

## Research Article

# Dynamic Behaviors of a Discrete Lotka-Volterra Competition System with Infinite Delays and Single Feedback Control

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A nonautonomous discrete two-species Lotka-Volterra competition system with infinite delays and single feedback control is considered in this paper. By applying the discrete comparison theorem, a set of sufficient conditions which guarantee the permanence of the system is obtained. Also, by constructing some suitable discrete Lyapunov functionals, some sufficient conditions for the global attractivity and extinction of the system are obtained. It is shown that if the discrete Lotka-Volterra competitive system with infinite delays and without feedback control is permanent, then, by choosing some suitable feedback control variable, the permanent species will be driven to extinction. That is, the feedback control variable, which represents the biological control or some harvesting procedure, is the unstable factor of the system. Such a finding overturns the previous scholars' recognition on feedback control variables.

## 1. Introduction

During the last decade, the study of the dynamic behaviors of discrete time models governed by difference equation has become one of the most important topics in mathematics biology; many interesting results concerned with permanence, extinction, and existence of positive periodic solution (almost periodic solution) and so forth have been extensively studied by many scholars; see [1–30] and the references cited therein.

As far as two-species discrete competition model is concerned, Chen and Zhou [1] proposed and studied the following discrete two-species Lotka-Volterra system:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left[ r_1(n) \left( 1 - \frac{x_1(n)}{K_1(n)} - \mu_2(n) x_2(n) \right) \right], \\ x_2(n+1) &= x_2(n) \exp \left[ r_2(n) \left( 1 - \mu_1(n) x_1(n) - \frac{x_2(n)}{K_2(n)} \right) \right], \end{aligned} \quad (1)$$

where  $K_i(n)$ ,  $i = 1, 2$  represent the environmental carrying capacity of species  $x_1$  and  $x_2$ , respectively;  $r_i(n)$ ,  $i = 1, 2$  are the intrinsic growth rate of two species; and  $x_1(n)$ ,  $x_2(n)$  represent the density of species  $x_1$  and  $x_2$  at the  $n$ th generation, respectively. The authors obtained a set of sufficient conditions which ensure the persistence of system (1). Also, for the periodic case, they gave a set of sufficient conditions which guarantee the existence of a globally stable periodic solution of the system. Chen [2] argued that it is more realistic to incorporate delays into system (1), and he proposed and investigated the following model:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left\{ a_1(n) - b_{11}(n) x_1(n) - b_{12}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\infty} k_{12}(s) x_2(n-s) \right\}, \\ x_2(n+1) &= x_2(n) \exp \left\{ a_2(n) - b_{22}(n) x_2(n) - b_{21}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\infty} k_{21}(s) x_1(n-s) \right\}. \end{aligned} \quad (2)$$

Concerned with the persistent property of system (2), the author obtained the following result.

Throughout this paper, given a bounded nonnegative sequence  $f(n)$  defined on  $Z$ , let  $f^L$  and  $f^M$  denote  $\inf_{n \in Z} f(n)$  and  $\sup_{n \in Z} f(n)$ , respectively.

**Theorem A.** Assume that

$$\frac{b_{11}^L}{b_{21}^M} > \frac{\exp(a_1^M - 1)}{a_2^L}, \quad \frac{a_1^L}{\exp(a_2^M - 1)} > \frac{b_{12}^M}{b_{22}^L} \quad (A_1)$$

hold; then system (2) is permanent.

However, the author did not investigate the stability and extinction property of the system (2), which are two of the most important topics on the study of population dynamics.

On the other hand, it is well known that, in the real world, ecosystems are continuously disturbed by unpredictable forces which can result in some changes of the biological parameters such as survival rates ([3]). For having a more accurate description of such a system, scholars introduced feedback controls into ecosystems and studied a variety of systems with feedback controls. Based on the work of Chen and Zhou [1], X. X. Chen and F. D. Chen [4] proposed and studied the following nonautonomous two-species discrete competitive system with feedback controls:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) \left[ 1 - \frac{x_1(n)}{K_1(n)} - \mu_2(n) x_2(n) \right. \right. \\ &\quad \left. \left. - b_1(n) u_1(n) \right] \right\}, \\ x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) \left[ 1 - \mu_1(n) x_1(n) \right. \right. \\ &\quad \left. \left. - \frac{x_2(n)}{K_2(n)} - b_2(n) u_2(n) \right] \right\}, \\ u_1(n+1) &= u_1(n) (1 - a_1(n)) + c_1(n) x_1(n), \\ u_2(n+1) &= u_2(n) (1 - a_2(n)) + c_2(n) x_2(n). \end{aligned} \quad (3)$$

Some sufficient conditions for the persistence and global stability of system (3) were obtained. Xu et al. [5] further considered the following two-species nonautonomous Lotka-Volterra competitive system with delays and feedback controls:

$$\begin{aligned} X_1(n+1) &= X_1(n) \exp \left\{ a_1(n) - b_{11}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\tau_{11}} k_{11}(s) X_1(n-s) - b_{12}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\tau_{12}} k_{12}(s) X_2(n-s) - d_1(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\delta_1} h_1(s) u_1(n-s) \right\}, \\ X_2(n+1) &= X_2(n) \exp \left\{ a_2(n) - b_{22}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\tau_{22}} k_{22}(s) X_2(n-s) - b_{21}(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\tau_{21}} k_{21}(s) X_1(n-s) - d_2(n) \right. \\ &\quad \left. \times \sum_{s=0}^{\delta_2} h_2(s) u_2(n-s) \right\}, \\ u_1(n+1) &= u_1(n) (1 - e_1(n)) + c_1(n) \sum_{s=0}^{\sigma_1} l_1(s) X_1(n-s), \\ u_2(n+1) &= u_2(n) (1 - e_2(n)) + c_2(n) \sum_{s=0}^{\sigma_2} l_2(s) X_2(n-s). \end{aligned} \quad (4)$$

By using the comparison theorem of discrete differential equation and constructing a suitable discrete type Lyapunov functional, they obtained new sufficient conditions on the permanence of species and global attractivity for system (4). Their results show that feedback controls are harmless to the permanence of system (4); that is, feedback controls have no influence on the permanence of system (4). X. Chen and F. Chen [29] and Liao et al. [30] also proposed a discrete time periodic  $n$ -species Lotka-Volterra competition system with feedback controls and deviating arguments; some sufficient conditions which ensure the existence of unique globally asymptotically stable periodic solution were obtained. Recently, Wu and Zhang [19] proposed a discrete autonomous Lotka-Volterra competition system with infinite delays and feedback controls; by using the iterative method, sufficient conditions which ensure the global attractivity of the system were obtained.

As we can see, those models considered in [4, 5, 19, 29, 30] contain two or more feedback control variables, which means that, for the different species, different control strategy is adopted. But, in the real world, the strategy adopted for one species may also affect the other species; in other words, such a strategy has influence on both species. For instance, in the agricultural system, spraying pesticide can reduce the number of weeds, but pesticide can also have a negative impact on the growth of crops or beneficial animals [6, 7].

In the medical system, when doctor takes chemotherapeutic drugs as tools to cure the cancer patients, cancer cells will decrease rapidly, but at the same time, drugs also do harm to normal cells and body's regulatory immune function. Yao et al. [8] studied the effect of chemotherapeutic drugs on cellular immunity in patients with lung cancer; they found that cell immunity is inhibited in patients with lung cancer; moreover, it is impaired considerably by chemotherapy. So how to keep the negative effect caused by the single strategy adopted for the weeds or cancer cells to a minimum?

The above phenomenons motivated us to propose and study the discrete Lotka-Volterra competition system with infinite delays and single feedback control variable as follows:

$$\begin{aligned}
 &x_1(n+1) \\
 &= x_1(n) \exp \left\{ a_1(n) - b_{11}(n)x_1(n) - b_{12}(n) \right. \\
 &\quad \times \sum_{s=0}^{\infty} k_{12}(s)x_2(n-s) - d_1(n) \\
 &\quad \left. \times \sum_{s=0}^{\infty} h_1(s)u(n-s) \right\}, \\
 &x_2(n+1) \\
 &= x_2(n) \exp \left\{ a_2(n) - b_{22}(n)x_2(n) - b_{21}(n) \right. \\
 &\quad \times \sum_{s=0}^{\infty} k_{21}(s)x_1(n-s) - d_2(n) \\
 &\quad \left. \times \sum_{s=0}^{\infty} h_2(s)u(n-s) \right\}, \\
 &u(n+1) \\
 &= u(n)(1 - e(n)) + c_1(n) \\
 &\quad \times \sum_{s=0}^{\infty} l_1(s)x_1(n-s) + c_2(n) \\
 &\quad \times \sum_{s=0}^{\infty} l_2(s)x_2(n-s).
 \end{aligned} \tag{5}$$

In system (5),  $x_i(n)$  ( $i = 1, 2$ ) is the density of  $x_i$  species at the  $n$ th generation and  $u(n)$  is the single feedback control variable.

Throughout this paper, we assume the following.

