Research Article

Nontrivial Solutions of a Fully Fourth-Order Periodic Boundary Value Problem

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We investigate the solvability of a fully fourth-order periodic boundary value problem of the form $x^{(4)} = f(t, x, x', x'', x''')$, $x^{(i)}(0) = x^{(i)}(T)$, $i = 0, 1, 2, 3$, where $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies Carathéodory conditions. By using the coincidence degree theory, the existence of nontrivial solutions is obtained. Meanwhile, as applications, some examples are given to illustrate our results.

1. Introduction

In this paper, we consider a fully nonlinear fourth-order periodic boundary value problem of the form

$$x^{(4)} = f\left(t, x, x', x'', x'''\right), \tag{1}$$

subject to the boundary conditions

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, 2, 3, \tag{2}$$

where $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R} = (-\infty, +\infty)$ satisfies Carathéodory conditions; that is,

(i) for a.e. $t \in [0, T]$, the function $f(t, x_0, x_1, x_2, x_3) : \mathbb{R}^4 \to \mathbb{R}$ is continuous;

(ii) for every $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, the function $f(t, x_0, x_1, x_2, x_3) : [0, T] \to \mathbb{R}$ is measurable;

(iii) for each $\rho > 0$, there is a real valued function $h_\rho(\cdot) \in L^1[0, T]$ such that

$$\left| f\left(t, x_0, x_1, x_2, x_3\right) \right| \le h_\rho(t), \tag{3}$$

for a.e. $t \in [0, T]$ and $x_0^2 + x_1^2 + x_2^2 + x_3^2 < \rho^2$.

It is well known that fourth-order periodic boundary value problems are important research topics which arise in a variety of different areas, such as nonlinear oscillations, fluid

mechanical, and nonlinear elastic mechanical phenomena, and thus have been extensively studied; for instance, see [1–30] and references therein. However, most of the works in the above-mentioned references allow only having t, x or t, x, x'' in the right-hand side nonlinear function f; see [2–11, 13, 15–18, 20–30]. The works on the fully nonlinear cases of which f contains explicitly t and every derivative of x up to order three have been quite rarely seen; see [1, 12, 14, 19].

The aim of this paper is to establish the existence of solutions and nontrivial solutions for the fully nonlinear fourth-order PBVP (1), (2). Our main tool is the coincidence degree theory. The paper [31] motivated our study.

2. Preliminary

In this section, we present some lemmas which are needed for our main results.

At first, we will briefly recall some notations that are needed for our discussion.

Let X, Z be real Banach spaces. A linear mapping $L : \operatorname{dom} L \subset X \to Z$ will be called a *Fredholm mapping of index zero* if the following two conditions hold:

(i) $\operatorname{Im} L$ is a closed subspace of Z;

(ii) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Let $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping of index zero; then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \tag{4}$$

so that

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Z = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{5}$$

It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. Let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$; the map $N : X \to Z$ will be called L-compact on $\overline{\Omega}$, if $QN : \overline{\Omega} \to Z$ and $K_P(I - Q)N : \overline{\Omega} \to X$ are compact.

Lemma 1 (see [32]). *Let $L : \operatorname{dom} L \to Z$ be a linear Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. Let $N : \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$ and let $A : X \to Z$ be L-completely continuous such that*

(i) $\operatorname{Ker}(L - A) = \{0\}$;

(ii) *for every $(x, \lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times (0, 1)$,*

$$Lx - (1 - \lambda) Ax - \lambda Nx \neq 0, \tag{6}$$

and assume that $0 \in \Omega$. Then equation

$$Lx = Nx \tag{7}$$

has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Lemma 2 (see [33]). *Let $L : \operatorname{dom} L \to Z$ be a linear Fredholm mapping of index zero and let $\Omega \subset X$ be an open bounded set. Let $N : \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$ and the coincidence degree $D[(L, N), \Omega]$ is well defined. If there exists $z \in Z$ with $z \neq 0$ such that*

$$Lx - Nx \neq \lambda z, \quad \forall (x, \lambda) \in (\operatorname{dom} L \cap \partial\Omega) \times [0, +\infty), \tag{8}$$

then $D[(L, N), \Omega] = 0$.

