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Research Article

Multiple Solutions to Fractional Difference Boundary Value Problems

Huiqin Chen, Yaqiong Cui, and Xianglan Zhao

School of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, China

Correspondence should be addressed to Huiqin Chen; dtdxchq@126.com

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The following fractional difference boundary value problems $\triangle^{\nu} y(t) = -f(t+\nu-1, y(t+\nu-1))$, $y(\nu-2) = y(\nu+b+1) = 0$ are considered, where $1 < \nu \leq 2$ is a real number and $\triangle^{\nu} y(t)$ is the standard Riemann-Liouville fractional difference. Based on the Krasnosel'skiĭ theorem and the Schauder fixed point theorem, we establish some conditions on f which are able to guarantee that this FBVP has at least two positive solutions and one solution, respectively. Our results significantly improve and generalize those in the literature. A number of examples are given to illustrate our main results.

1. Introduction

Fractional difference equations have been of great interest recently. It is caused by the intensive development of the theory of discrete fractional calculus itself; see [1–8]. Diaz and Osler [1] introduced a fractional difference defined as an infinite series, a generalization of the binomial formula for the Nth order difference $\triangle^{N} f$. Gray and Zhang [2] developed a special case for one composition rule and Leibniz formula. They worked exclusively with the nabla operator. A recent interest in discrete fractional calculus has been shown by Atici et al.; see [3–12]. Atici and Eloe developed some of the basic theory of both discrete fractional IVPs and BVPs with delta derivative on the time scale $\mathbb{Z}$. In particular, Atıcı and Şengül [5] provided some analysis of discrete fractional variational problems. Their paper also provided some initial attempts at using the discrete fractional calculus to model biological processes. Similarly, Goodrich [7–12] has established some results on both discrete fractional IVPs and BVPs. Holm [13] introduced fractional sum and difference operators and presented a complete and precise theory for composing fractional sums and differences. In addition, Wu and Baleanu [14] mainly concentrated on the analytical aspects, and the variational iteration method is extended in a new way to solve an initial value problem of q-fractional difference equations. Following this trend, in [15, 16], the authors discussed the boundary value problems of fractional difference equations depending on parameters.

In this paper, we consider the following boundary value problems for a fractional difference equation (FBVP):

$$\begin{gathered}\triangle^{\nu} y(t) = -f(t+\nu-1, y(t+\nu-1)), \\ y(\nu-2) = y(\nu+b+1) = 0,\end{gathered} \tag{1}$$

where $t \in [0, b+1]_{\mathbb{N}_0}$, $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous and f is not identically zero, $1 < \nu \leqslant 2$, $b > 2$ is an integer, and $\triangle^{\nu} y(t)$ is the standard Riemann-Liouville fractional difference. In this paper, we will use properties of Green's function of the FBVP (1) and the Krasnosel'skiĭ fixed point theorem to show that the FBVP (1) has at least one or two positive solutions. Our results significantly improve and generalize the results in [6, 8].

The plan of this paper is as follows. In Section 2, we will present some necessary lemmas. By using the Krasnosel'skiĭ theorem, Section 3 proves the existence of two positive solutions for the FBVP (1). Section 4 deduces the existence of one solution by using Schauder's fixed point theorem.

2. Preliminaries

In this section, we first review some basic notations and lemmas about fractional sums and differences in [6–8, 13].

For any t and ν, we define

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \quad (2)$$

for which the right-hand side is defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

The νth fractional sum of a function f is

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s) \quad (3)$$

for $\nu > 0$ and $t \in \{a+\nu, a+\nu+1, \ldots\} = \mathbb{N}_{a+\nu}$. We also define the νth fractional difference for $\nu > 0$ by $\Delta^{\nu} f(t) = \Delta^{n} \Delta^{-(n-\nu)} f(t)$, where $t \in \mathbb{N}_{a+n-\nu}$, and $n \in \mathbb{N}$ is chosen such that $0 \leqslant n-1 < \nu \leqslant n$.

Let $0 \leqslant n-1 < \nu \leqslant n$. Then

$$\Delta^{-\nu}\Delta^{\nu} f(t) = f(t) + c_1 (t-a)^{\underline{\nu-1}} + c_2 (t-a)^{\underline{\nu-2}} + \cdots + c_n (t-a)^{\underline{\nu-n}}, \quad c_i \in \mathbb{R},\ 1 \leqslant i \leqslant n. \quad (4)$$

In order to prove our results, we now provide some properties on Green's function associated with the problem (1).