(H<sub>1</sub>)  $a_i(n)$ ,  $b_{ij}(n)$ ,  $c_i(n)$ ,  $d_i(n)$ , and  $e_i(n)$  ( $i, j = 1, 2$ ) are bounded sequences of real numbers defined on  $Z$  such that

$$\begin{aligned}
 &a_i^L > 0, \quad b_{ij}^L > 0, \quad c_i^L > 0, \\
 &d_i^L > 0, \quad 0 < e^L < e^M < 1, \\
 &i, j = 1, 2.
 \end{aligned} \tag{6}$$

(H<sub>2</sub>)  $k_{ij}(s)$ ,  $h_i(s)$ , and  $l_i(s)$  ( $i, j = 1, 2$ ) are nonnegative bounded sequences such that

$$\begin{aligned}
 &\sum_{s=0}^{\infty} k_{ij}(s) = 1, \quad \sum_{s=0}^{\infty} h_i(s) = 1, \quad \sum_{s=0}^{\infty} l_i(s) = 1, \\
 &\quad i, j = 1, 2; \\
 &\Theta_{ij} = \sum_{s=0}^{\infty} k_{ij}(s) s < +\infty, \quad \Lambda_i = \sum_{s=0}^{\infty} h_i(s) s < +\infty, \\
 &\Upsilon_i = \sum_{s=0}^{\infty} l_i(s) s < +\infty.
 \end{aligned} \tag{7}$$

According to the biological background of system (5), we only consider the solution of system (5) with the following initial conditions:

$$\begin{aligned}
 &x_i(s) = \Psi_i(s) \geq 0, \quad \Psi_i(0) > 0, \quad \sup_{k \in Z^-} \Psi_i(k) < +\infty, \\
 &\quad i = 1, 2; \\
 &u(s) = \Phi(s) \geq 0, \quad \Phi(0) > 0, \\
 &\quad \sup_{k \in Z^-} \Phi(k) < +\infty,
 \end{aligned} \tag{8}$$

where  $s = \dots, -n, -n+1, \dots, -1, 0$ . It is easy to prove that the solution of system (5) which satisfies initial conditions (8) is positive.

We mention here that this is the first time such kind of model is proposed and studied, and, as far as system (5) is concerned, *whether the single feedback control variable has influence on the persistent property of the system or not* is an interesting problem. The aim of this paper is to investigate the dynamic behaviors of the system (5); in particular, we will find out the answer to the above problem.

The organization of this paper is as follows. We introduce some useful lemmas in the next section and then state and prove the main results in Sections 3, 4, and 5, respectively. Three examples together with their numeric simulations are presented to show the feasibility of the main results in Section 6. We end this paper by a brief discussion.

## 2. Lemmas

Now, let us consider the following difference equation:

$$y(k+1) = ay(k) + b, \tag{9}$$

where  $a, b$  are positive constants.

**Lemma 1** (see [9]). *Assume that  $|a| < 1$ ; for any initial value  $y(0)$ , there exists a unique solution  $y(k)$  of (9), which can be expressed as follows:*

$$y(k) = a^k(y(0) - y^*) + y^*, \tag{10}$$

where  $y^* = b/(1 - a)$ . Thus, for any solution  $y(k)$  of the system (10), we have

$$\lim_{k \rightarrow +\infty} y(k) = y^*. \tag{11}$$

**Lemma 2** (see [9]). Let  $k \in N_{k_0}^+ = \{k_0, k_0 + 1, \dots, k_0 + l, \dots\}$ ,  $r \geq 0$ . For any fixed  $k$ ,  $g(k, r)$  is nondecreasing function with respect to  $r$ , and, for  $k \geq k_0$ , the following inequalities hold:

$$y(k + 1) \leq g(k, y(k)), \quad u(k + 1) \geq g(k, u(k)). \quad (12)$$

If  $y(k_0) \leq u(k_0)$ , then  $y(k) \leq u(k)$  for all  $k \geq k_0$ .

**Lemma 3** (see [2]). Assume that  $r(n) > 0$ ,  $x(n)$  satisfies  $x(n) > 0$ , and

$$x(n + 1) \leq x(n) \exp\{r(n)(1 - ax(n))\}, \quad (13)$$

for  $n \in [n_1, \infty)$ , where  $a$  is a positive constant. Then

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{1}{ar^M} \exp(r^M - 1). \quad (14)$$

**Lemma 4** (see [2]). Assume that  $r(n) > 0$ ,  $x(n)$  satisfies  $x(n) > 0$ , and

$$x(n + 1) \geq x(n) \exp\{r(n)(1 - ax(n))\}, \quad (15)$$

for  $n \in [n_1, \infty)$ ,  $\limsup_{n \rightarrow +\infty} x(n) \leq x^*$ , and  $x(n_1) > 0$ , where  $a$  and  $x^*$  are positive constants such that  $ax^* > 1$ . Then

$$\liminf_{n \rightarrow +\infty} x(n) \geq \frac{1}{a} \exp(r^M(1 - ax^*)). \quad (16)$$

**Lemma 5** (see [2]). Let  $x : Z \rightarrow R$  be a nonnegative bounded sequence, and let  $H : Z_+ \rightarrow R$  be a nonnegative sequence such that  $\sum_{n=0}^{\infty} H(n) = 1$ ; then

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x(n) &\leq \liminf_{n \rightarrow +\infty} \sum_{s=-\infty}^n H(n-s)x(s) \\ &\leq \limsup_{n \rightarrow +\infty} \sum_{s=-\infty}^n H(n-s)x(s) \\ &\leq \limsup_{n \rightarrow +\infty} x(n). \end{aligned} \quad (17)$$

### 3. Permanence

Concerned with the persistent property of the system (5), we have the following result.

**Theorem 6.** Assume that

$$a_i^L - b_{ij}^M M_j - d_i^M B > 0, \quad i, j = 1, 2, \quad i \neq j, \quad (18)$$

holds; then, for any positive solution  $(x_1(n), x_2(n), u(n))$  of the system (5), we have

$$\begin{aligned} m_i &\leq \liminf_{n \rightarrow +\infty} x_i(n) < \limsup_{n \rightarrow +\infty} x_i(n) \leq M_i, \\ A &\leq \liminf_{n \rightarrow +\infty} u(n) < \limsup_{n \rightarrow +\infty} u(n) \leq B, \end{aligned} \quad (19)$$

where

$$\begin{aligned} m_i &= \frac{a_i^L - b_{ij}^M M_j - d_i^M B}{b_{ii}^M} \exp(a_i^L - b_{ij}^M M_j - d_i^M B - b_{ii}^M M_i), \\ M_i &= \frac{1}{b_{ii}^L} \exp(a_i^M - 1), \quad i, j = 1, 2, \quad i \neq j, \\ A &= \frac{c_1^L m_1 + c_2^L m_2}{e^M}, \quad B = \frac{c_1^M M_1 + c_2^M M_2}{e^L}. \end{aligned} \quad (20)$$

*Proof.* From the first and second equations of system (5), we have

$$x_i(n + 1) \leq x_i(n) \exp\left[a_i(n) \left(1 - \frac{b_{ii}^L}{a_i^M} x_i(n)\right)\right]. \quad (21)$$

And so, from Lemma 3, we can obtain

$$\limsup_{n \rightarrow +\infty} x_i(n) \leq \frac{1}{b_{ii}^L} \exp(a_i^M - 1) \stackrel{\text{def}}{=} M_i. \quad (22)$$

According to Lemma 5, from the above inequality we have

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \sum_{s=0}^{\infty} l_i(s) x_i(n-s) \\ &= \limsup_{n \rightarrow +\infty} \sum_{s=-\infty}^n l_i(n-s) x_i(s) \\ &\leq \limsup_{n \rightarrow +\infty} x_i(n) \leq M_i. \end{aligned} \quad (23)$$

For any  $\varepsilon > 0$ , there exists a positive integer  $n_1$  such that

$$\sum_{s=0}^{\infty} l_i(s) x_i(n-s) \leq M_i + \varepsilon \quad \forall n > n_1. \quad (24)$$

By the third equation of system (5), we have

$$u(n + 1) \leq (1 - e^L) u(n) + c_1^M (M_1 + \varepsilon) + c_2^M (M_2 + \varepsilon). \quad (25)$$

Hence, by applying Lemmas 1 and 2 to (25), we obtain

$$\limsup_{n \rightarrow +\infty} u(n) \leq \frac{c_1^M (M_1 + \varepsilon) + c_2^M (M_2 + \varepsilon)}{e^L}. \quad (26)$$

Setting  $\varepsilon \rightarrow 0$ , it follows that

$$\limsup_{n \rightarrow +\infty} u(n) \leq \frac{c_1^M M_1 + c_2^M M_2}{e^L} \stackrel{\text{def}}{=} B. \quad (27)$$

Condition (18) implies that, for enough small positive constant  $\varepsilon_1$ , the following inequalities hold:

$$a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1) > 0, \quad i, j = 1, 2, \quad i \neq j. \quad (28)$$

For the above  $\varepsilon_1$ , it follows from (22) and (27) that there exists a positive integer  $n_2 \geq n_1$  such that

$$\sum_{s=0}^{\infty} h_i(s) u(n-s) \leq B + \varepsilon_1, \tag{29}$$

$$\sum_{s=0}^{\infty} k_{ij}(s) x_i(n-s) \leq M_i + \varepsilon_1 \quad \forall n > n_2.$$

Thus, for all  $n > n_2$ , from (28), (29), and the first two equations of system (5), we have

$$x_i(n+1) \geq x_i(n) \exp \left\{ (a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1)) \times (1 - D_i^\varepsilon x_i(n)) \right\}, \tag{30}$$

where  $D_i^\varepsilon = b_{ii}^M / (a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1))$  for  $i, j = 1, 2, i \neq j$ .

