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Research Article

On Eigenvalues of the Generator of a C_0-Semigroup Appearing in Queueing Theory

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We describe the point spectrum of the generator of a C_0-semigroup associated with the M/M/1 queueing model that is governed by an infinite system of partial differential equations with integral boundary conditions. Our results imply that the essential growth bound of the C_0-semigroup is 0 and, therefore, that the semigroup is not quasi-compact. Moreover, our result also shows that it is impossible that the time-dependent solution of the M/M/1 queueing model exponentially converges to its steady-state solution.

1. Introduction

In 1955, by considering service time of customers, Cox [1] first established the M/G/1 queueing model which was described by an infinite system of partial differential equations with integral boundary conditions and studied the steady-state solution of the model under the following hypothesis: the time-dependent solution of the model converges to its steady-state solution. In 2001, Gupur et al. [2] have proved that the underlying operator, which corresponds to the M/G/1 queueing model, generates a positive contraction C_0-semigroup that is isometric for the initial value. Hence, they deduced that the model has a unique nonnegative time-dependent solution which satisfies the probability condition (i.e., its norm is 1). In 2011, by studying spectral properties of the underlying operator on the imaginary axis, Gupur [3] obtained that all points on the imaginary axis except 0 belong to the resolvent set of the underlying operator and 0 is an eigenvalue of the underlying operator and its adjoint operator with geometric multiplicity one. Thus, by using Theorem 14 in Gupur et al. [2] (Theorem 1.96 in Gupur [4]) it follows that the time-dependent solution of the model strongly converges to its steady-state solution; that is, Cox's hypothesis holds in the sense of strong convergence. When the service rate is a constant, the M/G/1 queueing model is called M/M/1 queueing model. Well-posedness of the M/M/1 queueing model and asymptotic behavior of its time-dependent solution can be found in Gupur et al. [2] (see also Radl [5]). In 2008, Zhang and Gupur [6] have found that the underlying operator, which corresponds to the M/M/1 queueing model, has one negative real eigenvalue. In 2011, Kasim and Gupur [7] discovered that the underlying operator has uncountable negative real eigenvalues and therefore suggested that it is impossible that the time-dependent solution of the model exponentially converges to its steady-state solution. So far, no other results concerning this model can be found in the literature.

In this paper, we study eigenvalues of the underlying operator associated with the M/M/1 queueing model and obtain that if the mean arrival rate of customers λ and the mean service rate of the server μ satisfy $\lambda < \mu$, then all points in the set

$$\left\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma + \mu > 0, \quad \left|\gamma + \lambda + \mu \pm \sqrt{(\gamma + \lambda + \mu)^2 - 4\lambda\mu}\right| < 2\mu\right\} \cup \{0\} \tag{1}$$

are eigenvalues of the underlying operator with geometric multiplicity one. In particular, the interval $(-\mu, 0]$ belongs to its point spectrum. These results together with the spectral mapping theorem for the point spectrum ([8], p. 277) imply that the C_0-semigroup generated by the underlying operator has uncountable eigenvalues and therefore it is not compact, even not eventually compact. Moreover, by combining the result in this paper and the results in Gupur et al. [2] with Corollary 2.11 in Engel and Nagel [8], p. 258, we deduce that the essential growth bound of the C_0-semigroup is 0 and therefore it is not quasi-compact ([8], p. 332). Hence, queueing models are essentially different from population equations (see [9, 10]) and the reliability models that are described by finite partial differential equations with integral boundary conditions (see [4, 11]). In addition, we show that the essential spectral radius of the C_0-semigroup is 1 and it is impossible that the time-dependent solution of the M/M/1 queueing model exponentially converges to its steady-state solution.

If we do not consider service time of customers, then the M/M/1 queueing model becomes an infinite system of ordinary differential equations. Its research can be found in Gupur et al. [2] and Zhao et al. [12].

2. The M/M/1 Queueing Model and Related Results

According to Cox [1] the M/M/1 queueing model can be described by the following system of partial differential equations with integral boundary conditions:

$$\begin{gathered}
\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu \int_0^\infty p_1(x,t)\,dx, \\
\frac{\partial p_1(x,t)}{\partial t} + \frac{\partial p_1(x,t)}{\partial x} = -(\lambda+\mu)\,p_1(x,t), \\
\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} = -(\lambda+\mu)\,p_n(x,t) + \lambda p_{n-1}(x,t), \\
\forall n \geqslant 2, \\
p_1(0,t) = \mu \int_0^\infty p_2(x,t)\,dx + \lambda p_0(t), \\
p_n(0,t) = \mu \int_0^\infty p_{n+1}(x,t)\,dx, \quad \forall n \geqslant 2, \\
p_0(0) = \phi_0 \ge 0, \qquad p_n(x,0) = \phi_n(x) \ge 0, \quad \forall n \ge 1.
\end{gathered} \tag{2}$$

Here $(x,t) \in [0,\infty)\times[0,\infty)$; $\phi_0+\sum_{n=1}^{\infty}\int_0^\infty \phi_n(x)dx = 1$; λ is the mean arrival rate of customers; μ is the mean service rate of the server; $p_0(t)$ is the probability that the system is empty at time t; $p_n(x,t)$ is the probability that at time t there are n customers in the system and the service time of the customer undergoing service is x.