In the following, we take Banach space $X = C^3[0, T]$ with the norm $\|x\|_{C^3} = \sum_{i=0}^{3} \|x^{(i)}\|_\infty$, and $Z = L^1[0, T]$. Define a linear map $L : \operatorname{dom} L \subset X \to Z$ by

$$(Lx)(t) = x^{(4)}(t), \quad \forall x \in \operatorname{dom} L, \tag{9}$$

where $\operatorname{dom} L = \{x \in W^{4,1}(0, T) : x^{(i)}(0) = x^{(i)}(T), i = 0, 1, 2, 3\}$ and $W^{4,1}(0, T)$ is the usual Sobolev space. It is easy to see that L is a Fredholm mapping of index zero. Also define a nonlinear map $N : X \to Z$ by

$$(Nx)(t) = f\left(t, x, x', x'', x'''\right), \quad \forall x \in X. \tag{10}$$

Define two projects $P : X \to X$ and $Q : Z \to Z$ as follows:

$$Px = \frac{1}{T} \int_0^T x(t)\, dt, \quad \forall x \in X;$$

$$\tag{11}$$

$$Qz = \frac{1}{T} \int_0^T z(t)\, dt, \quad \forall z \in Z.$$

Let $G(t, s)$ be Green function for the homogeneous BVP

$$x^{(4)}(t) = 0, \quad 0 < t < T,$$

$$\tag{12}$$

$$\int_0^T x(t)\, dt = 0, \qquad x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, 2, 3.$$

Then $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be given by

$$(K_P z)(t) = \int_0^1 G(t, s) z(s)\, ds, \quad \forall z \in \operatorname{Im} L. \tag{13}$$

Hence the map $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ is continuous. We note that if $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfying Carathéodory conditions, then $N : X \to Z$ is bounded and continuous by Lebesgue's dominated convergence theorem. Furthermore, N is L-compact on every bounded set $\overline{\Omega} \subset X$.

3. Main Results

For $h \in L^1[0, T]$ and $r \in (0, +\infty)$ we put

$$h_0 = \exp\left(2 \int_0^T h(t)\, dt\right), \qquad r_0 = r + e h_0 T^3;$$

$$\varepsilon \in \left(0, \min\left\{\frac{1}{2 r_0 T}, \left(\frac{2\pi}{T}\right)^4\right\}\right), \qquad r_3 = h_0 \exp\left(2\varepsilon r_0 T\right);$$

$$r_2 = r_3 T, \qquad r_1 = r_2 T.$$

$$\tag{14}$$

In order to introduce our main theorem, we need some lemmas.

Lemma 3. *Let $h \in L^1[0, T]$ be a nonnegative function and $r \in (0, +\infty)$. Let $r_0, r_1, r_2, r_3,$ and ε fulfil (14). Then for any $x \in \operatorname{dom} L$, the inequalities*

$$|x(t)| \leq r_0, \qquad |x'(t)| \leq r_1, \qquad |x''(t)| \leq r_2,$$

$$\forall t \in [0, T], \tag{15}$$

$$x^{(4)}(t)\, \operatorname{sign} x'''(t) \leq h(t) |x'''(t)| + \varepsilon |x(t)|$$

$$\text{for a.e. } t \in \{t \in [0, T] : |x'''(t)| \geq 1\} \tag{16}$$

imply

$$|x'''(t)| < r_3, \quad \forall t \in [0, T]. \tag{17}$$

Proof. Since $x \in \operatorname{dom} L$, there exists $t_0 \in (0, T)$ such that

$$x'''(t_0) = 0. \tag{18}$$

We will show that

$$|x'''(t)| < \sqrt{r_3}, \quad \forall t \in [t_0, T]. \tag{19}$$

By contradiction, assume that there exists $t_1 \in (t_0, T]$ such that

$$|x'''(t_1)| \geq \sqrt{r_3}. \tag{20}$$

Then there exists $t_2 \in (t_0, t_1)$ such that

$$\left|x'''(t_2)\right| = 1, \qquad \left|x'''(t)\right| \geq 1, \quad \forall t \in [t_2, t_1]. \qquad (21)$$

There are two cases to consider.