Noticing that

$$\frac{b_{ii}^M}{b_{ii}^L} \geq 1, \quad \frac{\exp(a_i^M - 1)}{a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1)} > 1, \tag{31}$$

then

$$D_i^\varepsilon \cdot M_i = \frac{b_{ii}^M}{a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1)} \cdot \frac{1}{b_{ii}^L} \exp(a_i^M - 1) > 1. \tag{32}$$

Hence, according to Lemma 4, we have

$$\liminf_{n \rightarrow +\infty} x_i(n) \geq \frac{1}{D_i^\varepsilon} \cdot \exp(a_i^L - b_{ij}^M (M_j + \varepsilon_1) - d_i^M (B + \varepsilon_1) - b_{ii}^M M_i). \tag{33}$$

Setting  $\varepsilon_1 \rightarrow 0$ , it follows that

$$\liminf_{n \rightarrow +\infty} x_i(n) \geq \frac{1}{D_i} \cdot \exp(a_i^L - b_{ij}^M M_j - d_i^M B - b_{ii}^M M_i) \stackrel{\text{def}}{=} m_i, \tag{34}$$

where  $D_i = b_{ii}^M / (a_i^L - b_{ij}^M M_j - d_i^M B)$ ,  $i = 1, 2$ .

According to Lemma 5, from (34) we have that, for any  $\varepsilon_2 > 0$  small enough (without loss of generality, assume that  $\varepsilon < (1/2) \min_i \{m_i\}$ ), there exists an  $n_3 > n_2$ , such that

$$\sum_{s=0}^{\infty} l_i(s) x_i(n-s) \geq m_i - \varepsilon_2 \quad \forall n \geq n_3. \tag{35}$$

For  $n \geq n_3$ , from (35) and the last equation of system (5), we have

$$u(n+1) \geq (1 - e^M) u(n) + c_1^L (m_1 - \varepsilon_2) + c_2^L (m_2 - \varepsilon_2). \tag{36}$$

Hence, by applying Lemmas 1 and 2 to (36), we obtain

$$\limsup_{n \rightarrow +\infty} u(n) \geq \frac{c_1^L (m_1 - \varepsilon_2) + c_2^L (m_2 - \varepsilon_2)}{e^M}. \tag{37}$$

Setting  $\varepsilon_2 \rightarrow 0$ , it follows that

$$\limsup_{n \rightarrow +\infty} u(n) \geq \frac{c_1^L m_1 + c_2^L m_2}{e^M} \stackrel{\text{def}}{=} A. \tag{38}$$

This ends the proof Theorem 6.  $\square$

### 4. Global Attractivity

Concerned with the stability property of the system (5), we have the following result.

**Theorem 7.** Assume that there exist positive constants  $\alpha_1, \alpha_2$ , and  $\alpha_3$  such that

$$\begin{aligned} \alpha_1 A_{11} - \alpha_2 b_{21}^M - \alpha_3 c_1^M &> 0, \\ \alpha_2 A_{22} - \alpha_1 b_{12}^M - \alpha_3 c_2^M &> 0, \\ \alpha_3 e^L - \alpha_1 d_1^M - \alpha_2 d_2^M &> 0 \end{aligned} \tag{39}$$

hold; then, for any two positive solutions  $(x_1(n), x_2(n), u(n))$  and  $(x_1^*(n), x_2^*(n), u^*(n))$  of system (5), we have

$$\lim_{n \rightarrow +\infty} (x_i(n) - x_i^*(n)) = 0, \quad \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) = 0, \tag{40}$$

where

$$A_{ii} = \min \left\{ b_{ii}^L, \frac{2}{M_i} - b_{ii}^M \right\}, \quad i = 1, 2. \tag{41}$$

*Proof.* By (39), we can choose enough small positive constants  $\delta$  and  $\varepsilon$  such that

$$\begin{aligned} \alpha_1 A_{11}^\varepsilon - \alpha_2 b_{21}^M - \alpha_3 c_1^M &> \delta, \\ \alpha_2 A_{22}^\varepsilon - \alpha_1 b_{12}^M - \alpha_3 c_2^M &> \delta, \\ \alpha_3 e^L - \alpha_1 d_1^M - \alpha_2 d_2^M &> \delta, \end{aligned} \tag{42}$$

where

$$A_{ii}^\varepsilon = \min \left\{ b_{ii}^L, \frac{2}{M_i + \varepsilon} - b_{ii}^M \right\}, \quad i = 1, 2. \tag{43}$$

Let  $(x_1(n), x_2(n), u(n))$  be any positive solution of system (5). For the above  $\varepsilon$ , from (22) and (27), there exists an enough large  $n^* > n_1$ , such that

$$x_i(n) < M_i + \varepsilon, \quad u(n) < B + \varepsilon \quad \forall n \geq n^*, \quad i = 1, 2. \tag{44}$$

Now, let us define a Lyapunov functional

$$V(n) = \alpha_1 V_1(n) + \alpha_2 V_2(n) + \alpha_3 V_3(n), \tag{45}$$

where  $\alpha_i, i = 1, 2, 3$ , are positive constants:

$$\begin{aligned}
 V_1(n) &= |\ln x_1(n) - \ln x_1^*(n)| + \sum_{s=0}^{\infty} k_{12}(s) \\
 &\quad \times \sum_{q=n-s}^{n-1} b_{12}(q+s) |x_2(q) - x_2^*(q)| + \sum_{s=0}^{\infty} h_1(s) \\
 &\quad \times \sum_{q=n-s}^{n-1} d_1(q+s) |u(q) - u^*(q)|; \\
 V_2(n) &= |\ln x_2(n) - \ln x_2^*(n)| + \sum_{s=0}^{\infty} k_{21}(s) \\
 &\quad \times \sum_{q=n-s}^{n-1} b_{21}(q+s) |x_1(q) - x_1^*(q)| \\
 &\quad + \sum_{s=0}^{\infty} h_2(s) \sum_{q=n-s}^{n-1} d_2(q+s) |u(q) - u^*(q)|; \\
 V_3(n) &= |u(n) - u^*(n)| + \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \\
 &\quad \times \sum_{q=n-s}^{n-1} c_i(q+s) |x_i(q) - x_i^*(q)|.
 \end{aligned} \tag{46}$$

Then, from the definition of  $V_i(n), i = 1, 2, 3$ , one could easily see that  $V(n) \geq 0$  for all  $n \in \mathbb{Z}^+$ . Also, for any fixed  $n^* \in \mathbb{Z}^+$ ,

$$\begin{aligned}
 V(n^*) &= \alpha_1 \left( |\ln x_1(n^*) - \ln x_1^*(n^*)| \right. \\
 &\quad + \sum_{s=0}^{\infty} k_{12}(s) \sum_{q=n^*-s}^{n^*-1} b_{12}(q+s) |x_2(q) - x_2^*(q)| \\
 &\quad + \left. \sum_{s=0}^{\infty} h_1(s) \sum_{q=n^*-s}^{n^*-1} d_1(q+s) |u(q) - u^*(q)| \right) \\
 &\quad + \alpha_2 \left( |\ln x_2(n^*) - \ln x_2^*(n^*)| \right. \\
 &\quad + \sum_{s=0}^{\infty} k_{21}(s) \sum_{q=n^*-s}^{n^*-1} b_{21}(q+s) |x_1(q) - x_1^*(q)| \\
 &\quad + \left. \sum_{s=0}^{\infty} h_2(s) \sum_{q=n^*-s}^{n^*-1} d_2(q+s) |u(q) - u^*(q)| \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \alpha_3 \left( |u(n^*) - u^*(n^*)| \right. \\
 &\quad + \left. \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \sum_{q=n^*-s}^{n^*-1} c_i(q+s) |x_i(q) - x_i^*(q)| \right) \\
 &\leq \alpha_1 \left( |\ln x_1(n^*) - \ln x_1^*(n^*)| \right. \\
 &\quad + b_{12}^M \sup_{q \in \mathbb{Z}, q \leq n^*} |x_2(q) - x_2^*(q)| \sum_{s=0}^{\infty} k_{12}(s) s \\
 &\quad + d_1^M \sup_{q \in \mathbb{Z}, q \leq n^*} |u(q) - u^*(q)| \sum_{s=0}^{\infty} h_1(s) s \left. \right) \\
 &\quad + \alpha_2 \left( |\ln x_2(n^*) - \ln x_2^*(n^*)| \right. \\
 &\quad + b_{21}^M \sup_{q \in \mathbb{Z}, q \leq n^*} |x_1(q) - x_1^*(q)| \\
 &\quad \times \sum_{s=0}^{\infty} k_{21}(s) s + d_2^M \sup_{q \in \mathbb{Z}, q \leq n^*} |u(q) - u^*(q)| \\
 &\quad \times \sum_{s=0}^{\infty} h_2(s) s \left. \right) \\
 &\quad + \alpha_3 \left( |u(n^*) - u^*(n^*)| \right. \\
 &\quad + c_i^M \sup_{q \in \mathbb{Z}, q \leq n^*} |x_i(q) - x_i^*(q)| \\
 &\quad \times \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) s \left. \right) < +\infty.
 \end{aligned} \tag{47}$$