In this paper, we use the notations in [2, 6, 7]. Select a state space as follows:

$$\begin{aligned}
X = \Big\{ y \mid y \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\
\|y\| = |y_0| + \sum_{n=1}^{\infty} \|y_n\|_{L^1[0,\infty)} < \infty \Big\}.
\end{aligned} \tag{3}$$

It is obvious that X is a Banach space. Moreover, X is a Banach lattice under the following order relation:

$$\begin{aligned}
p^{(1)} \le p^{(2)} \Longleftrightarrow p_0^{(1)} \le p_0^{(2)}, \qquad p_n^{(1)}(x) \le p_n^{(2)}(x), \\
p^{(1)}, p^{(2)} \in X.
\end{aligned} \tag{4}$$

For simplicity, we introduce

$$\Gamma = \begin{pmatrix}
e^{-x} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\lambda e^{-x} & 0 & \mu & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \mu & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \mu & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \mu & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{5}$$

In the following we define operators and their domains:

$$\begin{aligned}
A\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}(x)
&= \begin{pmatrix}
-\lambda & 0 & 0 & \cdots \\
0 & -\dfrac{d}{dx} - (\lambda+\mu) & 0 & \cdots \\
0 & 0 & -\dfrac{d}{dx} - (\lambda+\mu) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\times \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}, \\
D(A) &= \Big\{ p \in X \mid \frac{dp_n(x)}{dx} \in L^1[0,\infty), \\
&\qquad p_n(x)\ (n \ge 1) \text{ are absolutely continuous and} \\
&\qquad p(0) = \int_0^\infty \Gamma p(x)\,dx, \\
&\qquad |p_0| + \sum_{n=1}^{\infty} \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} < \infty \Big\};
\end{aligned}$$

$$U\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \end{pmatrix}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ \vdots \end{pmatrix}, \quad D(U) = X;$$

$$E\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \end{pmatrix}(x) = \begin{pmatrix} \mu\int_0^\infty p_1(x)\,dx \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad D(E) = X. \tag{6}$$

Then (2) can be rewritten as a Cauchy problem in X:

$$\frac{dp(t)}{dt} = (A+U+E)\,p(t), \quad t \in (0,\infty)$$
$$p(0) = (\phi_0, \phi_1, \phi_2, \ldots), \tag{7}$$

where $A+U+E$ is called M/M/1 operator.

The following results can be found in Gupur et al. [2].

Theorem 1. *$A+U+E$ generates a positive contraction C_0-semigroup $T(t)$. $T(t)$ is isometric for $D(A^2)$. Hence, the system (7) has a unique positive time-dependent solution $p(x,t) = T(t)p(0)$ for $p(0) \in D(A^2)$ satisfying*

$$\|p(\cdot,t)\| = 1, \quad \forall t \in [0,\infty). \tag{8}$$

The set

$$\Big\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma + \lambda + \mu > 0,\ |\gamma+\lambda+\mu| > \mu,$$
$$\sup\left\{\frac{\lambda}{|\gamma+\lambda|}, \frac{\lambda|\gamma+\lambda+\mu|}{(\operatorname{Re}\gamma+\lambda+\mu)(|\gamma+\lambda+\mu|-\mu)}\right\} < 1\Big\} \tag{9}$$

belongs to the resolvent set of $(A+U+E)^*$, the adjoint operator of $A+U+E$. In particular, all points on the imaginary axis except 0 belong to the resolvent set of $A+U+E$. When $\lambda < \mu$, 0 is an eigenvalue of $A+U+E$ and $(A+U+E)^*$ with geometric multiplicity 1. Therefore, the time-dependent solution of the system (7) strongly converges to its steady-state solution:

$$\lim_{t\to\infty}\|p(\cdot,t) - p(\cdot)\| = 0; \tag{10}$$

here $p(x)$ is the eigenvector with respect to 0.

In 2008, Zhang and Gupur [6] obtained the following result.

Theorem 2. *If $\lambda < \mu$, then $2\sqrt{\lambda\mu} - \lambda - \mu$ is an eigenvalue of $A+U+E$ with geometric multiplicity 1.*

In 2011, Kasim and Gupur [7] proved the following result.

Theorem 3. *If $\lambda < \mu$, then all points in $(2\sqrt{\lambda\mu} - \lambda - \mu, 0)$ are eigenvalues of $A+U+E$ with geometric multiplicity 1.*

3. Main Results

Theorem 4. *If $\lambda < \mu$, then all points in the set*

$$\Big\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma + \mu > 0,$$
$$\left|\gamma + \lambda + \mu \pm \sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| < 2\mu\Big\} \cup \{0\} \tag{11}$$

are eigenvalues of $A+U+E$ with geometric multiplicity 1. In particular, the interval $(-\mu, 0]$ belongs to the point spectrum of $A+U+E$.

