \left. + P_{b,n+1}(\vec{\xi}_{n+1})(t)) dm_L(t) \right|^m d\omega_{\varphi}(x) \\
 &= \int_{C[0,T]} \exp \left\{ \left| \int_{t_{j-1}}^{t_j} (Z_{b,n+1}(x, t) \right. \right. \\
 &\quad \left. \left. + P_{b,n+1}(\vec{\xi}_{n+1})(t)) dm_L(t) \right| \right\} d\omega_{\varphi}(x) \\
 &= E \left[\exp \left\{ \left| \int_{t_{j-1}}^{t_j} Z(x, t) dm_L(t) \right| \right\} | Z_{n+1} \right](\vec{\xi}_{n+1}) < \infty.
 \end{aligned} \tag{52}$$

Now by Theorems 5, 6, 14, the Fubini's theorem, and the dominated convergence theorem we have for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}
 & E [H_Z | Z_{n+1}] (\vec{\xi}_{n+1}) \\
 &= \int_{C[0,T]} \exp \left\{ \int_0^T (Z_{b,n+1} (x, t) \right. \\
 &\quad \left. + P_{b,n+1} (\vec{\xi}_{n+1}) (t)) dm_L (t) \right\} dw_\varphi (x) \\
 &= \prod_{j=1}^{n+1} \left[\int_{C[0,T]} \exp \left\{ \int_{t_{j-1}}^{t_j} (Z_{b,n+1} (x, t) \right. \right. \\
 &\quad \left. \left. + P_{b,n+1} (\vec{\xi}_{n+1}) (t)) dm_L (t) \right\} dw_\varphi (x) \right] \\
 &= \prod_{j=1}^{n+1} \left[1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} E [(H_m (\vec{s}, \cdot))_Z | Z_{n+1}] (\vec{\xi}_{n+1}) d(m_L)^m (\vec{s}) \right]
 \end{aligned} \tag{53}$$

which proves the first equality of the theorem. Furthermore

$$\begin{aligned}
 & E [H_Z | Z_{n+1}] (\vec{\xi}_{n+1}) \\
 &= \exp \left\{ \int_0^T P_{b,n+1} (\vec{\xi}_{n+1}) (t) dm_L (t) \right\} \\
 &\quad \times \left[\prod_{j=1}^{n+1} \int_{C[0,T]} \exp \left\{ \int_{t_{j-1}}^{t_j} Z_{b,n+1} (x, t) dm_L (t) \right\} dw_\varphi (x) \right].
 \end{aligned} \tag{54}$$

Since (52) holds for $\vec{\xi}_{n+1} = \vec{0}$, we have by Theorem 13 and the dominated convergence theorem

$$\begin{aligned}
 & \int_{C[0,T]} \exp \left\{ \int_{t_{j-1}}^{t_j} Z_{b,n+1} (x, t) dm_L (t) \right\} dw_\varphi (x) \\
 &= 1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} \int_{C[0,T]} \prod_{l=1}^m Z_{b,n+1} (x, s_l) dw_\varphi (x) d(m_L)^m (\vec{s}) \\
 &= 1 + \sum_{m=1}^{\infty} \int_{\Delta_{m,j}} \int_{\mathbb{R}^m} u_1 \cdots u_m \Psi_m (\vec{s}, \vec{u}) d(m_L)^m (\vec{u}) d(m_L)^m (\vec{s})
 \end{aligned} \tag{55}$$

which proves the second equality of the theorem. \square

Theorem 21. *Let the notations and the first part of the assumptions in Theorem 20 be given. Then for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ $E[H_Z | Z_n](\vec{\xi}_n)$ is given by the first equality in Theorem 20 replacing $E[(H_m(\vec{s}, \cdot))_Z | Z_{n+1}](\vec{\xi}_{n+1})$ by $E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n)$, where the $E[(H_m(\vec{s}, \cdot))_Z | Z_n](\vec{\xi}_n)$ s are as given in Theorems 16 and 18. Moreover if the second part of the assumptions in Theorem 20 holds, then $E[H_Z | Z_n](\vec{\xi}_n)$*

is given by the second equality in Theorem 20 replacing $\exp\{\int_0^T P_{b,n+1}(\vec{\xi}_{n+1})(t)dm_L(t)\}$ by

$$\begin{aligned}
 & \exp \left\{ \int_0^{t_n} P_{b,n} (\vec{\xi}_n) (t) dm_L (t) \right. \\
 &\quad \left. + \frac{1}{2} (b(T) - b(t_n)) (B(T) - B(t_n))^2 \right. \\
 &\quad \left. + (a(T) - a(t_n)) (B(T) - B(t_n)) + \xi_n (T - t_n) \right\},
 \end{aligned} \tag{56}$$

where $(d/dt)B(t) = \beta_{n+1}^b(t) = (b(t) - b(t_n))/(b(T) - b(t_n))$ and $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n)$.