Also, from the first equation of system (5) and using the Mean Value Theorem, we can obtain

$$\begin{aligned}
 \Delta V_1(n) &\leq |\ln x_1(n) - \ln x_1^*(n) - b_{11}(n) (x_1(n) - x_1^*(n))| \\
 &\quad - |\ln x_1(n) - \ln x_1^*(n)| + b_{12}(n) \\
 &\quad \times \sum_{s=0}^{\infty} k_{12}(s) |x_2(n-s) - x_2^*(n-s)| \\
 &\quad + d_1(n) \sum_{s=0}^{\infty} h_1(s) |u(n-s) - u^*(n-s)| \\
 &\quad + \sum_{s=0}^{\infty} k_{12}(s) (b_{12}(n+s) |x_2(n) - x_2^*(n)| \\
 &\quad - b_{12}(n) |x_2(n-s) - x_2^*(n-s)|)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{s=0}^{\infty} h_1(s) (d_1(n+s) |u(n) - u^*(n)| \\
 & \quad - d_1(n) |u(n-s) - u^*(n-s)|) \\
 & \leq - \left( \frac{1}{\varphi_1(n)} - \left| \frac{1}{\varphi_1(n)} - b_{11}(n) \right| \right) |x_1(n) - x_1^*(n)| \\
 & \quad + \sum_{s=0}^{\infty} k_{12}(s) b_{12}(n+s) |x_2(n) - x_2^*(n)| \\
 & \quad + \sum_{s=0}^{\infty} h_1(s) d_1(n+s) |u(n) - u^*(n)| \\
 & \leq - \left( \frac{1}{\varphi_1(n)} - \left| \frac{1}{\varphi_1(n)} - b_{11}(n) \right| \right) |x_1(n) - x_1^*(n)| \\
 & \quad + b_{12}^M |x_2(n) - x_2^*(n)| + d_1^M |u(n) - u^*(n)|, \tag{48}
 \end{aligned}$$

where  $\varphi_1(n)$  lies between  $x_1(n)$  and  $x_1^*(n)$ .  
 Similarly to the analysis of (48), we can obtain

$$\begin{aligned}
 \Delta V_2(n) & \leq - \left( \frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - b_{22}(n) \right| \right) |x_2(n) - x_2^*(n)| \tag{49} \\
 & \quad + b_{21}^M |x_1(n) - x_1^*(n)| + d_2^M |u(n) - u^*(n)|,
 \end{aligned}$$

where  $\varphi_2(n)$  lies between  $x_2(n)$  and  $x_2^*(n)$ :

$$\begin{aligned}
 \Delta V_3(n) & = V_3(n+1) - V_3(n) \\
 & \leq -e(n) |u(n) - u^*(n)| + \sum_{i=1}^2 c_i(n) \\
 & \quad \times \sum_{s=0}^{\infty} l_i(s) |x_i(n-s) - x_i^*(n-s)| \\
 & \quad + \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) (c_i(n+s) \\
 & \quad \times |x_i(n) - x_i^*(n)| - c_i(n) \\
 & \quad \times |x_i(n-s) - x_i^*(n-s)|) \tag{50} \\
 & = -e(n) |u(n) - u^*(n)| \\
 & \quad + \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) c_i(n+s) |x_i(n) - x_i^*(n)| \\
 & \leq -e^L |u(n) - u^*(n)| \\
 & \quad + \sum_{i=1}^2 c_i^M |x_i(n) - x_i^*(n)|.
 \end{aligned}$$

From (42)–(44) and (48)–(50), for any  $n \geq n^*$ , we have

$$\begin{aligned}
 \Delta V(n) & \leq \alpha_1 \left\{ - \left( \frac{1}{\varphi_1(n)} - \left| \frac{1}{\varphi_1(n)} - b_{11}(n) \right| \right) \right. \\
 & \quad \times |x_1(n) - x_1^*(n)| \\
 & \quad + b_{12}^M |x_2(n) - x_2^*(n)| \\
 & \quad \left. + d_1^M |u(n) - u^*(n)| \right\} \\
 & \quad + \alpha_2 \left\{ - \left( \frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - b_{22}(n) \right| \right) \right. \\
 & \quad \times |x_2(n) - x_2^*(n)| \\
 & \quad + b_{21}^M |x_1(n) - x_1^*(n)| \\
 & \quad \left. + d_2^M |u(n) - u^*(n)| \right\} \\
 & \quad + \alpha_3 \left\{ -e^L |u(n) - u^*(n)| \right. \\
 & \quad \left. + \sum_{i=1}^2 c_i^M |x_i(n) - x_i^*(n)| \right\} \\
 & \leq - \left\{ \alpha_1 \min \left[ b_{11}^L, \frac{2}{M_1 + \varepsilon} - b_{11}^M \right] - \alpha_2 b_{21}^M - \alpha_3 c_1^M \right\} \\
 & \quad \times |x_1(n) - x_1^*(n)| \\
 & \quad - \left\{ \alpha_2 \min \left[ b_{22}^L, \frac{2}{M_2 + \varepsilon} - b_{22}^M \right] - \alpha_1 b_{12}^M - \alpha_3 c_2^M \right\} \\
 & \quad \times |x_2(n) - x_2^*(n)| \\
 & \quad - \left\{ \alpha_3 e^L - \alpha_1 d_1^M - \alpha_2 d_2^M \right\} |u(n) - u^*(n)| \\
 & = - \left\{ \alpha_1 A_{11}^\varepsilon - \alpha_2 b_{21}^M - \alpha_3 c_1^M \right\} |x_1(n) - x_1^*(n)| \\
 & \quad - \left\{ \alpha_2 A_{22}^\varepsilon - \alpha_1 b_{12}^M - \alpha_3 c_2^M \right\} |x_2(n) - x_2^*(n)| \\
 & \quad - \left\{ \alpha_3 e^L - \alpha_1 d_1^M - \alpha_2 d_2^M \right\} |u(n) - u^*(n)| \\
 & \leq -\delta \left( \sum_{i=1}^2 |x_i(n) - x_i^*(n)| + |u(n) - u^*(n)| \right). \tag{51}
 \end{aligned}$$

Summating both sides of the above inequalities from  $n^*$  to  $n$ , we have

$$\begin{aligned}
 & \sum_{p=n^*}^n (V(p+1) - V(p)) \\
 & \leq -\delta \sum_{p=n^*}^n \left( \sum_{i=1}^2 |x_i(p) - x_i^*(p)| + |u(p) - u^*(p)| \right). \tag{52}
 \end{aligned}$$

Hence

$$\begin{aligned}
 &V(n+1) + \delta \sum_{p=n^*}^n \left( \sum_{i=1}^2 |x_i(p) - x_i^*(p)| + |u(p) - u^*(p)| \right) \\
 &\leq V(n^*) < +\infty.
 \end{aligned} \tag{53}$$

Then, we have

$$\sum_{p=n^*}^n \left( \sum_{i=1}^2 |x_i(p) - x_i^*(p)| + |u(p) - u^*(p)| \right) \leq \frac{V(n^*)}{\delta}. \tag{54}$$

Therefore

$$\sum_{p=n^*}^{\infty} \left( \sum_{i=1}^2 |x_i(p) - x_i^*(p)| + |u(p) - u^*(p)| \right) < +\infty, \tag{55}$$

which means that

$$\lim_{n \rightarrow +\infty} \left( \sum_{i=1}^2 |x_i(n) - x_i^*(n)| + |u(n) - u^*(n)| \right) = 0. \tag{56}$$

Consequently

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} (x_i(n) - x_i^*(n)) &= 0, \quad i = 1, 2, \\
 \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) &= 0.
 \end{aligned} \tag{57}$$

This completes the proof of Theorem 7 □

### 5. Extinction

Concerned with the extinction property of the system (5), when the coefficients of the third equation are all constants, we could establish the following results.

**Theorem 8.** Assume that

$$\frac{a_1^M}{a_2^L} < \frac{b_{11}^L e + d_1^L c_1}{b_{21}^M e + d_2^M c_1}, \quad \frac{a_1^M}{a_2^L} < \frac{b_{12}^L e + d_1^L c_2}{b_{22}^M e + d_2^M c_2} \tag{58}$$

hold; let  $(x_1(n), x_2(n), u(n))$  be any positive solution of system (5); then

$$\lim_{n \rightarrow +\infty} x_1(n) = 0. \tag{59}$$

**Theorem 9.** Assume that

$$\frac{a_1^L}{a_2^M} > \frac{b_{11}^M e + d_1^M c_1}{b_{21}^L e + d_2^L c_1}, \quad \frac{a_1^L}{a_2^M} > \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2} \tag{60}$$

hold; let  $(x_1(n), x_2(n), u(n))$  be any positive solution of system (5); then

$$\lim_{n \rightarrow +\infty} x_2(n) = 0. \tag{61}$$

*Proof of Theorem 8.* By conditions (58), we can choose positive constants  $\alpha_{n1}, \alpha_{n2}$  such that

$$\begin{aligned}
 \frac{a_1^M}{a_2^L} &< \frac{\alpha_{n2}}{\alpha_{n1}} < \frac{b_{11}^L e + d_1^L c_1}{b_{21}^M e + d_2^M c_1}, \\
 \frac{a_1^M}{a_2^L} &< \frac{\alpha_{n2}}{\alpha_{n1}} < \frac{b_{12}^L e + d_1^L c_2}{b_{22}^M e + d_2^M c_2}.
 \end{aligned} \tag{62}$$

Thus, there exists a positive constant  $\eta$  such that

$$\begin{aligned}
 \alpha_{n1} a_1^M - \alpha_{n2} a_2^L &< -\eta < 0, \\
 \alpha_{n2} b_{21}^M - \alpha_{n1} b_{11}^L + \frac{(\alpha_{n2} d_2^M - \alpha_{n1} d_1^L) c_1}{e} &< 0, \\
 \alpha_{n2} b_{22}^M - \alpha_{n1} b_{12}^L + \frac{(\alpha_{n2} d_2^M - \alpha_{n1} d_1^L) c_2}{e} &< 0.
 \end{aligned} \tag{63}$$

There exists a constant  $\alpha_{n3}$  such that

$$\begin{aligned}
 \frac{\alpha_{n2} d_2^M - \alpha_{n1} d_1^L}{e} &< \alpha_{n3} \implies \alpha_{n2} d_2^M - \alpha_{n1} d_1^L - \alpha_{n3} e < 0, \\
 \alpha_{n2} b_{21}^M - \alpha_{n1} b_{11}^L + \alpha_{n3} c_1 &< 0, \\
 \alpha_{n2} b_{22}^M - \alpha_{n1} b_{12}^L + \alpha_{n3} c_2 &< 0.
 \end{aligned} \tag{64}$$