Lemma 1 (see [6, Theorem 3.1]). *Let $1 < \nu \leqslant 2$ and $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ be given. Then the solution of the FBVP (1) is given by*

$$y(t) = \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)), \quad (5)$$

where Green's function $G : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \dfrac{t^{\underline{\nu-1}} (\nu+b-s)^{\underline{\nu-1}}}{(\nu+b+1)^{\underline{\nu-1}}} - (t-s-1)^{\underline{\nu-1}}, & 0 \leqslant s < t-\nu+1 \leqslant b+1, \\ \dfrac{t^{\underline{\nu-1}} (\nu+b-s)^{\underline{\nu-1}}}{(\nu+b+1)^{\underline{\nu-1}}} & 0 \leqslant t-\nu+1 \leqslant s \leqslant b+1. \end{cases} \quad (6)$$

Remark 2. It is easy to see that $G(\nu-2, s) = G(\nu+b+1, s) = 0$. G could be extended to $[\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$; we only discuss $(t,s) \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$.

Lemma 3 (see [6, Theorem 3.2]). *The Green function $G(t,s)$ satisfies the following conditions.*

(i) $G(t,s) > 0$, $(t,s) \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$.

(ii) $\max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s) = G(s+\nu-1, s)$, $s \in [0, b+1]_{\mathbb{N}_0}$.

(iii) *There exists a number $\gamma \in (0,1)$ such that*

$$\min_{(\nu+b)/4 \leqslant t \leqslant 3(\nu+b)/4} G(t,s) \geqslant \gamma \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} G(t,s) = \gamma G(s+\nu-1, s), \quad s \in [0, b+1]_{\mathbb{N}_0}. \quad (7)$$

Denote

$$\mathscr{B} = \left\{ y : [\nu-2, \nu+b+1]_{\mathbb{N}_{\nu-2}} \longrightarrow \mathbb{R},\ y(\nu-2) = y(\nu+b+1) = 0 \right\}. \quad (8)$$

It is clear that $\mathscr{B}$ is a Banach space with the norm $\|y\| = \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_0}} |y(t)|$. We choose a cone

$$\mathscr{K} = \left\{ y \in \mathscr{B} : y(t) \geqslant 0,\ \min_{(\nu+b)/4 \leqslant t \leqslant 3(\nu+b)/4} y(t) \geqslant \gamma \|y\| \right\}. \quad (9)$$

Now consider the operator T defined by

$$(Ty)(t) = \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)). \quad (10)$$

Referring to Lemma 3.1 of [8], we have the following.

Lemma 4.

$$T(\mathscr{K}) \subseteq \mathscr{K}. \quad (11)$$

We notice that T is a summation operator on a discrete finite set. Hence, T is trivially completely continuous. And a fixed point of T is equivalent to a solution of the FBVP (1). We will obtain sufficient conditions on the nonlinear f for the existence of fixed points of T. In order to prove our results, we need the following well-known Krasnosel'skiĭ fixed point theorem.

Lemma 5 (see [17]). *Let $\mathscr{B}$ be a Banach space and let $\mathscr{K} \subseteq \mathscr{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathscr{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume $T : \mathscr{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathscr{K}$ is a completely continuous operator. If either*

(i) $\|Ty\| \leqslant \|y\|$ *for* $y \in \mathscr{K} \cap \partial\Omega_1$ *and* $\|Ty\| \geqslant \|y\|$ *for* $y \in \mathscr{K} \cap \partial\Omega_2$; *or*

(ii) $\|Ty\| \geqslant \|y\|$ *for* $y \in \mathscr{K} \cap \partial\Omega_1$ *and* $\|Ty\| \leqslant \|y\|$ *for* $y \in \mathscr{K} \cap \partial\Omega_2$.

Then the operator T has at least one fixed point in $\mathscr{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of Positive Solutions

In this section, we state and prove the multiplicity of positive solutions regarding FBVP (1). Then, we conclude this section with two examples to illustrate our main results. For this, we need to suppose that $f : [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, +\infty) \to [0, +\infty)$ is continuous and f is not identically zero. Denote

$$\begin{aligned} f_0 &= \liminf_{y\to 0} \min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y}, \\ f^0 &= \limsup_{y\to 0} \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y}, \\ f_\infty &= \liminf_{y\to +\infty} \min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y}, \\ f^\infty &= \limsup_{y\to +\infty} \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{f(t,y)}{y}, \\ \eta &= \frac{1}{\sum_{s=0}^{b+1} G(s+\nu-1,s)}, \\ \lambda &= \frac{1}{\gamma \sum_{s=[(\nu+b)/4-\nu+1]}^{[3(\nu+b)/4-\nu+1]} G([(b-\nu)/2]+\nu,s)}, \end{aligned} \tag{12}$$

where γ is the constant in Lemma 3. In the sequel, let $\Omega_r = \{y \in \mathscr{K} : \|y\| < r\}$, for $r > 0$, and $\partial\Omega_r = \{y \in \mathscr{K} : \|y\| = r\}$. For convenience in what follows, we state these conditions of this section below.