Proof. The first part of the theorem immediately follows from Theorems 7, 16, 18, and 20. Suppose that the second part of the assumptions in Theorem 20 holds. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Then we have

$$\begin{aligned}
 & \left[\frac{1}{2\pi (b(T) - b(t_n))} \right]^{1/2} \\
 &\quad \times \int_{\mathbb{R}} \exp \left\{ \int_0^T P_{b,n+1} (\vec{\xi}_{n+1}) (t) dm_L (t) \right\} \\
 &\quad \times \exp \left\{ -(\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2 \right. \\
 &\quad \left. \times (2 (b(T) - b(t_n)))^{-1} \right\} dm_L (\xi_{n+1}) \\
 &= \left[\frac{1}{2\pi (b(T) - b(t_n))} \right]^{1/2} \\
 &\quad \times \int_{\mathbb{R}} \exp \left\{ \int_0^{t_n} P_{b,n} (\vec{\xi}_n) (t) dm_L (t) \right. \\
 &\quad \left. + \int_{t_n}^T (\beta_{n+1} (t) (\xi_{n+1} - \xi_n) + \xi_n) dm_L (t) \right. \\
 &\quad \left. - ((\xi_{n+1} - a(T) - (\xi_n - a(t_n)))^2 \right. \\
 &\quad \left. \times (2 (b(T) - b(t_n)))^{-1}) \right\} dm_L (\xi_{n+1}) \\
 &= \left[\frac{1}{2\pi (b(T) - b(t_n))} \right]^{1/2} \\
 &\quad \times \exp \left\{ \int_0^{t_n} P_{b,n} (\vec{\xi}_n) (t) dm_L (t) + (T - t_n) \xi_n \right\} \\
 &\quad \times \int_{\mathbb{R}} \exp \left\{ (a(T) - a(t_n) + u) \right. \\
 &\quad \left. \times \int_{t_n}^T \beta_{n+1}^b (t) dm_L (t) \right. \\
 &\quad \left. - \frac{u^2}{2 (b(T) - b(t_n))} \right\} dm_L (u)
 \end{aligned}$$

$$\begin{aligned}
 &= \exp \left\{ \int_0^{t_n} P_{b,n}(\tilde{\xi}_n)(t) dm_L(t) + \xi_n(T - t_n) \right. \\
 &\quad + (a(T) - a(t_n))(B(T) - B(t_n)) \\
 &\quad \left. + \frac{1}{2}(B(T) - B(t_n))^2 (b(T) - b(t_n)) \right\}.
 \end{aligned} \tag{57}$$

Now the second part of the theorem follows from Theorems 7 and 20. \square

Example 22. (1) Let $h \equiv 1$. Then $X(x, t) = x(t) - x(0)$ so that for $j = 1, \dots, n + 1$

$$\begin{aligned}
 &E \left[\exp \left\{ \left| \int_{t_{j-1}}^{t_j} X(\cdot, t) dm_L(t) \right| \right\} \right] \\
 &\leq E \left[\exp \left\{ T \sup_{0 \leq t \leq T} |x(t) - x(0)| \right\} \right]
 \end{aligned} \tag{58}$$

which is finite by Theorem 1.4 of [19]. Hence $E[H_Z | Z_{n+1}]$ and $E[H_Z | Z_n]$ are given by Theorems 20 and 21, respectively, with $b(t) = t$ and $(d/dt)B(t) = (t - t_n)/(T - t_n)$ for $t \in [0, T]$.