Consider the following discrete Lyapunov functional:

$$\begin{aligned}
 &V_4(n) = x_1^{\alpha_{n1}}(n) x_2^{-\alpha_{n2}}(n) \\
 &\times \exp \left\{ \alpha_{n3} u(n) - \alpha_{n1} \sum_{s=0}^{\infty} k_{12}(s) \right. \\
 &\times \sum_{q=n-s}^{n-1} b_{12}(q+s) x_2(q) \\
 &- \alpha_{n1} \sum_{s=0}^{\infty} h_1(s) \sum_{q=n-s}^{n-1} d_1(q+s) u(q) \\
 &+ \alpha_{n2} \sum_{s=0}^{\infty} k_{21}(s) \sum_{q=n-s}^{n-1} b_{21}(q+s) x_1(q) \\
 &+ \alpha_{n2} \sum_{s=0}^{\infty} h_2(s) \sum_{q=n-s}^{n-1} d_2(q+s) u(q) \\
 &\left. + \alpha_{n3} \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \sum_{q=n-s}^{n-1} c_i x_i(q) \right\}.
 \end{aligned} \tag{65}$$



From (65), we obtain

$$\begin{aligned} & \frac{V_4(n+1)}{V_4(n)} \\ &= \exp \left\{ \alpha_{n1} \left( a_1(n) - b_{11}(n) x_1(n) \right. \right. \\ & \quad \left. \left. - b_{12}(n) \sum_{s=0}^{\infty} k_{12}(s) x_2(n-s) \right. \right. \\ & \quad \left. \left. - d_1(n) \sum_{s=0}^{\infty} h_1(s) u(n-s) \right) \right. \\ & \quad \left. - \alpha_{n2} \left( a_2(n) - b_{22}(n) x_2(n) \right. \right. \\ & \quad \left. \left. - b_{21}(n) \sum_{s=0}^{\infty} k_{21}(s) x_1(n-s) \right. \right. \\ & \quad \left. \left. - d_2(n) \sum_{s=0}^{\infty} h_2(s) u(n-s) \right) \right. \\ & \quad \left. + \alpha_{n3} \left( -eu(n) + c_1 \sum_{s=0}^{\infty} l_1(s) x_1(n-s) \right. \right. \\ & \quad \left. \left. + c_2 \sum_{s=0}^{\infty} l_2(s) x_2(n-s) \right) \right. \\ & \quad \left. - \alpha_{n1} \sum_{s=0}^{\infty} k_{12}(s) (b_{12}(n+s) x_2(n) \right. \\ & \quad \quad \left. - b_{12}(n) x_2(n-s)) \right. \\ & \quad \left. - \alpha_{n1} \sum_{s=0}^{\infty} h_1(s) (d_1(n+s) u(n) \right. \\ & \quad \quad \left. - d_1(n) u(n-s)) \right. \\ & \quad \left. + \alpha_{n2} \sum_{s=0}^{\infty} k_{21}(s) (b_{21}(n+s) x_1(n) \right. \\ & \quad \quad \left. - b_{21}(n) x_1(n-s)) \right. \\ & \quad \left. + \alpha_{n2} \sum_{s=0}^{\infty} h_2(s) (d_2(n+s) u(n) \right. \\ & \quad \quad \left. - d_2(n) u(n-s)) \right. \\ & \quad \left. + \alpha_{n3} \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \right. \\ & \quad \left. \times (c_i x_i(n) - c_i x_i(n-s)) \right\} \\ & \leq \exp \left\{ (\alpha_{n1} a_1^M - \alpha_{n2} a_2^L) \right. \end{aligned}$$

$$\begin{aligned} & \left. + (\alpha_{n2} b_{21}^M - \alpha_{n1} b_{11}^L + \alpha_{n3} c_1) x_1(n) \right. \\ & \left. + (\alpha_{n2} b_{22}^M - \alpha_{n1} b_{12}^L + \alpha_{n3} c_2) x_2(n) \right. \\ & \left. + (\alpha_{n2} d_2^M - \alpha_{n1} d_1^L - \alpha_{n3} e) u(n) \right\}. \end{aligned} \tag{66}$$

From inequalities (63) and (64), we can obtain

$$V_4(n+1) \leq V_4(n) \exp(-\eta). \tag{67}$$

Therefore

$$V_4(n) \leq V_4(0) \exp(-n\eta). \tag{68}$$

From (22) and (27) we know that there exists an  $M > 0$  such that

$$x_i(k) < M, \quad u(k) < M \quad \forall k \in Z, \tag{69}$$

and so

$$\begin{aligned} & V_4(0) \\ &= x_1^{\alpha_{n1}}(0) x_2^{-\alpha_{n2}}(0) \\ & \quad \times \exp \left\{ \alpha_{n3} u(0) - \alpha_{n1} \sum_{s=0}^{\infty} k_{12}(s) \right. \\ & \quad \times \sum_{q=-s}^{-1} b_{12}(q+s) x_2(q) \\ & \quad \left. - \alpha_{n1} \sum_{s=0}^{\infty} h_1(s) \sum_{q=-s}^{-1} d_1(q+s) u(q) \right. \\ & \quad \left. + \alpha_{n2} \sum_{s=0}^{\infty} k_{21}(s) \sum_{q=-s}^{-1} b_{21}(q+s) x_1(q) \right. \\ & \quad \left. + \alpha_{n2} \sum_{s=0}^{\infty} h_2(s) \sum_{q=n-s}^{n-1} d_2(q+s) u(q) \right. \\ & \quad \left. + \alpha_{n3} \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \sum_{q=-s}^{-1} c_i x_i(q) \right\} \\ & < x_1^{\alpha_{n1}}(0) x_2^{-\alpha_{n2}}(0) \\ & \quad \times \exp \left\{ |\alpha_{n3}| M + \alpha_{n2} M b_{21}^M \sum_{s=0}^{\infty} k_{21}(s) s \right. \\ & \quad \left. + \alpha_{n2} M d_2^M \sum_{s=0}^{\infty} h_2(s) s \right. \\ & \quad \left. + \alpha_{n3} M c_i \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) s \right\} \end{aligned}$$

$$\begin{aligned}
 &= x_1^{\alpha_{n1}}(0) x_2^{-\alpha_{n2}}(0) \\
 &\times \exp \left\{ M \left( |\alpha_{n3}| + \alpha_{n2} b_{21}^M \Theta_{21} + \alpha_{n2} d_2^M \Lambda_2 \right. \right. \\
 &\quad \left. \left. + \alpha_{n3} \sum_{i=1}^2 c_i Y_i \right) \right\} < +\infty.
 \end{aligned} \tag{70}$$

On the other hand, we also have

$$\begin{aligned}
 &V_4(n) \\
 &\geq x_1^{\alpha_{n1}}(n) x_2^{-\alpha_{n2}}(n) \\
 &\times \exp \left\{ -\alpha_{n1} \sum_{s=0}^{\infty} k_{12}(s) \sum_{q=n-s}^{n-1} b_{12}(q+s) x_2(q) \right. \\
 &\quad \left. - \alpha_{n1} \sum_{s=0}^{\infty} h_1(s) \sum_{q=n-s}^{n-1} d_1(q+s) u(q) \right\} \\
 &\geq x_1^{\alpha_{n1}}(n) M^{-\alpha_{n2}} \\
 &\times \exp \left\{ -\alpha_{n1} M b_{12}^M \sum_{s=0}^{\infty} k_{12}(s) s \right. \\
 &\quad \left. - \alpha_{n1} M d_1^M \sum_{s=0}^{\infty} h_1(s) s \right\} \\
 &= x_1^{\alpha_{n1}}(n) M^{-\alpha_{n2}} \\
 &\times \exp \left\{ -\alpha_{n1} M (b_{12}^M \Theta_{12} + d_1^M \Lambda_1) \right\}.
 \end{aligned} \tag{71}$$

Combining inequalities (68), (70), and (71), we have

$$x_1(n) \leq \lambda \exp \left\{ -\frac{\eta}{\alpha_{n1}} n \right\}, \tag{72}$$

where

$$\begin{aligned}
 \lambda &= (V_4(0))^{1/\alpha_{n1}} M^{\alpha_{n2}/\alpha_{n1}} \exp \left\{ M (b_{12}^M \Theta_{12} + d_1^M \Lambda_1) \right\} \\
 &< +\infty.
 \end{aligned} \tag{73}$$

Hence we obtain that

$$\lim_{n \rightarrow +\infty} x_1(n) = 0. \tag{74}$$

This ends the proof of Theorem 8.  $\square$

*Proof of Theorem 9.* By (60), we can choose positive constants  $\beta_{n1}$ ,  $\beta_{n2}$ , and  $\beta_{n3}$  and constant  $\gamma$  such that

$$\begin{aligned}
 \frac{a_1^M}{a_2^L} &> \frac{\beta_{n2}}{\beta_{n1}} > \frac{b_{11}^M e + d_1^M c_1}{b_{21}^L e + d_2^L c_1}, \\
 \frac{a_1^M}{a_2^L} &> \frac{\beta_{n2}}{\beta_{n1}} > \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2}. \\
 \beta_{n2} a_2^M - \beta_{n1} a_1^L &< -\gamma < 0, \\
 \beta_{n1} b_{11}^M - \beta_{n2} b_{21}^L + \beta_{n3} c_1 &< 0, \\
 \beta_{n1} b_{12}^M - \beta_{n2} b_{22}^L + \beta_{n3} c_2 &< 0, \\
 \beta_{n1} d_1^M - \beta_{n2} d_2^L - \beta_{n3} e &< 0.
 \end{aligned} \tag{75}$$