(C1) There is a $p > 0$ such that $f(t,y) < \eta p$ for $0 \leqslant y \leqslant p$ and $\nu - 2 \leqslant t \leqslant \nu + b$.

(C2) There is a $p > 0$ such that $f(t,y) > \lambda p$ for $\gamma p \leqslant y \leqslant p$ and $(\nu+b)/4 \leqslant t \leqslant 3(\nu+b)/4$.

(C3) $f_0 > \lambda$, $f_\infty > \lambda$.

(C4) $f^0 < \eta$, $f^\infty < \eta$.

Lemma 6 (see [8]). *Suppose that there exist two different positive numbers r and R such that f satisfies condition (C1) at r and condition (C2) at R. Then FBVP (1) has at least one positive solution $y_0 \in \mathscr{K}$ satisfying $\min\{r,R\} \leqslant \|y_0\| \leqslant \max\{r,R\}$.*

Theorem 7. *Assume that f satisfies conditions (C1) and (C3). Then FBVP (1) has at least two positive solutions y_1 and y_2 with $0 < \|y_1\| < p \leqslant \|y_2\|$.*

Proof. Suppose that (C3) holds. Since $f_0 > \lambda$, there exist $\varepsilon > 0$ and $0 < r_0 < p$ such that $f(t,y) \geqslant (\lambda+\varepsilon)y$, $0 \leqslant y \leqslant r_0$, $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. Let $r_1 \in (0,r_0)$ and note that $[(b-\nu)/2]+\nu \in [(b+\nu)/4, 3(b+\nu)/4]$. Thus for $y \in \partial\Omega_{r_1}$, we get

$$\begin{aligned} &(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) \\ &= \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) f(s+\nu-1, y(s+\nu-1)) \\ &\geqslant \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right)(\lambda+\varepsilon) y \\ &\geqslant (\lambda+\varepsilon)\cdot\gamma\|y\| \sum_{s=[((\nu+b)/4)-\nu+1]}^{[(3(\nu+b)/4)-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) \\ &> \lambda\cdot\gamma\|y\| \sum_{s=[((\nu+b)/4)-\nu+1]}^{[(3(\nu+b)/4)-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) \\ &= r_1; \end{aligned} \tag{13}$$

that is, there is $\|Ty\| > \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_{r_1}$.

On the other hand, since $f_\infty > \lambda$, there exist $\sigma > 0$ and $R_0 > 0$ such that $f(t,y) \geqslant (\lambda+\sigma)y$, $y \geqslant R_0$, $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. Choose $R_1 > \max\{(1/\gamma)R_0, p\}$. If $y \in \partial\Omega_{R_1}$, then $y(t) \geqslant \gamma\|y\| > R_0$ for $(\nu+b)/4 \leqslant t \leqslant 3(\nu+b)/4$. So it follows that

$$\begin{aligned} &(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) \\ &= \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) f(s+\nu-1, y(s+\nu-1)) \\ &\geqslant \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right)(\lambda+\sigma) y \\ &\geqslant (\lambda+\sigma)\cdot\gamma\|y\| \sum_{s=[((\nu+b)/4)-\nu+1]}^{[(3(\nu+b)/4)-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) \\ &> \lambda\cdot\gamma\|y\| \sum_{s=[((\nu+b)/4)-\nu+1]}^{[(3(\nu+b)/4)-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu,s\right) \\ &= R_1, \end{aligned} \tag{14}$$

from which we see that $\|Ty\| > \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_{R_1}$.

For any $y \in \partial\Omega_p$, from (C1), we have $f(t,y) \leqslant \eta p$, $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$.

Consider

$$\begin{aligned} (Ty)(t) &= \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)\eta p \\ &= p = \|y\|; \end{aligned} \tag{15}$$

that is, there is $\|Ty\| \leqslant \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_p$.