Case 1. Consider $x'''(t) \geq 1$ on $[t_2, t_1]$. In this case, integrating (16) from t_2 to t_1 and using (15) and (21) we infer that

$$\int_{t_2}^{t_1} \frac{x^{(4)}(t)}{x'''(t)} \, dt \leq \int_{t_2}^{t_1} \left(h(t) + \varepsilon r_0\right) dt < \int_0^T h(t) \, dt + \varepsilon r_0 T. \qquad (22)$$

Thus $x'''(t_1) < \sqrt{r_3}$, which contradicts (20).

Case 2. Consider $x'''(t) \leq -1$ on $[t_2, t_1]$. Similar to Case 1, we have

$$\int_{t_2}^{t_1} \frac{-x^{(4)}(t)}{-x'''(t)} \, dt \leq \int_{t_2}^{t_1} \left(h(t) + \varepsilon r_0\right) dt < \int_0^T h(t) \, dt + \varepsilon r_0 T. \qquad (23)$$

Thus, $x'''(t_1) > -\sqrt{r_3}$, which contradicts (20). Therefore (19) is true. Furthermore, from the fact $x'''(0) = x'''(T)$ it follows that $|x'''(0)| < \sqrt{r_3}$.

Finally, we show that

$$\left|x'''(t)\right| < r_3, \quad \forall t \in [0, t_0]. \qquad (24)$$

Suppose on the contrary that there exists $t_3 \in (0, t_0)$ satisfying

$$\left|x'''(t_3)\right| \geq r_3. \qquad (25)$$

Then there exists $t_4 \in (0, t_3)$ such that

$$\left|x'''(t_4)\right| = \sqrt{r_3}, \qquad \left|x'''(t)\right| \geq \sqrt{r_3}, \quad \forall t \in [t_4, t_3]. \quad (26)$$

There are two cases to consider.

Case 1′. Consider $x'''(t) \geq \sqrt{r_3}$ on $[t_4, t_3]$. Similar to Case 1, we have $x'''(t_3) < r_3$, which contradicts (25).

Case 2′. Consider $x'''(t) \leq -\sqrt{r_3}$ on $[t_4, t_3]$. Similar to Case 2, one has $x'''(t_3) > -r_3$, which also contradicts (25). Hence (24) is true.

In summary, from (19) and (24) it follows that estimate (17) holds. This completes the proof of the lemma. □

Lemma 4. *Let* $r \in (0, +\infty)$ *and let* $h \in L^1[0, T]$ *be a nonnegative function. Let* $r_0, r_1, r_2, r_3,$ *and* ε *fulfil* (14). *Then for any function* $x \in \text{dom } L$ *the inequalities*

$$\left|x'''(t)\right| < r_3, \quad \forall t \in [0, T], \qquad (27)$$

$$\mu x^{(4)}(t) \, \text{sign} \, x(t) > 0 \quad \text{for a.e. } t \in \{t \in [0, T] : |x(t)| \geq r\} \qquad (28)$$

imply

$$|x(t)| < r_0, \qquad \left|x'(t)\right| < r_1, \qquad \left|x''(t)\right| < r_2, \\ \forall t \in [0, T]. \qquad (29)$$

Proof. For every $x \in \text{dom } L$, from (28), there exist $t_0, t_1, t_2 \in [0, T]$ such that

$$\left|x(t_0)\right| < r, \qquad x'(t_1) = 0, \qquad x''(t_2) = 0. \qquad (30)$$

Integrating (27) by (14) and (30) we get

$$\left|x''(t)\right| < r_3 T = r_2, \qquad \left|x'(t)\right| < r_2 T = r_1 \quad \text{on } [0, T],$$

$$|x(t)| < r + r_2 T^2 < r_0 \quad \text{on } [0, T]. \qquad (31)$$

This completes the proof of the lemma. □

Now, we apply Lemma 1 to establish the existence results of solutions for the fourth-order PBVP (1), (2).