Define the following Lyapunov functional:

$$\begin{aligned}
 &V_5(n) \\
 &= x_1^{-\beta_{n1}}(n) x_2^{\beta_{n2}}(n) \\
 &\times \exp \left\{ \beta_{n3} u(n) + \beta_{n1} \sum_{s=0}^{\infty} k_{12}(s) \right. \\
 &\quad \times \sum_{q=n-s}^{n-1} b_{12}(q+s) x_2(q) \\
 &\quad + \beta_{n1} \sum_{s=0}^{\infty} h_1(s) \sum_{q=n-s}^{n-1} d_1(q+s) u(q) \\
 &\quad - \beta_{n2} \sum_{s=0}^{\infty} k_{21}(s) \sum_{q=n-s}^{n-1} b_{21}(q+s) x_1(q) \\
 &\quad - \beta_{n2} \sum_{s=0}^{\infty} h_2(s) \sum_{q=n-s}^{n-1} d_2(q+s) u(q) \\
 &\quad \left. + \beta_{n3} \sum_{i=1}^2 \sum_{s=0}^{\infty} l_i(s) \sum_{q=n-s}^{n-1} c_i x_i(q) \right\}.
 \end{aligned} \tag{76}$$

From (76), we have

$$\begin{aligned}
 &\frac{V_5(n+1)}{V_5(n)} \\
 &\leq \exp \left\{ (\beta_{n2} a_2^M - \beta_{n1} a_1^L) \right. \\
 &\quad + (\beta_{n1} b_{11}^M - \beta_{n2} b_{21}^L + \beta_{n3} c_1) x_1(n) \\
 &\quad + (\beta_{n1} b_{12}^M - \beta_{n2} b_{22}^L + \beta_{n3} c_2) x_2(n) \\
 &\quad \left. + (\beta_{n1} d_1^M - \beta_{n2} d_2^L - \beta_{n3} e) u(n) \right\} \\
 &\leq \exp \{-\gamma\}.
 \end{aligned} \tag{77}$$

Similarly to the analysis of (68)–(73), we have

$$\lim_{n \rightarrow +\infty} x_2(n) = 0. \tag{78}$$

This ends the proof of Theorem 9.  $\square$

From Theorems 8 and 9 we know that, under some suitable assumption, one of the species in the system may be driven to extinction; in this case, one interesting problem is to investigate the stability property of the rest of the species.

Consider the following discrete equations:

$$\begin{aligned} x_2(n+1) &= x_2(n) \exp \left\{ a_2(n) - b_{22}(n) x_2(n) \right. \\ &\quad \left. - d_2(n) \sum_{s=0}^{\infty} h_2(s) u(n-s) \right\}, \\ u(n+1) &= u(n) (1-e) + c_2 \sum_{s=0}^{\infty} l_2(s) x_2(n-s). \end{aligned} \tag{79}$$

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left\{ a_1(n) - b_{11}(n) x_1(n) \right. \\ &\quad \left. - d_1(n) \sum_{s=0}^{\infty} h_1(s) u(n-s) \right\}, \\ u(n+1) &= u(n) (1-e) + c_1 \sum_{s=0}^{\infty} l_1(s) x_1(n-s). \end{aligned} \tag{80}$$

**Theorem 10.** Assume that (58) holds and also

$$d_2^M < \frac{A_{22}e}{c_2} \tag{81}$$

holds; then, for any positive solution  $(x_1(n), x_2(n), u(n))$  of system (5) and any positive solution  $(x_2^*(n), u^*(n))$  of system (79), we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} x_1(n) &= 0, & \lim_{n \rightarrow +\infty} (x_2(n) - x_2^*(n)) &= 0, \\ \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) &= 0, \end{aligned} \tag{82}$$

where  $A_{22}$  is defined in Theorem 7.

**Theorem 11.** Assume that (60) holds and also

$$d_1^M < \frac{A_{11}e}{c_1} \tag{83}$$

holds; then, for any positive solution  $(x_1(n), x_2(n), u(n))$  of system (5) and any positive solution  $(x_1^*(n), u^*(n))$  of system (80), we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} (x_1(n) - x_1^*(n)) &= 0, & \lim_{n \rightarrow +\infty} x_2(n) &= 0, \\ \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) &= 0, \end{aligned} \tag{84}$$

where  $A_{11}$  is defined in Theorem 7.

*Proof of Theorem 10.* By conditions (81), we can choose positive constants  $\beta_1, \beta_2$  such that

$$\frac{e}{d_2^M} > \frac{\beta_1}{\beta_2} > \frac{c_2}{A_{22}}. \tag{85}$$

Thus, there exist enough small positive constants  $\delta$  and  $\varepsilon$  such that

$$\begin{aligned} \beta_1 A_{22}^\varepsilon - \beta_2 c_2 &> \delta, \\ \beta_2 e - \beta_1 d_2^M &> \delta, \end{aligned} \tag{86}$$

where  $A_{22}^\varepsilon$  is defined in (43)

Now, we define a Lyapunov functional

$$Q_1(n) = \beta_1 G_1(n) + \beta_2 H_1(n), \tag{87}$$

where

$$\begin{aligned} G_1(n) &= |\ln x_2(n) - \ln x_2^*(n)| + \sum_{s=0}^{\infty} k_{21}(s) \\ &\quad \times \sum_{q=n-s}^{n-1} b_{21}(q+s) x_1(q) + \sum_{s=0}^{\infty} h_2(s) \\ &\quad \times \sum_{q=n-s}^{n-1} d_2(q+s) |u(q) - u^*(q)|; \end{aligned} \tag{88}$$

$$\begin{aligned} H_1(n) &= |u(n) - u^*(n)| + \sum_{s=0}^{\infty} l_1(s) \\ &\quad \times \sum_{q=n-s}^{n-1} c_1 x_1(q) + \sum_{s=0}^{\infty} l_2(s) \\ &\quad \times \sum_{q=n-s}^{n-1} c_2 |x_2(q) - x_2^*(q)|. \end{aligned}$$

Then, from the definition of  $G_1(n), H_1(n)$ , one could easily see that  $Q_1(n) \geq 0$  for all  $n \in \mathbb{Z}^+$ . Also, for any fixed  $n^* \in \mathbb{Z}^+$ , from (69) one could see that

$$\begin{aligned} Q_1(n^*) &= \beta_1 |\ln x_2(n^*) - \ln x_2^*(n^*)| \\ &\quad + \beta_1 M b_{21}^M \sum_{s=0}^{\infty} k_{21}(s) s \\ &\quad + \beta_1 d_2^M \sup_{q \in \mathbb{Z}^+, q \leq n^*} |u(q) - u^*(q)| \sum_{s=0}^{\infty} h_2(s) s \\ &\quad + \beta_2 |u(n^*) - u^*(n^*)| + \beta_2 M c_1 \sum_{s=0}^{\infty} l_1(s) s \\ &\quad + \beta_2 c_2 \sup_{q \in \mathbb{Z}^+, q \leq n^*} |x_2(q) - x_2^*(q)| \sum_{s=0}^{\infty} l_2(s) s \\ &< +\infty. \end{aligned} \tag{89}$$

It follows from the second equation of system (5) and the Mean Value Theorem that

$$\begin{aligned}
& \Delta G_1(n) \\
& \leq |\ln x_2(n) - \ln x_2^*(n) - b_{22}(n)(x_2(n) - x_2^*(n))| \\
& \quad - |\ln x_2(n) - \ln x_2^*(n)| \\
& \quad + b_{21}(n) \sum_{s=0}^{\infty} k_{21}(s) x_1(n-s) + d_2(n) \\
& \quad \times \sum_{s=0}^{\infty} h_2(s) |u(n-s) - u^*(n-s)| \\
& \quad + \sum_{s=0}^{\infty} k_{21}(s) (b_{21}(n+s) x_1(n) \\
& \quad \quad - b_{21}(n) x_1(n-s)) \\
& \quad + \sum_{s=0}^{\infty} h_2(s) (d_2(n+s) |u(n) - u^*(n)| \\
& \quad \quad - d_2(n) |u(n-s) - u^*(n-s)|) \\
& \leq - \left( \frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - b_{22}(n) \right| \right) |x_2(n) - x_2^*(n)| \\
& \quad + \sum_{s=0}^{\infty} k_{21}(s) b_{21}(n+s) x_1(n) \\
& \quad + \sum_{s=0}^{\infty} h_2(s) d_2(n+s) |u(n) - u^*(n)| \\
& \leq - \left( \frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - b_{22}(n) \right| \right) |x_2(n) - x_2^*(n)| \\
& \quad + b_{12}^M x_1(n) + d_2^M |u(n) - u^*(n)|.
\end{aligned} \tag{90}$$

Also, from the third equation of system (5) we have

$$\begin{aligned}
\Delta H_1(n) &= H_1(n+1) - H_1(n) \\
&\leq -e |u(n) - u^*(n)| \\
&\quad + c_1 \sum_{s=0}^{\infty} l_1(s) x_1(n-s) \\
&\quad + c_2 \sum_{s=0}^{\infty} l_2(s) |x_2(n-s) - x_2^*(n-s)| \\
&\quad + \sum_{s=0}^{\infty} l_1(s) (c_1 x_1(n) - c_1 x_1(n-s)) \\
&\quad + \sum_{s=0}^{\infty} l_2(s) (c_2 |x_2(n) - x_2^*(n)| \\
&\quad \quad - c_2 |x_2(n-s) - x_2^*(n-s)|)
\end{aligned}$$

$$\begin{aligned}
&= -e |u(n) - u^*(n)| + \sum_{s=0}^{\infty} l_1(s) c_1 x_1(n) \\
&\quad + \sum_{s=0}^{\infty} l_2(s) c_2 |x_2(n) - x_2^*(n)| \\
&\leq -e |u(n) - u^*(n)| + c_1 x_1(n) \\
&\quad + c_2 |x_2(n) - x_2^*(n)|.
\end{aligned} \tag{91}$$