Consequently, Lemma 5 implies that there are two fixed points y_1 and y_2 of operator T such that $0 < \|y_1\| < p < \|y_2\|$. And this completes the proof. ☐

Remark 8. By the proof of Theorem 7, we know that the conclusion of Theorem 7 is valid if (C3) is replaced by $f_0 = +\infty$ and $f_\infty = +\infty$. Namely, our result in this paper improve Theorem 3.4 in [8].

Theorem 9. *Suppose that conditions (C2) and (C4) hold, $f > 0$ for $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. Then FBVP (1) has at least two positive solutions y_1 and y_2 with $0 < \|y_1\| < p < \|y_2\|$.*

Proof. From the assumption $f^0 < \eta$, one can find $\varepsilon > 0 (\varepsilon < \eta)$ and $0 < r_0 < p$ such that $f(t,y) \leqslant (\eta - \varepsilon)y$, $0 \leqslant y \leqslant r_0$, $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$. Let $r_2 \in (0, r_0)$; then for $y \in \partial\Omega_{r_2}$, we have

$$\begin{aligned}(Ty)(t) &= \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)(\eta-\varepsilon) r_2 \\ &< \eta r_2 \sum_{s=0}^{b+1} G(s+\nu-1,s) \\ &= r_2 = \|y\|,\end{aligned} \tag{16}$$

from which we see that $\|Ty\| < \|y\|$ for $y \in \partial\Omega_{r_2}$.

On the other hand, since $f^\infty < \eta$, there exist $0 < \sigma < \eta$ and $R_0 > 0$ such that

$$f(t,y) \leqslant \sigma y, \qquad y \geqslant R_0, \quad t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}. \tag{17}$$

Denote $M = \max_{(t,y)\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}\times[0,R_0]} f(t,y)$; then $0 \leqslant f(t,y) \leqslant \sigma y + M$, $0 \leqslant y < +\infty$. Let $R_2 > \max\{2p, M/(\eta-\sigma)\}$. For $y \in \partial\Omega_{R_2}$, we have

$$\begin{aligned}\|Ty\| &= \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, y(s+\nu-1)) \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s) f(s+\nu-1, y(s+\nu-1)) \\ &\leqslant (\sigma\|y\| + M) \sum_{s=0}^{b+1} G(s+\nu-1,s) \\ &= (\sigma R_2 + M)\cdot\frac{1}{\eta} < R_2.\end{aligned} \tag{18}$$

Therefore, we have $\|Ty\| < \|y\|$ for $y \in \partial\Omega_{R_2}$.

Finally, for any $y \in \partial\Omega_p$, since $\gamma p \leqslant y(t) \leqslant p$ for $t \in [(\nu+b)/4, 3(\nu+b)/4]$, we estimate

$$\begin{aligned}&(Ty)\left(\left[\frac{b-\nu}{2}\right]+\nu\right) \\ &= \sum_{s=0}^{b+1} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) f(s+\nu-1, y(s+\nu-1)) \\ &> \lambda\gamma p \sum_{s=[((\nu+b)/4)-\nu+1]}^{[(3(\nu+b)/4)-\nu+1]} G\left(\left[\frac{b-\nu}{2}\right]+\nu, s\right) \\ &= p = \|y\|.\end{aligned} \tag{19}$$

Hence $\|Ty\| > \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_p$.

By Lemma 5, the proof is complete. □

Remark 10. From the proof of Theorem 9, we know that the conclusion of Theorem 9 is valid if the condition (C4) is replaced by $f^0 = 0$ and $f^\infty = 0$.

Remark 11. Theorem 9 is not included in [6, 8].

From the proof of Theorems 7 and 9, we have the following.

Theorem 12. *Suppose that $f_0 > \lambda$, $f^\infty < \eta$. Then FBVP (1) has at least one positive solution.*

Theorem 13. *Suppose that $f^0 < \eta$, $f_\infty > \lambda$. Then FBVP (1) has at least one positive solution.*

Remark 14. Theorems 12 and 13 in this paper significantly generalize Theorems 4.1 and 4.2 in [6].