Remark 5. The condition $\lambda < \mu$ in Theorem 4, which was used by Cox [1], means that the service rate of the server is larger than the arrival rate of customers, thus preventing long queues of customers and therefore ensuring that the queueing system exists. In other words, the condition is a necessary condition for the existence of the M/M/1 queueing system.

Proof. We consider the equation $(\gamma I - A - U - E)p = 0$; that is,

$$(\gamma+\lambda)\,p_0 = \mu\int_0^\infty p_1(x)\,dx, \tag{12}$$

$$\frac{dp_1(x)}{dx} = -(\gamma+\lambda+\mu)\,p_1(x), \tag{13}$$

$$\frac{dp_n(x)}{dx} = -(\gamma+\lambda+\mu)\,p_n(x) + \lambda p_{n-1}(x), \quad n \geq 2, \tag{14}$$

$$p_1(0) = \mu\int_0^\infty p_2(x)\,dx + \lambda p_0, \tag{15}$$

$$p_n(0) = \mu\int_0^\infty p_{n+1}(x)\,dx, \quad n \geq 2. \tag{16}$$

By solving (13) and (14) we have

$$p_1(x) = a_1 e^{-(\gamma+\lambda+\mu)x}, \tag{17}$$

$$p_n(x) = a_n e^{-(\gamma+\lambda+\mu)x} + \lambda e^{-(\gamma+\lambda+\mu)x}\int_0^x p_{n-1}(\tau)\,e^{(\gamma+\lambda+\mu)\tau}d\tau, \quad n \geq 2. \tag{18}$$

By using (17) and (18) repeatedly we deduce

$$\begin{aligned} p_2(x) &= a_2 e^{-(\gamma+\lambda+\mu)x} + \lambda e^{-(\gamma+\lambda+\mu)x}\int_0^x p_1(\tau)\,e^{(\gamma+\lambda+\mu)\tau}d\tau \\ &= a_2 e^{-(\gamma+\lambda+\mu)x} + \lambda e^{-(\gamma+\lambda+\mu)x}\int_0^x a_1\,d\tau \\ &= a_2 e^{-(\gamma+\lambda+\mu)x} + \lambda x a_1 e^{-(\gamma+\lambda+\mu)x} \\ &= [a_2 + \lambda x a_1]\,e^{-(\gamma+\lambda+\mu)x} \end{aligned}$$

$$= \sum_{k=1}^{2} \frac{(\lambda x)^{2-k}}{(2-k)!} e^{-(\gamma+\lambda+\mu)x} a_k,$$

$$p_3(x) = \left[a_3 + \lambda x a_2 + \frac{(\lambda x)^2}{2!} a_1\right] e^{-(\gamma+\lambda+\mu)x} = \sum_{k=1}^{3} \frac{(\lambda x)^{3-k}}{(3-k)!} e^{-(\gamma+\lambda+\mu)x} a_k, \tag{19}$$

$$\vdots$$

$$p_n(x) = \left[a_n + \lambda x a_{n-1} + \frac{(\lambda x)^2}{2!} a_{n-2} + \cdots + \frac{(\lambda x)^{n-1}}{(n-1)!} a_1\right] \times e^{-(\gamma+\lambda+\mu)x} = \sum_{k=1}^{n} \frac{(\lambda x)^{n-k}}{(n-k)!} e^{-(\gamma+\lambda+\mu)x} a_k, \quad n \geq 1. \tag{20}$$

Equation (20) and $\int_0^\infty x^{n-k} e^{-(\gamma+\lambda+\mu)x} dx = (n-k)!/(\gamma+\lambda+\mu)^{n+1-k}$ for $\operatorname{Re}\gamma+\lambda+\mu > 0$ imply

$$\int_0^\infty p_n(x)\,dx = \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(n-k)!} a_k \int_0^\infty x^{n-k} e^{-(\gamma+\lambda+\mu)x} dx = \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(\gamma+\lambda+\mu)^{n+1-k}} a_k, \quad n \geq 1. \tag{21}$$

Combining (15), (16), and (21) gives

$$a_1 = \mu \int_0^\infty p_2(x)\,dx + \lambda p_0 = \frac{\mu}{\gamma+\lambda+\mu} a_2 + \frac{\lambda\mu}{(\gamma+\lambda+\mu)^2} a_1 + \lambda p_0, \tag{22}$$

$$a_n = \mu \int_0^\infty p_{n+1}(x)\,dx = \mu \sum_{k=1}^{n+1} \frac{\lambda^{n+1-k}}{(\gamma+\lambda+\mu)^{n+2-k}} a_k, \quad n \geq 2 \tag{23}$$

$$\Longrightarrow$$

$$a_{n+1} = \mu \sum_{k=1}^{n+2} \frac{\lambda^{n+2-k}}{(\gamma+\lambda+\mu)^{n+3-k}} a_k, \quad n \geq 1. \tag{24}$$