Theorem 5. *Assume that there exist* $\mu \in \{-1, 1\}$, $r \in (0, +\infty)$, *and a nonnegative function* $h \in L^1[0, T]$. *Suppose further that*

(H_0) $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ *satisfies the Carathéodory conditions;*

(H_1) *if* $|x_0| \geq r$, $|x_1| \leq r_1$, $|x_2| \leq r_2$, $|x_3| \leq r_3$, *then*

$$\mu f(t, x_0, x_1, x_2, x_3) \, \text{sign} \, x_0 \geq 0 \quad \text{for a.e. } t \in [0, T]; \quad (32)$$

(H_2) *if* $|x_0| \leq r_0$, $|x_1| \leq r_1$, $|x_2| \leq r_2$, $|x_3| \geq 1$, *then*

$$f(t, x_0, x_1, x_2, x_3) \, \text{sign} \, x_3 \leq h(t) \left|x_3\right| \quad \text{for a.e. } t \in [0, T], \qquad (33)$$

where r_0, r_1, r_2, r_3 *fulfil* (14). *Then PBVP* (1), (2) *has at least one solution* x *such that*

$$\left|x^{(i)}(t)\right| \leq r_i, \quad i = 0, 1, 2, 3, \, \forall t \in [0, T]. \qquad (34)$$

Proof. Let

$$\Omega = \left\{x \in X : \left|x^{(i)}(t)\right| < r_i, \, i = 0, 1, 2, 3, \, t \in [0, T]\right\}. \quad (35)$$

Then $x \in \partial \Omega$ iff there exist some $j \in \{0, 1, 2, 3\}$ such that

$$\max_{0 \leq t \leq T} \left|x^{(j)}(t)\right| = r_j, \qquad \left|x^{(i)}(t)\right| \leq r_i, \quad i = 0, 1, 2, 3,$$

$$t \in [0, T]. \qquad (36)$$

Now, we show that

$$\text{Ker}(L - A) = \{0\}, \qquad (37)$$

where $A : X \to Z$, $x \mapsto \mu \varepsilon x$. To do this, we assume that $x(t)$ is the solution of the following periodic boundary value problem:

$$x^{(4)}(t) = \mu \varepsilon x(t), \quad \text{a.e. } t \in (0, T),$$

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, 2, 3. \qquad (38)$$

Integrating the equation as above on $[0, T]$, we obtain

$$\int_0^T x(t) \, dt = 0. \qquad (39)$$

Thus, by Wirtinger inequality,

$$\left\| x^{(i)} \right\|_2 \le \frac{T}{2\pi} \left\| x^{(i+1)} \right\|_2, \quad i = 0, 1, 2, 3. \tag{40}$$

Hence from (38) it follows that

$$\|x\|_2 \le \left(\frac{T}{2\pi} \right)^4 \left\| x^{(4)} \right\|_2 \le \varepsilon \left(\frac{T}{2\pi} \right)^4 \|x\|_2. \tag{41}$$

If $\|x\|_2 > 0$, then, from $\varepsilon < (2\pi/T)^4$, the following contradiction holds:

$$\|x\|_2 < \|x\|_2. \tag{42}$$

Therefore, $\mathrm{Ker}(L - A) = \{0\}$.