Example 15. Consider the following boundary value problems:

$$\begin{aligned}&\triangle^{9/8} y(t) = -\frac{1}{100} e^{t-(57/8)} \left(y^{1/2}\left(t+\frac{1}{8}\right) + \frac{1}{4} y^2\left(t+\frac{1}{8}\right)\right), \\ &y\left(-\frac{7}{8}\right) = y\left(\frac{65}{8}\right) = 0,\end{aligned} \tag{20}$$

where $\nu = 9/8$ and $b = 6$, and $f(t,y) = (1/100)e^{t-(29/4)}(y^{1/2} + (1/4)y^2)$. A simple computation shows that $\eta > 0.0126$, $\lambda = 5/18\gamma$ (γ is the constant in Lemma 3(iii)), and $f_0 = f_\infty = +\infty$. Taking $p = 1$, we get

$$\begin{aligned}f(t,y) &= \frac{1}{100} e^{t-(29/4)} \left(y^{1/2} + \frac{1}{4} y^2\right) \\ &\leqslant \frac{1}{100}\left(p^{1/2} + \frac{1}{4} p^2\right) \\ &= 0.0125 < \eta p,\end{aligned} \tag{21}$$

$0 \leqslant y \leqslant p$, $\nu - 2 \leqslant t \leqslant b + \nu$. All conditions in Theorem 7 are satisfied. Therefore FBVP (20) has at least two positive solutions y_1 and y_2 such that $0 < \|y_1\| < 1 < \|y_2\|$.

Example 16. Consider the following boundary value problems:

$$\begin{aligned}&\triangle^{9/8} y(t) = -\frac{1}{100} y\left(t+\frac{1}{8}\right)\left[1 + \frac{28-\gamma}{\gamma(1+y^2(t+(1/8)))}\right], \\ &y\left(-\frac{7}{8}\right) = y\left(\frac{65}{8}\right) = 0,\end{aligned} \tag{22}$$

where $\nu = 9/8$ and $b = 6$, and $f(t, y) = (1/100)y[1 + ((28 - \gamma)/\gamma(1 + y^2))]$ (γ is the constant in Lemma 3(iii)); it is easy to compute that

$$\begin{aligned} f^{\infty} &= \limsup_{y\to+\infty} \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{1}{100}\left(1 + \frac{28-\gamma}{\gamma(1+y^2)}\right) \\ &= \frac{1}{100} < 0.0126 < \eta, \\ f_0 &= \liminf_{y\to 0^+} \min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \frac{1}{100}\left(1 + \frac{28-\gamma}{\gamma(1+y^2)}\right) \\ &= \frac{7}{25\gamma} > \frac{5}{18\gamma} = \lambda, \end{aligned} \tag{23}$$

which yields the condition of Theorem 12. By Theorem 12, FBVP (22) has at least one positive solution.

4. Existence of Solutions

In this section, we give the existence of solutions for problem (1). We will prove this result by using Schauder's fixed point theorem and provide an example to illustrate our results.

Theorem 17. *Let $f : [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that one of the following conditions is satisfied.*

(H_1) *There exist a nonnegative function $a(t) \in \mathscr{C}[\nu-1, \nu+b]$ and a constant c such that $|f(t, y)| \leqslant a(t)+c|y|^{\rho}$, where $c \geqslant 0$, $0 < \rho < 1$.*

(H_2) *$|f(t, y)| \leqslant c|y|^{\rho}$, where $c > 0$, $\rho > 1$.*

Then problem (1) *has at least one solution.*

Proof. First, suppose the condition (H_1) can be satisfied. Let

$$E = \left\{y(t) \in \mathscr{B} : \|y\| \leqslant R, t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}\right\}, \tag{24}$$

where

$$R \geqslant \max\left\{2\sum_{s=0}^{b+1} G(s+\nu-1, s)\,|a(s+\nu-1)|,\ \left[4c\frac{\Gamma(\nu+b+2)}{\Gamma(b+2)\Gamma(\nu+1)}\right]^{1/(1-\rho)}\right\}. \tag{25}$$

Obviously, E is a closed ball in the Banach space $\mathscr{B}$.