From (70), we have

$$\sum_{p=n^*}^{\infty} x_1(p) \leq \sum_{p=n^*}^{\infty} \lambda \exp \left\{ -\frac{\eta}{\alpha_{n1}} p \right\} < +\infty. \tag{92}$$

Therefore, from (44), (88), and (90), for all  $n > n^*$ , we have

$$\begin{aligned}
& \Delta Q_1(n) \\
& \leq \beta_1 \left\{ - \left( \frac{1}{\varphi_2(n)} - \left| \frac{1}{\varphi_2(n)} - b_{22}(n) \right| \right) |x_2(n) - x_2^*(n)| \right. \\
& \quad \left. + b_{21}^M x_1(n) + d_2^M |u(n) - u^*(n)| \right\} \\
& \quad + \beta_2 \{ -e |u(n) - u^*(n)| + c_1 x_1(n) + c_2 |x_2(n) - x_2^*(n)| \} \\
& \leq - \left\{ \beta_1 \min \left[ b_{22}^L, \frac{2}{M_2 + \varepsilon} - b_{22}^M \right] - \beta_2 c_2 \right\} |x_2(n) - x_2^*(n)| \\
& \quad - \{ \beta_2 e - \beta_1 d_2^M \} |u(n) - u^*(n)| + (\beta_1 b_{21}^M + \beta_2 c_1) x_1(n) \\
& = - \{ \beta_1 A_{22}^e - \beta_2 c_2 \} |x_2(n) - x_2^*(n)| \\
& \quad - \{ \beta_2 e - \beta_1 d_2^M \} |u(n) - u^*(n)| + (\beta_1 b_{21}^M + \beta_2 c_1) x_1(n) \\
& \leq -\delta (|x_2(n) - x_2^*(n)| + |u(n) - u^*(n)|) + \rho x_1(n),
\end{aligned} \tag{93}$$

where  $\rho = \beta_1 b_{21}^M + \beta_2 c_1$ .

Summating both sides of the above inequality from  $n^*$  to  $n$ , we have

$$\begin{aligned}
& \sum_{p=n^*}^n (Q_1(p+1) - Q_1(p)) \\
& \leq -\delta \sum_{p=n^*}^n (|x_2(p) - x_2^*(p)| + |u(p) - u^*(p)|) \\
& \quad + \rho \sum_{p=n^*}^n x_1(p).
\end{aligned} \tag{94}$$

Hence

$$\begin{aligned}
& Q_1(n+1) + \delta \sum_{p=n^*}^n (|x_2(p) - x_2^*(p)| + |u(p) - u^*(p)|) \\
& \leq Q_1(n^*) + \rho \sum_{p=n^*}^n x_1(p).
\end{aligned} \tag{95}$$

Then, from (89) and (92) we have

$$\begin{aligned} & \sum_{p=n^*}^n (|x_2(p) - x_2^*(p)| + |u(p) - u^*(p)|) \\ & \leq \frac{Q_1(n^*) + \rho \sum_{p=n^*}^n x_1(p)}{\delta} < +\infty. \end{aligned} \tag{96}$$

Therefore

$$\sum_{p=n^*}^{\infty} (|x_2(p) - x_2^*(p)| + |u(p) - u^*(p)|) < +\infty, \tag{97}$$

which means that

$$\lim_{n \rightarrow +\infty} (|x_2(n) - x_2^*(n)| + |u(n) - u^*(n)|) = 0. \tag{98}$$

Consequently

$$\begin{aligned} \lim_{n \rightarrow +\infty} (x_2(n) - x_2^*(n)) &= 0, \\ \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) &= 0. \end{aligned} \tag{99}$$

This completes the proof of Theorem 10.  $\square$

*Proof of Theorem 11.* The proof of Theorem 11 is similar to that of Theorem 10, and we omit the details here.  $\square$

*Remark 12.* One of the purposes of this paper is to find out the influence of feedback control variable on the persistent property of the system. Obviously, the answer lies in the relations among conditions (A<sub>1</sub>), (58), and (60). Now let us consider conditions (A<sub>1</sub>) and (60); there may exist the following subcase:

$$\frac{b_{11}^L}{b_{21}^M} > \frac{\exp(a_1^M - 1)}{a_2^L} > \frac{a_1^M}{a_2^L} > \frac{a_1^L}{a_2^M} > \frac{b_{11}^M e + d_1^M c_1}{b_{21}^L e + d_2^L c_1}. \tag{100}$$

If  $b_{12}^M/b_{22}^L > d_1^M/d_2^L$ , then we have

$$\frac{a_1^L}{a_2^M} > \frac{a_1^L}{\exp(a_2^M - 1)} > \frac{b_{12}^M}{b_{22}^L} > \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2}. \tag{101}$$

If  $b_{12}^M/b_{22}^L < d_1^M/d_2^L$ , then there may exist the following two subcases:

$$\begin{aligned} \frac{a_1^L}{a_2^M} &> \frac{a_1^L}{\exp(a_2^M - 1)} > \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2} > \frac{b_{12}^M}{b_{22}^L}, \\ \frac{a_1^L}{a_2^M} &> \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2} > \frac{a_1^L}{\exp(a_2^M - 1)} > \frac{b_{12}^M}{b_{22}^L}. \end{aligned} \tag{102}$$

(101) and (100) (or (102) and (100)) show that conditions (A<sub>1</sub>) and (60) could be satisfied together; that is, for the original permanent system (2), by choosing suitable feedback control variable, species  $x_2$  will be driven to extinction.

*Remark 13.* Similarly to the above analysis, if system (2) is permanent, by choosing suitable feedback control variable, the first species  $x_1$  will be driven to extinction.

*Remark 14.* From Theorem 7, we can find that feedback control variable has influence on the global attractivity of system (5); that is, only the feedback control variable is very low such that the third inequality of (39) holds; in other words, when inequality  $d = \max\{d_1^M, d_2^M\} < \alpha_3 e^L / (\alpha_1 + \alpha_2)$  holds, the two species can be coexist.

### 6. Examples

In this section we will give three examples to illustrate the feasibility of main results.

*Example 1.* Consider the following equations:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left\{ 0.9 + 0.1 \sin(\sqrt{2}n) \right. \\ &\quad - (0.25 + 0.05 \cos(\sqrt{3}n)) x_1(n) \\ &\quad - (0.08 + 0.02 \cos(\sqrt{3}n)) \\ &\quad \times \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_2(n-s) \\ &\quad \left. - (0.45 + 0.03 \sin(\sqrt{2}n)) \right. \\ &\quad \left. \times \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\}, \\ x_2(n+1) &= x_2(n) \exp \left\{ 0.9 + 0.1 \sin(\sqrt{2}n) \right. \\ &\quad - (0.6 + 0.1 \cos(\sqrt{3}n)) x_2(n) \\ &\quad - (0.15 + 0.05 \cos(\sqrt{3}n)) \\ &\quad \times \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_1(n-s) \\ &\quad \left. - (0.42 + 0.04 \sin(\sqrt{2}n)) \right. \\ &\quad \left. \times \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\}, \\ u(n+1) &= u(n) \left( 1 - (0.92 + 0.03 \cos(\sqrt{3}n)) \right) \\ &\quad + (0.01 + 0.005 \cos(2n)) \\ &\quad \times \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_1(n-s) \\ &\quad + (0.008 + 0.002 \cos(\sqrt{3}n)) \\ &\quad \times \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_2(n-s). \end{aligned} \tag{103}$$

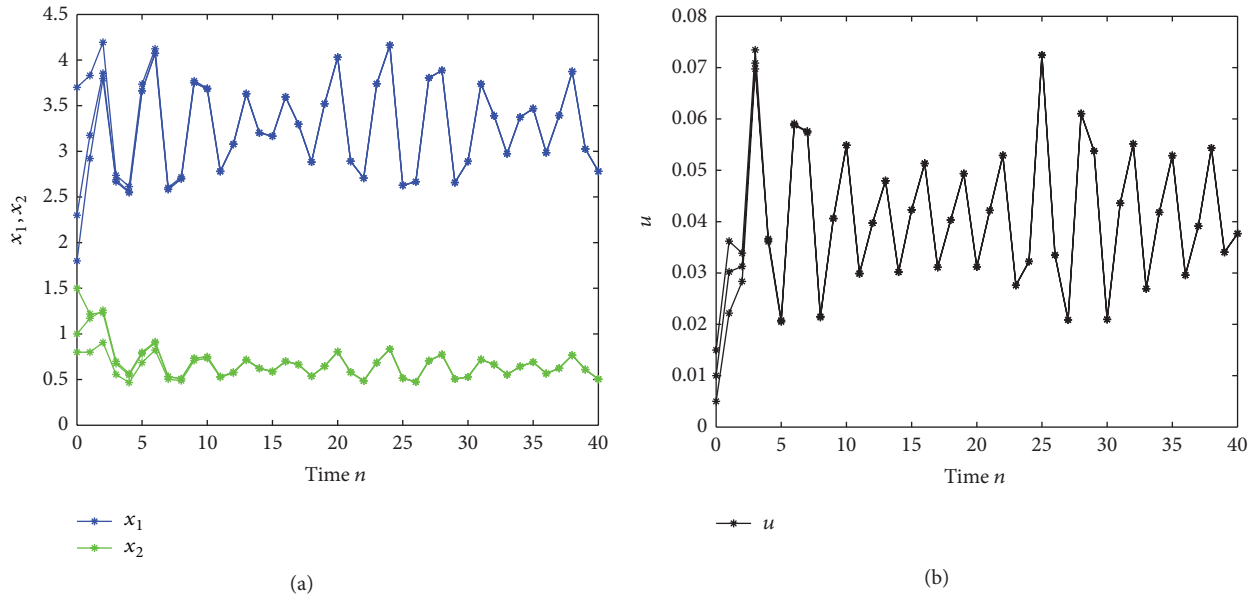


FIGURE 1: Numeric simulations of the solutions  $(x_1(n), x_2(n), u(n))$  of system (103), with the initial conditions  $(x_1(s), x_2(s), u(s)) = (1.8, 1, 0.005), (2.3, 1.5, 0.01),$  and  $(3.7, 0.8, 0.015), s = \dots, -n, -n + 1, \dots, -1, 0,$  respectively.