Multiplying (23) by $\lambda/(\gamma+\lambda+\mu)$ and subtracting it from (24) yield

$$a_{n+1} - \frac{\lambda}{\gamma+\lambda+\mu} a_n = \mu \sum_{k=1}^{n+2} \frac{\lambda^{n+2-k}}{(\gamma+\lambda+\mu)^{n+3-k}} a_k - \mu \sum_{k=1}^{n+1} \frac{\lambda^{n+2-k}}{(\gamma+\lambda+\mu)^{n+3-k}} a_k = \frac{\mu}{\gamma+\lambda+\mu} a_{n+2}$$

$$\Longrightarrow$$

$$a_{n+2} = \frac{\gamma+\lambda+\mu}{\mu} a_{n+1} - \frac{\lambda}{\mu} a_n, \quad n \geq 2. \tag{25}$$

If we assume

$$a_{n+2} - \xi a_{n+1} = \eta (a_{n+1} - \xi a_n) \Longleftrightarrow a_{n+2} - (\xi+\eta) a_{n+1} + \xi\eta a_n = 0, \quad n \geq 2, \tag{26}$$

then this together with (25) yields

$$\xi + \eta = \frac{\gamma+\lambda+\mu}{\mu}, \qquad \xi\eta = \frac{\lambda}{\mu}. \tag{27}$$

From (27) we determine

$$\xi = \frac{(\gamma+\lambda+\mu)/\mu + \sqrt{((\gamma+\lambda+\mu)/\mu)^2 - 4\lambda/\mu}}{2} = \frac{\gamma+\lambda+\mu + \sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu},$$
$$\eta = \frac{(\gamma+\lambda+\mu)/\mu - \sqrt{((\gamma+\lambda+\mu)/\mu)^2 - 4\lambda/\mu}}{2} = \frac{\gamma+\lambda+\mu - \sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}. \tag{28}$$

Equation (26) implies

$$a_{n+2} - \xi a_{n+1} = \eta (a_{n+1} - \xi a_n) = \eta^2 (a_n - \xi a_{n-1}) = \eta^3 (a_{n-1} - \xi a_{n-2}) = \cdots = \eta^{n-1} (a_3 - \xi a_2), \quad n \geq 2. \tag{29}$$

From (29) we know

$$a_{n+2} - \xi a_{n+1} = \eta^{n-1} (a_3 - \xi a_2),$$
$$a_{n+1} - \xi a_n = \eta^{n-2} (a_3 - \xi a_2) \Longrightarrow \xi a_{n+1} - \xi^2 a_n = \xi \eta^{n-2} (a_3 - \xi a_2),$$
$$a_n - \xi a_{n-1} = \eta^{n-3} (a_3 - \xi a_2) \Longrightarrow \xi^2 a_n - \xi^3 a_{n-1} = \xi^2 \eta^{n-3} (a_3 - \xi a_2),$$
$$a_{n-1} - \xi a_{n-2} = \eta^{n-4} (a_3 - \xi a_2) \Longrightarrow \xi^3 a_{n-1} - \xi^4 a_{n-2} = \xi^3 \eta^{n-4} (a_3 - \xi a_2),$$
$$\vdots$$

$$
\begin{aligned}
& a_5-\xi a_4=\eta^2\left(a_3-\xi a_2\right) \\
& \qquad \Longrightarrow \xi^{n-3} a_5-\xi^{n-2} a_4=\xi^{n-3} \eta^2\left(a_3-\xi a_2\right), \\
& a_4-\xi a_3=\eta\left(a_3-\xi a_2\right) \\
& \qquad \Longrightarrow \xi^{n-2} a_4-\xi^{n-1} a_3=\xi^{n-2} \eta\left(a_3-\xi a_2\right) .
\end{aligned} \tag{30}
$$

By using (30) we derive, for $n \geq 2$,

$$
\begin{aligned}
& a_{n+2}-\xi^{n-1} a_3 \\
& =\eta^{n-1}\left(a_3-\xi a_2\right)+\xi \eta^{n-2}\left(a_3-\xi a_2\right) \\
& \quad+\xi^2 \eta^{n-3}\left(a_3-\xi a_2\right) \\
& \quad+\cdots+\xi^{n-3} \eta^2\left(a_3-\xi a_2\right)+\xi^{n-2} \eta\left(a_3-\xi a_2\right) \\
& =\left(\eta^{n-1}+\xi \eta^{n-2}+\xi^2 \eta^{n-3}+\cdots+\xi^{n-3} \eta^2+\xi^{n-2} \eta\right) \\
& \quad \times\left(a_3-\xi a_2\right) \\
& \Longrightarrow \\
& a_{n+2} \\
& =\xi^{n-1} a_3+\left(\eta^{n-1}+\xi \eta^{n-2}+\xi^2 \eta^{n-3}\right. \\
& \qquad \left.+\cdots+\xi^{n-3} \eta^2+\xi^{n-2} \eta\right)\left(a_3-\xi a_2\right) \\
& =\left(\eta^{n-1}+\xi \eta^{n-2}+\xi^2 \eta^{n-3}\right. \\
& \qquad \left.+\cdots+\xi^{n-3} \eta^2+\xi^{n-2} \eta+\xi^{n-1}\right) a_3 \\
& \quad -\xi\left(\eta^{n-1}+\xi \eta^{n-2}+\xi^2 \eta^{n-3}\right. \\
& \qquad \left.+\cdots+\xi^{n-3} \eta^2+\xi^{n-2} \eta\right) a_2 \\
& =\begin{cases} n \xi^{n-1} a_3-(n-1) \xi^n a_2 & \text{if } \xi=\eta \\ \dfrac{\xi^n-\eta^n}{\xi-\eta} a_3-\xi\left(\dfrac{\xi^n-\eta^n}{\xi-\eta}-\xi^{n-1}\right) a_2 & \text{if } \xi \neq \eta \end{cases} \\
& =\begin{cases} n \xi^{n-1}\left[a_3-\dfrac{n-1}{n} \xi a_2\right] & \text{if } \xi=\eta \\ \dfrac{\xi^n-\eta^n}{\xi-\eta}\left(a_3-\xi a_2\right)+\xi^n a_2 & \text{if } \xi \neq \eta . \end{cases}
\end{aligned} \tag{31}
$$