Finally, we show that, for every $(x, \lambda) \in (\mathrm{dom}\, L \cap \partial\Omega) \times (0, 1)$,

$$Lx - (1 - \lambda) Ax - \lambda Nx \ne 0. \tag{43}$$

To do this, let $\lambda \in (0, 1)$ and let $x_\lambda \in \overline{\Omega}$ be a solution of the following PBVP:

$$\begin{aligned} x^{(4)} &= \lambda f\left(t, x, x', x'', x''' \right) + (1 - \lambda) \mu \varepsilon x, \\ x^{(i)}(0) &= x^{(i)}(T), \quad i = 0, 1, 2, 3. \end{aligned} \tag{44}$$

Then $x_\lambda \notin \partial\Omega$. In fact, let

$$\left| x_\lambda(t) \right| \le r_0, \qquad \left| x_\lambda'(t) \right| \le r_1, \qquad \left| x_\lambda''(t) \right| \le r_2, \\ \forall t \in [0, T]. \tag{45}$$

Then, by (33),

$$\begin{aligned} x_\lambda^{(4)} \,\mathrm{sign}\, x_\lambda''' &= \lambda f \,\mathrm{sign}\, x_\lambda''' \\ &\quad + (1 - \lambda) \mu \varepsilon x_\lambda \,\mathrm{sign}\, x_\lambda''' \le h(t) \left| x_\lambda''' \right| + \varepsilon \left| x_\lambda \right|, \end{aligned} \tag{46}$$

for a.e. $t \in \{t \in [0, T] : |x_\lambda'''(t)| \ge 1\}$. Applying Lemma 3, we obtain

$$\left| x_\lambda'''(t) \right| < r_3, \quad \forall t \in [0, T]. \tag{47}$$

Thus according to (32), we have

$$\mu x_\lambda^{(4)} \,\mathrm{sign}\, x_\lambda = \lambda \mu f \,\mathrm{sign}\, x_\lambda + \mu^2 (1 - \lambda) \varepsilon \left| x_\lambda \right| > 0, \tag{48}$$

for a.e. $t \in \{t \in [0, T] : |x_\lambda(t)| \ge r\}$. It follows from Lemma 4 that

$$\left| x_\lambda(t) \right| < r_0, \qquad \left| x_\lambda'(t) \right| < r_1, \qquad \left| x_\lambda''(t) \right| < r_2, \\ \forall t \in [0, T]. \tag{49}$$

Thus $x_\lambda \notin \partial\Omega$. This implies that condition (ii) of Lemma 1 is valid.

In summary, all conditions of Lemma 1 are satisfied. Therefore the conclusion of Theorem 5 holds. This completes the proof of the theorem. $\qquad \square$

Next, we establish the existence result of nontrivial solutions for the fourth-order PBVP (1), (2) by means of Lemma 2.

Theorem 6. *Assume that all conditions in Theorem 5 hold with the exception of (H_1), which is replaced by the following:*

(H_1') *there exists a constant $r^* \in (0, r)$ such that if $x_0 > -r^*$, $|x_1| \le r_1$, $|x_2| \le r_2$, $|x_3| \le r_3$, then*

$$\mu f\left(t, x_0, x_1, x_2, x_3 \right) \ge 0 \quad \textit{for a.e. } t \in [0, T], \tag{50}$$

and if $x_0 < -r$, $|x_1| \le r_1$, $|x_2| \le r_2$, $|x_3| \le r_3$, then

$$\mu f\left(t, x_0, x_1, x_2, x_3 \right) \le 0 \quad \textit{for a.e. } t \in [0, T]. \tag{51}$$

Then PBVP (1), (2) has at least one nontrivial solution x satisfying (34).

Proof. From the proof of Theorem 5 and Lemma 1, it follows that $Lx = Nx$ has a solution in

$$\Omega = \left\{ x \in X : \left| x^{(i)}(t) \right| < r_i, \ i = 0, 1, 2, 3, \ t \in [0, T] \right\}, \tag{52}$$

and $|D[(L, N), \Omega]| = 1$. Without loss of generality, we assume that $\mu = 1$ and $B_{r^*} = \{x : \|x\|_{C^3} < r^*\} \subset \Omega$. We also assume that $Lx \ne Nx$ for all $x \in \partial B_{r^*}$.