Now we prove that $T : E \to E$. For any $y \in E$, then

$$\begin{aligned} |(Ty)(t)| &\leqslant \sum_{s=0}^{b+1} \left|G(t,s)\left[a(s+\nu-1) + c|y(s+\nu-1)|^{\rho}\right]\right| \\ &\leqslant \sum_{s=0}^{b+1} |G(t,s)\,a(s+\nu-1)| + \sum_{s=0}^{b+1} c\|y(s+\nu-1)\|^{\rho} \cdot |G(t,s)| \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)\,|a(s+\nu-1)| + cR^{\rho}\left(\sum_{s=0}^{t-\nu}|G(t,s)| + \sum_{s=t-\nu+1}^{b+1}|G(t,s)|\right) \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)\,|a(s+\nu-1)| + \frac{cR^{\rho}}{\Gamma(\nu)}\left(\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}} + \sum_{s=0}^{b+1}\frac{t^{\underline{\nu-1}}(\nu+b-s)^{\underline{\nu-1}}}{(\nu+b+1)^{\underline{\nu-1}}}\right) \\ &\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)\,|a(s+\nu-1)| + cR^{\rho}\max_{t\in[\nu-2,\nu+b]_{N_{\nu-2}}}\left(\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}}\right) \\ &\quad + cR^{\rho}\max_{t\in[\nu-2,\nu+b]_{N_{\nu-2}}}\left(\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(\nu+b+1)^{\underline{\nu-1}}} \times \sum_{s=0}^{b+1}(\nu+b-s)^{\underline{\nu-1}}\right). \end{aligned} \tag{26}$$

Notice that

$$\begin{aligned} &\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{\underline{\nu-1}} = \frac{1}{\Gamma(\nu)}\left[-\frac{1}{\nu}(t-s)^{\underline{\nu}}\right]_{s=0}^{t-\nu+1} = \frac{1}{\Gamma(\nu+1)}t^{\underline{\nu}} \leq \frac{\Gamma(\nu+b+1)}{\Gamma(b+1)\Gamma(\nu+1)}, \\ &\frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(\nu+b+1)^{\underline{\nu-1}}}\sum_{s=0}^{b+1}(\nu+b-s)^{\underline{\nu-1}} \leqslant \frac{1}{\Gamma(\nu)}\sum_{s=0}^{b+1}(\nu+b-s)^{\underline{\nu-1}} \\ &\quad = \frac{1}{\Gamma(\nu)}\left[-\frac{1}{\nu}(\nu+b-s+1)^{\underline{\nu}}\right]_{s=0}^{b+2} = \frac{\Gamma(\nu+b+2)}{\Gamma(b+2)\Gamma(\nu+1)}. \end{aligned} \tag{27}$$

Thus

$$\begin{aligned}
&\left|(Ty)(t)\right| \\
&\leqslant \sum_{s=0}^{b+1} G(s+\nu-1,s)\left|a(s+\nu-1)\right| \\
&\quad + 2cR^{\rho}\frac{\Gamma(\nu+b+2)}{\Gamma(b+2)\Gamma(\nu+1)} \\
&\leqslant \frac{R}{2}+\frac{R}{2}=R;
\end{aligned} \tag{28}$$

we get $\|Ty\| \leqslant R$.

Second, suppose the condition (H_2) can be satisfied; we choose

$$0 < R \leqslant \left(\frac{\Gamma(b+2)\Gamma(\nu+1)}{2c\Gamma(\nu+b+2)}\right)^{1/(\rho-1)}. \tag{29}$$

With a similar argument as the above, we obtain $\|Ty\| \leqslant R$. Consequently, we get $T: E \rightarrow E$.

Note that T is a summation operator on a discrete finite set. Hence, T is trivially completely continuous. Therefore, according to the Schauder fixed point theorem, T has a fixed point y. Namely, y is a solution of problem (1). The theorem is proved. □

Remark 18. In this section, f is only a continuous function, without nonnegative assumptions on function f.

Remark 19. If $\rho = 1$ in (H_1), we need the condition $c(\Gamma(\nu+b+2)/\Gamma(b+2)\Gamma(\nu+1)) \leqslant 1/4$. Then, choose

$$R \geqslant 2\sum_{s=0}^{b} G(s+\nu-1,s)\left|a(s+\nu-1)\right|. \tag{30}$$

If $\rho = 1$ in (H_2), we only need the condition $c(\Gamma(\nu+b+2)/\Gamma(b+2)\Gamma(\nu+1)) \leqslant 1/2$. Then the conclusion of Theorem 17 remains true.

Example 20. Consider the fractional difference equation

$$\begin{aligned}
\triangle^{3/2}y(t) &= -\left(t+\frac{1}{2}\right)^4 y^{\rho}\left(t+\frac{1}{2}\right), \\
y\left(-\frac{1}{2}\right) &= y\left(\frac{5}{2}+b\right)=0,
\end{aligned} \tag{31}$$

where $t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$, $0<\rho<1$ or $\rho>1$, $b \in \mathbb{N}_+$, and $a(t)=0$, $f(t,y)=t^4y^{\rho}$. By using Theorem 17, the existence of solutions is obvious for $0<\rho<1$ or $\rho>1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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