By inserting (17) into (12) and noting $\operatorname{Re} \gamma+\lambda+\mu>0$ we have

$$
\begin{aligned}
& (\gamma+\lambda) p_0=\mu \int_0^{\infty} p_1(x) d x=\frac{\mu}{\gamma+\lambda+\mu} a_1 \\
& \qquad \Longrightarrow \\
& \qquad a_1=\frac{(\gamma+\lambda)(\gamma+\lambda+\mu)}{\mu} p_0 .
\end{aligned} \tag{32}
$$

By combining (22) with (32) and $\operatorname{Re} \gamma+\lambda+\mu>0$ it follows that

$$
\begin{aligned}
& a_1=\frac{\mu}{\gamma+\lambda+\mu} a_2+\frac{\lambda \mu}{(\gamma+\lambda+\mu)^2} a_1+\lambda p_0 \\
& \quad \Longrightarrow \\
& \frac{\mu}{\gamma+\lambda+\mu} a_2 \\
& \quad =\left[1-\frac{\lambda \mu}{(\gamma+\lambda+\mu)^2}\right] a_1-\lambda p_0 \\
& \quad =\frac{(\gamma+\lambda+\mu)^2-\lambda \mu}{(\gamma+\lambda+\mu)^2} a_1-\lambda p_0=\frac{(\gamma+\lambda+\mu)^2-\lambda \mu}{(\gamma+\lambda+\mu)^2} \\
& \quad \times \frac{(\gamma+\lambda)(\gamma+\lambda+\mu)}{\mu} p_0-\lambda p_0 \\
& \quad =\frac{(\gamma+\lambda)\left[(\gamma+\lambda+\mu)^2-\lambda \mu\right]}{\mu(\gamma+\lambda+\mu)} p_0-\lambda p_0 \\
& \quad =\frac{(\gamma+\lambda)\left[(\gamma+\lambda+\mu)^2-\lambda \mu\right]-\lambda \mu(\gamma+\lambda+\mu)}{\mu(\gamma+\lambda+\mu)} p_0 \\
& \quad \Longrightarrow \\
& a_2=\frac{(\gamma+\lambda)\left[(\gamma+\lambda+\mu)^2-\lambda \mu\right]-\lambda \mu(\gamma+\lambda+\mu)}{\mu^2} p_0 .
\end{aligned} \tag{33}
$$

Combining (23), (32), and (33) gives

$$
\begin{aligned}
& a_2=\frac{\mu}{\gamma+\lambda+\mu} a_3+\frac{\lambda \mu}{(\gamma+\lambda+\mu)^2} a_2+\frac{\lambda^2 \mu}{(\gamma+\lambda+\mu)^3} a_1 \\
& \quad \Longrightarrow \\
& \frac{\mu}{\gamma+\lambda+\mu} a_3 \\
& \quad =\left[1-\frac{\lambda \mu}{(\gamma+\lambda+\mu)^2}\right] a_2-\frac{\lambda^2 \mu}{(\gamma+\lambda+\mu)^3} a_1 \\
& \quad \Longrightarrow \\
& a_3=\frac{(\gamma+\lambda+\mu)^2-\lambda \mu}{\mu(\gamma+\lambda+\mu)} a_2-\frac{\lambda^2}{(\gamma+\lambda+\mu)^2} a_1 \\
& \quad =\frac{\left[(\gamma+\lambda+\mu)^2-\lambda \mu\right]}{\mu^3(\gamma+\lambda+\mu)} \\
& \quad \times\left\{(\gamma+\lambda)\left[(\gamma+\lambda+\mu)^2-\lambda \mu\right]\right. \\
& \qquad \left.-\lambda \mu(\gamma+\lambda+\mu)\right\} p_0-\frac{\lambda^2(\gamma+\lambda)}{\mu(\gamma+\lambda+\mu)} p_0
\end{aligned}
$$

$$= \{[(\gamma+\lambda+\mu)^2-\lambda\mu] \times \{(\gamma+\lambda)[(\gamma+\lambda+\mu)^2-\lambda\mu] - \lambda\mu(\gamma+\lambda+\mu)\} - \lambda^2\mu^2(\gamma+\lambda)\} \cdot (\{\mu^3(\gamma+\lambda+\mu)\} p_0)^{-1}. \quad (34)$$