Now we assert that

$$D\left[(L, N), B_{r^*} \right] = 0. \tag{53}$$

In fact, suppose that there exist $x_0 \in \partial B_{r^*}$ and $\lambda_0 > 0$ such that

$$Lx_0 - Nx_0 = \lambda_0. \tag{54}$$

Applying Q to both sides of above equality, it follows that

$$-QNx_0 = \lambda_0 Q(1); \tag{55}$$

that is,

$$-\frac{1}{T} \int_0^T f\left(t, x_0(t), x_0'(t), x_0''(t), x_0'''(t) \right) \mathrm{d}t = \lambda_0. \tag{56}$$

Notice that $\|x_0\|_{C^3} = r^*$ and $B_{r^*} \subset \Omega$; we have

$$x_0(t) \ge -\|x_0\|_\infty \ge -\|x_0\|_{C^3} = -r^*, \quad \forall t \in [0, T], \\ \left| x_0'(t) \right| \le r_1, \qquad \left| x_0''(t) \right| \le r_2, \qquad \left| x_0'''(t) \right| \le r_3, \tag{57} \\ \forall t \in [0, T].$$

Consequently, from assumption (H_1') one has

$$f\left(t, x_0(t), x_0'(t), x_0''(t), x_0'''(t) \right) \ge 0 \quad \text{for a.e. } t \in [0, T]. \tag{58}$$

This together with (56) it follows that

$$\lambda_0 \le 0, \tag{59}$$

which is a contradiction. This implies that

$$Lx - Nx \neq \lambda, \quad \forall x \in \partial B_{r^*}, \ \forall \lambda \geq 0. \tag{60}$$

Thus from Lemma 2 it follows that

$$D\left[(L, N), B_{r^*}\right] = 0. \tag{61}$$

Hence

$$D\left[(L, N), \Omega \setminus \overline{B}_{r^*}\right] = D\left[(L, N), \Omega\right] - D\left[(L, N), B_{r^*}\right] \neq 0. \tag{62}$$

Therefore $Lx = Nx$ has a solution in $\Omega \setminus \overline{B}_{r^*}$; that is, PBVP (1), (2) has at least one nontrivial solution. This completes the proof of the theorem. □

Finally, we give some examples to illustrate our results.

Example 7. Consider the fourth-order periodic boundary value problem

$$x^{(4)}(t) = \lambda h(t)(x(t) + 1) e^{x(t)x'(t)} \left(x''(t)\right)^2 \left(x'''(t) + c\right),$$

$$\text{a.e. } t \in [0, T],$$

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, 2, 3, \tag{63}$$

where λ is a parameter, $h \in L^1[0, T]$ is nonnegative, and

$$c \in \left(\exp\left(2\int_0^T h(t)\, dt + 1\right), +\infty\right) \tag{64}$$

is a constant.

Let

$$f(t, x_0, x_1, x_2, x_3) = \lambda h(t)(x_0 + 1) e^{x_0 x_1} x_2^2 (x_3 + c). \tag{65}$$

Then $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies Carathéodory conditions. Taking any $r \in (1, +\infty)$, then r_0, r_1, r_2, and r_3 are well defined by (14).

Now, we assert that all conditions of Theorem 5 are satisfied when

$$\lambda \in \left[-\frac{1}{(r_0 + 1) e^{r_0 r_1} r_2^2 (r_3 + c)}, \frac{1}{(r_0 + 1) e^{r_0 r_1} r_2^2 (r_3 + c)}\right]. \tag{66}$$

In fact, without loss of generality, we can assume $\lambda \in [-1/(r_0 + 1)e^{r_0 r_1} r_2^2 (r_3 + c), 0]$. In this case, we choose $\mu = -1$. It is easy to see that, for every $(x_0, x_1, x_2, x_3) \in [r, +\infty) \times \mathbb{R}^2 \times [-r_3, r_3]$,

$$\mu f(t, x_0, x_1, x_2, x_3) \operatorname{sign} x_0 \geq 0, \quad \text{a.e. } t \in [0, T], \tag{67}$$

and, for every $(x_0, x_1, x_2, x_3) \in (-\infty, -r] \times \mathbb{R}^2 \times [-r_3, r_3]$,

$$\mu f(t, x_0, x_1, x_2, x_3) \operatorname{sign} x_0 \geq 0, \quad \text{a.e. } t \in [0, T]. \tag{68}$$