(2) Let $h(u) = T - u$ for $u \in [0, T]$ and suppose that $\int_{\mathbb{R}} \exp\{M|u|\} d\varphi(u) < \infty$, where $M = \max\{T, 2T^2\}$. Then for $j = 1, \dots, n + 1$ and $x \in C[0, T]$ we have by the integration parts formula

$$\begin{aligned}
 &\left| \int_{t_{j-1}}^{t_j} X(x, t) m_L(t) \right| \\
 &= \left| \int_{t_{j-1}}^{t_j} \left[(T - t)x(t) - Tx(0) \right. \right. \\
 &\quad \left. \left. + \int_0^t x(s) dm_L(s) \right] dm_L(t) \right| \\
 &\leq \int_{t_{j-1}}^{t_j} \left[T \sup_{0 \leq s \leq T} |x(s) - x(0)| \right. \\
 &\quad + 2t \sup_{0 \leq s \leq T} |x(s) - x(0)| \\
 &\quad \left. + 2t|x(0)| \right] dm_L(t) \\
 &\leq 2T^2 \sup_{0 \leq s \leq T} |x(s) - x(0)| + T^2|x(0)|
 \end{aligned} \tag{59}$$

so that by Hölder's inequality

$$\begin{aligned}
 &\int_{C[0,T]} \exp \left\{ \left| \int_{t_{j-1}}^{t_j} X(x, t) m_L(t) \right| \right\} dw_\varphi(x) \\
 &\leq \left[\int_{\mathbb{R}} \exp \{2T^2|u|\} d\varphi(u) \right]^{1/2} \\
 &\quad \times \left[\int_{C[0,T]} \exp \left\{ 4T^2 \sup_{0 \leq s \leq T} |x(s) - x(0)| \right\} dw_\varphi(x) \right]^{1/2}
 \end{aligned} \tag{60}$$

which is finite by Theorem 1.4 of [19]. Now $E[H_Z | Z_{n+1}]$ and $E[H_Z | Z_n]$ are given by Theorems 20 and 21, respectively, with $b(t) = (1/3)[T^3 - (T - t)^3]$ and $(d/dt)B(t) = 1 - ((T - t)/(T - t_n))^3$ for $t \in [0, T]$.

5. Evaluation Formulas for Other Functions

In this section, using the simple formulas in Section 2, we derive evaluation formulas for generalized conditional Wiener integrals of various functions which are of interest in Feynman integration theories themselves and quantum mechanics.

Since $Z_{b,n+1}(x, t) = Y(x, t) - P_{b,n+1}(Y(x, \cdot))(t)$ for $x \in C[0, T]$ and $t \in [0, T]$ we have the following theorems from Theorem 3.2 of [11] and Theorems 21, 22, 23, 24, 25, and 26 of [10].

Theorem 23. Let $m \in \mathbb{N}$ and $F_m(x) = \int_0^T (x(t))^m dm_L(t)$ for $x \in C[0, T]$. Suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then for $P_{Z_{n+1}}$ a.e. $\tilde{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}
 &E[(F_m)_Z | Z_{n+1}](\tilde{\xi}_{n+1}) \\
 &= \sum_{j=1}^{n+1} \sum_{l=0}^{[m/2]} \frac{m!}{2^l!(m-2l)!} \int_{t_{j-1}}^{t_j} (A_t + P_{b,n+1}(\tilde{\xi}_{n+1})(t))^{m-2l} \\
 &\quad \times (\sigma_j^b(t, t))^l dm_L(t),
 \end{aligned} \tag{61}$$

where $[m/2]$ denotes the greatest integer less than or equal to $m/2$.

Theorem 24. Let the assumptions be as given in Theorem 23 and for $\tilde{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let

$$\begin{aligned}
 \Xi(\tilde{\xi}_n) &= \sum_{j=1}^n \sum_{l=0}^{[m/2]} \frac{m!}{2^l!(m-2l)!} \int_{t_{j-1}}^{t_j} (A_t + P_{b,n}(\tilde{\xi}_n)(t))^{m-2l} \\
 &\quad \times (\sigma_j^b(t, t))^l dm_L(t).
 \end{aligned} \tag{62}$$

Then for P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
 &E[(F_m)_Z | Z_n](\tilde{\xi}_n) \\
 &= \Xi(\tilde{\xi}_n) \\
 &\quad + m! \sum_{l=0}^{[m/2]} \sum_{k=0}^{m-2l} \sum_{p=0}^k \sum_{q=0}^{[p/2]} (\xi_n^{k-p}(a(T) - a(t_n)))^{p-2q} \\
 &\quad \times (b(T) - b(t_n))^q \\
 &\quad \times (2^{l+q}l!q!(k-p)!(p-2q)!) \\
 &\quad \times (m-2l-k!)^{-1} \\
 &\quad \times \int_{t_n}^T (\sigma_{n+1}^b(t, t))^l (\beta_{n+1}^b(t))^p A_t^{m-2l-k} dm_L(t).
 \end{aligned} \tag{63}$$

Theorem 25. Let $0 < s_1 < s_2 \leq T$ and let $s_1 \in [t_{l-1} - t_l]$, $s_2 \in [t_{j-1} - t_j]$ with $l \neq j$. For $x \in C[0, T]$ let $G(x) = x(s_1)x(s_2)$ and suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$. Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E[G_Z | Z_{n+1}](\vec{\xi}_{n+1}) = (A_{s_1} + P_{b,n+1}(\vec{\xi}_{n+1})(s_1))(A_{s_2} + P_{b,n+1}(\vec{\xi}_{n+1})(s_2)). \tag{64}$$

Theorem 26. Let the assumptions be as given in Theorem 25.