From (21), $\int_0^\infty x^{n-k}|e^{-(\gamma+\lambda+\mu)x}|dx = (n-k)!/(\operatorname{Re}\gamma+\lambda+\mu)^{n+1}$ for $\operatorname{Re}\gamma+\lambda+\mu > 0$, and the Cauchy product of series we estimate, when $\operatorname{Re}\gamma+\mu > 0$

$$\|p_n\|_{L^1[0,\infty)} \le \sum_{k=1}^{n} \frac{\lambda^{n-k}}{(\operatorname{Re}\gamma+\lambda+\mu)^{n+1-k}}|a_k|, \quad n \ge 1$$

$$\Longrightarrow$$

$$\sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} \le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n-k}}{(\operatorname{Re}\gamma+\lambda+\mu)^{n+1-k}}|a_k| = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(\operatorname{Re}\gamma+\lambda+\mu)^{n}}\sum_{n=1}^{\infty}|a_n| = \frac{1}{\operatorname{Re}\gamma+\mu}\sum_{n=1}^{\infty}|a_n|. \quad (35)$$

For simplicity, we introduce

$$\Lambda := \left\{\gamma\in\mathbb{C} \mid \operatorname{Re}\gamma+\mu>0, \left|\gamma+\lambda+\mu\pm\sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu}\right| < 2\mu\right\}. \quad (36)$$

It is easy to see that

$$\gamma\in\Lambda \Longleftrightarrow \operatorname{Re}\gamma+\mu>0, \quad |\xi|<1, \quad |\eta|<1. \quad (37)$$

Together with (35) and (31) to (34), this yields

$$\sum_{n=1}^{\infty}|a_n| = |a_1|+|a_2|+|a_3|+\sum_{n=2}^{\infty}|a_{n+2}| < \infty$$

$$\Longrightarrow$$

$$\|p\| = |p_0| + \sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} = |p_0| + \frac{1}{\operatorname{Re}\gamma+\mu}\sum_{n=1}^{\infty}|a_n| < \infty. \quad (38)$$

That is, all $\gamma\in\Lambda$ are eigenvalues of $A+U+E$. Moreover, from (20) and (31) to (34) it is easy to see that the eigenvectors corresponding to each γ span 1-dimensional linear space; that is, their geometric multiplicity is one.

In the following, we discuss the case that γ is a real number and obtain explicit results.

Since Theorem 1 implies that all $\gamma > 0$ belong to the resolvent set of $A+U+E$, $\gamma\in\mathbb{R}$ includes the following three cases.

(1) If $(\gamma+\lambda+\mu)^2 > 4\lambda\mu \Rightarrow \gamma+\lambda+\mu > 2\sqrt{\lambda\mu} \Rightarrow \gamma > 2\sqrt{\lambda\mu}-\lambda-\mu$, then by noting $\lambda<\mu$,

$$\begin{aligned}
\gamma<0 &\Longrightarrow \gamma+\lambda-\lambda<0 \\
&\Longrightarrow 4\mu(\gamma+\lambda)-4\lambda\mu<0 \\
&\Longrightarrow 2\mu(\gamma+\lambda)-4\lambda\mu < -2\mu(\gamma+\lambda) \\
&\Longrightarrow (\gamma+\lambda)^2+2\mu(\gamma+\lambda)+\mu^2-4\lambda\mu \\
&\quad < (\gamma+\lambda)^2-2\mu(\gamma+\lambda)+\mu^2 \\
&\Longrightarrow 0<(\gamma+\lambda+\mu)^2-4\lambda\mu<(\gamma+\lambda-\mu)^2 \\
&\Longrightarrow \sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu} < -(\gamma+\lambda-\mu) \\
&\Longrightarrow \gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu} < 2\mu \\
&\Longrightarrow \frac{\gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu}}{2\mu} < 1 \\
&\Longrightarrow |\xi| = \left|\frac{\gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu}}{2\mu}\right| < 1, \\
|\eta| &= \left|\frac{\gamma+\lambda+\mu-\sqrt{(\gamma+\lambda+\mu)^2-4\lambda\mu}}{2\mu}\right| < 1 \\
&\Longrightarrow \gamma\in\Lambda.
\end{aligned} \quad (39)$$

This together with (38) implies that all points in $(2\sqrt{\lambda\mu}-\lambda-\mu, 0)$ are eigenvalues of $A+U+E$, which is the main result in Kasim and Gupur [7] (see Theorem 3).