Hence condition (H_1) of Theorem 5 is satisfied. In addition, for $|x_0| \leq r_0, |x_1| \leq r_1, |x_2| \leq r_2, |x_3| \geq 1$, we have

$$f(t, x_0, x_1, x_2, x_3) \operatorname{sign} x_3$$

$$\leq |\lambda| h(t) (|x_0| + 1) e^{|x_0 x_1|} x_2^2 (|x_3| + c)$$

$$\leq h(t) \frac{|x_3| + c}{r_3 + c} \tag{69}$$

$$\leq h(t) |x_3|, \quad \text{a.e. } t \in [0, T].$$

Therefore, condition (H_2) of Theorem 5 is also satisfied. Hence, from Theorem 5, the fourth-order PBVP (63) has at least a solution $x = x(t)$, provided

$$\lambda \in \left[-\frac{1}{(r_0 + 1) e^{r_0 r_1} r_2^2 (r_3 + c)}, \frac{1}{(r_0 + 1) e^{r_0 r_1} r_2^2 (r_3 + c)}\right]. \tag{70}$$

Example 8. Consider the fourth-order periodic boundary value problem

$$x^{(4)}(t) = \frac{\lambda h(t)(x(t) + 1)^k |x'''(t)|}{1 + (x'(t))^n + (x''(t))^n}, \quad \text{a.e. } t \in [0, T], \tag{71}$$

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, 2, 3,$$

where λ is a parameter, $k, n \in \mathbb{N} = \{1, 2, \ldots\}$, k is odd, n is even, and $h \in L^1[0, T]$ is a nonnegative function.

Let

$$f(t, x_0, x_1, x_2, x_3) = \frac{\lambda h(t)(x_0 + 1)^k |x_3|}{1 + x_1^n + x_2^n}. \tag{72}$$

Then $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies Carathéodory conditions. We choose $r \in (1, +\infty)$; then r_0 is well defined by (14).

Now, we assert that f satisfies all conditions of Theorem 6 when

$$\lambda \in \left[-(r_0 + 1)^{-k}, (r_0 + 1)^{-k}\right]. \tag{73}$$

In fact, without loss of generality, we can assume that $\lambda \in [0, (r_0 + 1)^{-k}]$. Choose $\mu = 1$ and $r^* = 1$. Then it is easy to see that, for every $(x_0, x_1, x_2, x_3) \in [-r^*, +\infty) \times \mathbb{R}^3$,

$$\mu f(t, x_0, x_1, x_2, x_3) \geq 0, \quad \text{a.e. } t \in [0, T], \tag{74}$$

and, for every $(x_0, x_1, x_2, x_3) \in (-\infty, -r] \times \mathbb{R}^3$,

$$\mu f(t, x_0, x_1, x_2, x_3) \leq 0 \quad \text{a.e. } t \in [0, T]. \tag{75}$$

On the other hand, for every $(x_0, x_1, x_2, x_3) \in [-r_0, r_0] \times \mathbb{R}^3$,

$$f(t, x_0, x_1, x_2, x_3) \operatorname{sign} x_3$$

$$\leq \frac{\lambda |x_0 + 1|^k}{1 + x_1^n + x_2^n} h(t) |x_3|$$

$$\leq \frac{1}{1 + x_1^n + x_2^n} h(t) |x_3| \tag{76}$$

$$\leq h(t) |x_3|, \quad \text{a.e. } t \in [0, T].$$

In summary, all conditions of Theorem 6 are satisfied. Therefore, from Theorem 6, the fourth-order PBVP (71) has at least one nontrivial solution $x = x(t)$, provided $\lambda \in [-(r_0 + 1)^{-k}, (r_0 + 1)^{-k}]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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