(1) If $l \leq n$, $j \leq n$, and $l \neq j$, then for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$E[G_Z | Z_n](\vec{\xi}_n) = (A_{s_1} + P_{b,n}(\vec{\xi}_n)(s_1))(A_{s_2} + P_{b,n}(\vec{\xi}_n)(s_2)). \tag{65}$$

(2) If $l \leq n$ and $j = n + 1$, then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[G_Z | Z_n](\vec{\xi}_n) = (A_{s_2} + \xi_n + \beta_{n+1}^b(s_2)(a(T) - a(t_n))) \times (A_{s_1} + P_{b,n}(\vec{\xi}_n)(s_1)). \tag{66}$$

For $j = 1, \dots, n + 1$, let $g_j = (1/\sqrt{b(t_j) - b(t_{j-1})})\chi_{(t_{j-1}, t_j]}h$, let V be the subspace of $L_2[0, T]$ generated by $\{g_1, \dots, g_{n+1}\}$, let V^\perp be the orthogonal complement of V and let $\mathcal{P}^\perp : L_2[0, T] \rightarrow V^\perp$ be the orthogonal projection. Let $\mathcal{M}(L_2[0, T])$ be the class of all complex valued Borel measures of bounded variation over $L_2[0, T]$ and let \mathcal{S}_{w_φ} be the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0, T])$ have the form

$$F(x) = \int_{L_2[0, T]} \exp\{i(v, x)\} d\sigma(v) \tag{67}$$

for w_φ a.e. $x \in C[0, T]$. Note that \mathcal{S}_{w_φ} is a Banach algebra [6].

Theorem 27. Let a be absolutely continuous on $[0, T]$. Let $F \in \mathcal{S}_{w_\varphi}$ and $\sigma \in \mathcal{M}(L_2[0, T])$ be related by (67). Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $E[F_Z | Z_{n+1}](\vec{\xi}_{n+1})$ is given by

$$E[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) = \int_{L_2[0, T]} \exp\{i(v, A_t + P_{b,n+1}(\vec{\xi}_{n+1}))\} \times \exp\left\{-\frac{1}{2}\|\mathcal{P}^\perp(vh)\|_2^2\right\} d\sigma(v). \tag{68}$$

Theorem 28. Let the assumptions be as given in Theorem 27 and for $\vec{\xi}_n \in \mathbb{R}^{n+1}$ let

$$D(v, \vec{\xi}_n) = \exp\left\{i(v, A_t + P_{b,n}(\vec{\xi}_n)) - \frac{1}{2}\|\mathcal{P}^\perp(vh)\|_2^2\right\}. \tag{69}$$

Then for P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $E[F_Z | Z_n](\vec{\xi}_n)$ is given by

$$E[F_Z | Z_n](\vec{\xi}_n) = \int_{L_2[0, T]} D(v, \vec{\xi}_n) \exp\left\{i\frac{\langle vh, g_{n+1} \rangle_2 (a(T) - a(t_n))}{\sqrt{b(T) - b(t_n)}} - \frac{1}{2}\langle vh, g_{n+1} \rangle_2^2\right\} d\sigma(v). \tag{70}$$

Remark 29. Suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$, if necessary, in each lemma, theorem, and example of this paper. Then by Lemma 2.5 of [11] both $E[X_0]$ and $\text{Var}[X_0]$ exist. Let $m_Z(t) = a(t) + E[X_0]$ and $b_Z(t) = b(t) + \text{Var}[X_0]$ for $t \in [0, T]$. Since for $t_1, t_2 \in [0, T]$

$$m_Z(t_2) - m_Z(t_1) = a(t_2) - a(t_1), \tag{71}$$

$$b_Z(t_2) - b_Z(t_1) = b(t_2) - b(t_1),$$

a and b can be replaced by m_Z and b_Z , respectively, in each result of this paper.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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