By applying the condition $\lambda<\mu$ we have

$$\begin{aligned}
\gamma=0 &\Longrightarrow \xi = \frac{\lambda+\mu+\sqrt{(\lambda+\mu)^2-4\lambda\mu}}{2\mu} \\
&= \frac{\lambda+\mu+\sqrt{(\mu-\lambda)^2}}{2\mu} = \frac{\lambda+\mu+\mu-\lambda}{2\mu} = 1, \\
\eta &= \frac{\lambda+\mu-\sqrt{(\lambda+\mu)^2-4\lambda\mu}}{2\mu} = \frac{\lambda+\mu-(\mu-\lambda)}{2\mu} \\
&= \frac{\lambda}{\mu} < 1.
\end{aligned} \quad (40)$$

Since $\gamma = 0$, (33) and (34) are simplified as

$$a_2 = \frac{\lambda^3}{\mu^2} p_0 = \eta^2 \lambda p_0, \qquad a_3 = \frac{\lambda^4}{\mu^3} p_0 = \eta^3 \lambda p_0. \tag{41}$$

By combining (40) and (41) with (31) and using (35) we have

$$\begin{aligned} a_{n+2} &= \frac{1-\eta^n}{1-\eta}(a_3 - a_2) + a_2 \\ &= \frac{1-\eta^n}{1-\eta}\left(\eta^3\lambda p_0 - \eta^2 \lambda p_0\right) + \eta^2\lambda p_0 \\ &= -\eta^2(1-\eta^n)\lambda p_0 + \eta^2 \lambda p_0 = \eta^2 \lambda p_0 [\eta^n - 1 + 1] \\ &= \eta^{n+2}\lambda p_0, \quad n \geq 2 \\ &\Longrightarrow \\ \sum_{n=1}^{\infty} |a_n| &= |a_1| + |a_2| + |a_3| + \sum_{n=2}^{\infty} |a_{n+2}| < \infty \\ &\Longrightarrow \\ \|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} < \infty. \end{aligned} \tag{42}$$

This means that 0 is an eigenvalue of $A + U + E$, which is the result in Gupur et al. [2] (see Theorem 1).

(2) If $(\gamma + \lambda + \mu)^2 = 4\lambda\mu \Rightarrow \gamma + \lambda + \mu = 2\sqrt{\lambda\mu} \Rightarrow \gamma = 2\sqrt{\lambda\mu} - \lambda - \mu$, then the condition $\lambda < \mu$ implies

$$\begin{aligned} \xi = \eta = \frac{\gamma+\lambda+\mu}{2\mu} &\Longrightarrow |\xi| = |\eta| = \frac{|\gamma+\lambda+\mu|}{2\mu} = \frac{2\sqrt{\lambda\mu}}{2\mu} \\ &= \sqrt{\frac{\lambda}{\mu}} < 1 \\ &\Longrightarrow \gamma \in \Lambda. \end{aligned} \tag{43}$$

Hence, $\gamma = 2\sqrt{\lambda\mu} - \lambda - \mu$ is an eigenvalue of $A + U + E$, which is the main result in Zhang and Gupur [6] (see Theorem 2).

(3) If $(\gamma + \lambda + \mu)^2 < 4\lambda\mu \Rightarrow \gamma + \lambda + \mu < 2\sqrt{\lambda\mu} \Rightarrow \gamma < 2\sqrt{\lambda\mu} - \lambda - \mu$, then the condition $\lambda < \mu$ gives

$$\begin{aligned} \xi &= \frac{\gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu} \\ &= \frac{\gamma+\lambda+\mu+i\sqrt{4\lambda\mu - (\gamma+\lambda+\mu)^2}}{2\mu} \\ &\Longrightarrow \\ |\xi| = |\eta| &= \frac{\sqrt{(\gamma+\lambda+\mu)^2 + 4\lambda\mu - (\gamma+\lambda+\mu)^2}}{2\mu} \\ &= \frac{2\sqrt{\lambda\mu}}{2\mu} = \sqrt{\frac{\lambda}{\mu}} < 1 \\ &\Longrightarrow \gamma \in \Lambda. \end{aligned} \tag{44}$$

This yields that all points in $(-\mu, 2\sqrt{\lambda\mu} - \lambda - \mu)$ are eigenvalues of $A + U + E$.

By summarizing the above discussion we conclude that all points in

$$\begin{aligned} &\Lambda \cup \{0\} \\ &= \Big\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma + \mu > 0, \\ &\qquad \left|\gamma+\lambda+\mu \pm \sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| < 2\mu\Big\} \cup \{0\} \end{aligned} \tag{45}$$

are eigenvalues of $A + U + E$ with geometric multiplicity one. In particular, $(-\mu, 0]$ belongs to the point spectrum of $A + U + E$. □

Remark 6. From (31) it is easy to see that if $|\xi| > 1$ and $|\eta| < 1$ or $|\xi| < 1$ and $|\eta| > 1$, then

$$\sum_{n=2}^{\infty} |a_{n+2}| = \infty \Longrightarrow \sum_{n=1}^{\infty} |a_n| = \infty \Longrightarrow \|p\| = \infty. \tag{46}$$

That is, there are no eigenvalues in

$$\begin{aligned} \Big\{\gamma \in \mathbb{C} \mid &\left|\gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| > 2\mu, \\ &\left|\gamma+\lambda+\mu-\sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| < 2\mu \\ \text{or } &\left|\gamma+\lambda+\mu+\sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| < 2\mu, \\ &\left|\gamma+\lambda+\mu-\sqrt{(\gamma+\lambda+\mu)^2 - 4\lambda\mu}\right| > 2\mu\Big\}. \end{aligned} \tag{47}$$

4. Conclusion and Discussion

Let $\sigma_p(T(t))$ and $\sigma_p(A + U + E)$ be the point spectrum of $T(t)$ and $A + U + E$, respectively. From Theorem 4 and the spectral mapping theorem for the point spectrum ([8], p. 277)

$$\sigma_p(T(t)) = e^{t\sigma_p(A+U+E)} \cup \{0\} \tag{48}$$

we know that $T(t)$ has uncountable eigenvalues and therefore it is not compact, even not eventually compact ([8], p. 330).

Corollary 2.11 in Engel and Nagel [8], p. 258, states that if $T(t)$ is a C_0-semigroup on the Banach space X with generator $A+U+E$, then

(I) $\omega_0 = \max\{\omega_{ess}, s(A+U+E)\}$, where ω_0 is the growth bound of $T(t)$, ω_{ess} is the essential growth bound of $T(t)$, and $s(A+U+E)$ is the spectral bound of $A+U+E$.

(II) $\sigma(A+U+E)\cap\{\gamma\in\mathbb{C} \mid \operatorname{Re}\gamma \geq w\}$ is finite for each $w > \omega_{ess}$. Here, $\sigma(A+U+E)$ is the spectrum of $A+U+E$.

Theorem 1 implies that $\omega_0 = 0$ and $s(A+U+E) = 0$. These together with items (I) and (II) above yield $\omega_{ess} = 0$. From this and Proposition 3.5 in ([8], p. 332), we conclude that $T(t)$ is not quasi-compact. Hence, queueing models are essentially different from the population equations ([9, 10]) and the reliability models that are described by a finite number of partial differential equations with integral boundary conditions ([4, 11]).

Since $\omega_0 = 0$ and $\omega_{ess} = 0$, from Nagel ([13], p. 74), it follows that

$$r(T(t)) = r_{ess}(T(t)) = e^{\omega_{ess}t} = e^0 = 1, \quad (49)$$

where $r(T(t))$ and $r_{ess}(T(t))$ are the spectral radius and essential spectral radius of $T(t)$, respectively.

Let $p^{(0)}(x)$ be an eigenvector with respect to 0 in Theorem 1 and let $p^{(\epsilon)}(x)$ be eigenvectors with respect to $-\mu\epsilon$ for $\epsilon\in(0,1)$ in Theorem 4. Then, by using $(A+U+E)p^{(0)}(x) = 0$ and $(A+U+E)p^{(\epsilon)}(x) = -\mu\epsilon p^{(\epsilon)}(x)$, we have

$$\begin{aligned} T(t)\left(p^{(0)}(x) + (A+U+E)\,p^{(\epsilon)}(x)\right) &= T(t)\,p^{(0)}(x) + T(t)(A+U+E)\,p^{(\epsilon)}(x) \\ &= p^{(0)}(x) + T(t)\left[-\mu\epsilon p^{(\epsilon)}(x)\right] \\ &= p^{(0)}(x) - \mu\epsilon T(t)\,p^{(\epsilon)}(x) \\ &= p^{(0)}(x) - \mu\epsilon e^{-\mu\epsilon t}p^{(\epsilon)}(x) \\ &\Longrightarrow \\ \left\|T(t)\left(p^{(0)}(\cdot) + (A+U+E)\,p^{(\epsilon)}(\cdot)\right) - p^{(0)}(\cdot)\right\| &= \mu\epsilon e^{-\mu\epsilon t}\left\|p^{(\epsilon)}\right\|, \quad \forall t\geq 0,\ \forall\epsilon\in(0,1). \end{aligned} \quad (50)$$

This means that there are no positive constants $\mathscr{M} > 0$ and $\overline{\omega} > 0$ such that

$$\begin{aligned} &\left\|T(t)\left(p^{(0)}(\cdot) + (A+U+E)\,p(\cdot)\right) - p^{(0)}(\cdot)\right\| \\ &\leq \mathscr{M}e^{-\overline{\omega}t}\|p\|, \quad \forall t\geq 0,\ \forall p\in D(A). \end{aligned} \quad (51)$$

That is, it is impossible that the time-dependent solution of the system (7) exponentially converges to its steady-state solution. In other words, the convergence result given in Theorem 1 is optimal.

From Theorems 1 and 4 and Browder [14], Kato [15], and Schechter [16] we know that the set

$$\begin{aligned} \{\gamma\in\mathbb{C} \mid &\operatorname{Re}\gamma+\lambda+\mu\leq 0 \text{ or } |\gamma+\lambda+\mu|\leq\mu \\ &\text{or } \lambda\geq|\gamma+\lambda| \text{ or } \\ &\lambda|\gamma+\lambda+\mu| \\ &\geq(\operatorname{Re}\gamma+\lambda+\mu)\left(|\gamma+\lambda+\mu|-\mu\right)\}\setminus\{\Lambda\cup\{0\}\} \end{aligned} \quad (52)$$

probably implies essential spectrum of $A+U+E$, which is our next research topic.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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