One could easily see that conditions  $(H_1)$  and  $(H_2)$  are satisfied. Also, by calculating, one has

$$\begin{aligned}
 A_{11} &= \min \left\{ b_{11}^L, \frac{2}{M_1} - b_{11}^M \right\} = 0.1, \\
 A_{22} &= \min \left\{ b_{22}^L, \frac{2}{M_2} - b_{22}^M \right\} = 0.3.
 \end{aligned}
 \tag{104}$$

Now, let us take  $\alpha_1 = 1.3, \alpha_2 = 0.5,$  and  $\alpha_3 = 1;$  then

$$\begin{aligned}
 \alpha_1 A_{11} - \alpha_2 b_{21}^M - \alpha_3 c_1^M &= 0.015 > 0, \\
 \alpha_2 A_{22} - \alpha_1 b_{12}^M - \alpha_3 c_2^M &= 0.01 > 0, \\
 \alpha_3 e^L - \alpha_1 d_1^M - \alpha_2 d_2^M &= 0.046 > 0.
 \end{aligned}
 \tag{105}$$

Clearly, condition (39) is satisfied, and so from Theorem 7 we have  $\lim_{n \rightarrow +\infty} (x_i(n) - x_i^*(n)) = 0, \lim_{n \rightarrow +\infty} (u(n) - u^*(n)) = 0,$  where  $(x_1(n), x_2(n), u(n))$  and  $(x_1^*(n), x_2^*(n), u^*(n))$  are any two positive solutions of system (103).

Figure 1 shows the dynamic behaviors of system (103), which strongly supports the above assertions.

*Example 2.* Consider the following equations:

$$\begin{aligned}
 x_1(n+1) &= x_1(n) \exp \left\{ 0.8 + 0.2 \sin(\sqrt{2}n) - 3x_1(n) \right. \\
 &\quad \left. - 1.5 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_2(n-s) \right. \\
 &\quad \left. - 0.1 \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\},
 \end{aligned}$$

$$\begin{aligned}
 x_2(n+1) &= x_2(n) \exp \left\{ 1.8 + 0.2 \sin(\sqrt{2}n) - x_2(n) \right. \\
 &\quad \left. - 3 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_1(n-s) \right. \\
 &\quad \left. - 0.2 \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\}, \\
 u(n+1) &= u(n) (1 - 0.8) \\
 &\quad + 1.2 \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_1(n-s) \\
 &\quad + 3 \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_2(n-s).
 \end{aligned}
 \tag{106}$$

One could easily see that conditions  $(H_1)$  and  $(H_2)$  are satisfied. Also, by calculating, one has

$$\begin{aligned}
 \frac{a_1^M}{a_2^L} &= \frac{5}{8}, & \frac{b_{11}^L e + d_1^L c_1}{b_{21}^M e + d_2^M c_1} &= \frac{9}{10}, \\
 \frac{b_{12}^L e + d_1^L c_2}{b_{22}^M e + d_2^M c_2} &= \frac{15}{14}, \\
 A_{22} &= \min \left\{ b_{22}^L, \frac{2}{M_2} - b_{22}^M \right\} = 1, \\
 0.2 &= d_2^M < \frac{A_{22} e}{c_2} \approx 0.67.
 \end{aligned}
 \tag{107}$$

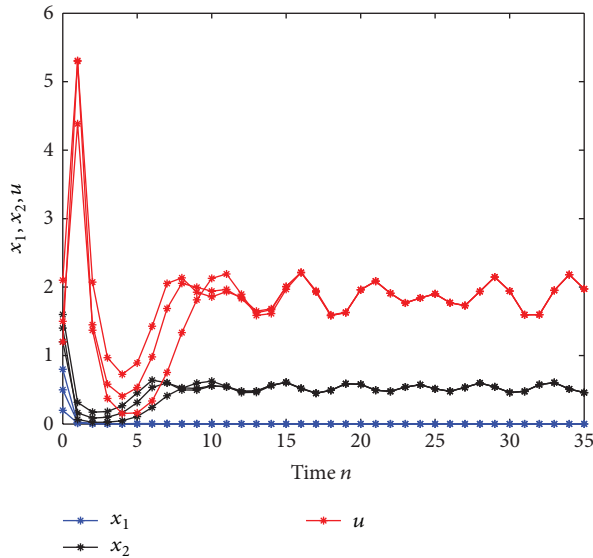


FIGURE 2: Numeric simulations of the solutions  $(x_1(n), x_2(n), u(n))$  of system (106), with the initial conditions  $(x_1(s), x_2(s), u(s)) = (0.8, 1.4, 1.2), (0.5, 1.2, 1.5),$  and  $(0.2, 1.6, 2.1), s = \dots, -n, -n + 1, \dots, -1, 0,$  respectively.

Clearly, conditions (58) and (81) are satisfied, and so from Theorems 8 and 10 we know that species  $x_1$  will be driven to extinction, while species  $x_2$  is globally attractive.

Figure 2 shows the dynamic behaviors of system (106), which strongly supports our results.

Example 3. Consider the following example:

$$\begin{aligned}
 x_1(n+1) &= x_1(n) \exp \left\{ 0.9 + 0.1 \sin(n) - 0.6x_1(n) \right. \\
 &\quad \left. - 0.4 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_2(n-s) \right\}, \\
 x_2(n+1) &= x_2(n) \exp \left\{ 0.8 + 0.2 \sin(n) - 0.8x_2(n) \right. \\
 &\quad \left. - 0.3 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_1(n-s) \right\}.
 \end{aligned}
 \tag{108}$$

By calculating, one has

$$\begin{aligned}
 \frac{b_{11}^L}{b_{21}^M} &= 2 > \frac{\exp(a_1^M - 1)}{a_2^L} = \frac{5}{3}, \\
 \frac{a_1^L}{\exp(a_2^M - 1)} &= 0.8 > \frac{b_{12}^M}{b_{22}^L} = 0.5.
 \end{aligned}
 \tag{109}$$

Clearly, condition  $(A_1)$  is satisfied; thus from Theorem A we know that system (108) is permanent. Figure 3 shows the dynamic behaviors of system (108).

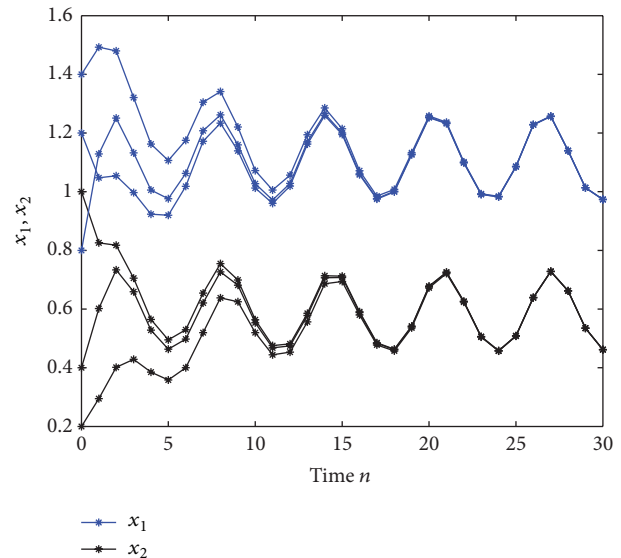


FIGURE 3: Dynamic behaviors of the solution  $(x_1(n), x_2(n))$  of system (110), with the initial conditions  $(x_1(s), x_2(s)) = (0.8, 0.4), (1.4, 1.2),$  and  $(1.2, 1.0), s = \dots, -n, -n + 1, \dots, -1, 0,$  respectively.

Now let us further incorporate the feedback control variable into system (108) and consider the following system:

$$\begin{aligned}
 x_1(n+1) &= x_1(n) \exp \left\{ 0.9 + 0.1 \sin(n) - 0.6x_1(n) \right. \\
 &\quad \left. - 0.4 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_2(n-s) \right. \\
 &\quad \left. - 0.2 \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\}, \\
 x_2(n+1) &= x_2(n) \exp \left\{ 0.8 + 0.2 \sin(n) - 0.8x_2(n) \right. \\
 &\quad \left. - 0.3 \sum_{s=0}^{\infty} \frac{e-1}{e} e^{-s} x_1(n-s) \right. \\
 &\quad \left. - 0.3 \sum_{s=0}^{\infty} \frac{e^2-1}{e^2} e^{-2s} u(n-s) \right\}, \\
 u(n+1) &= u(n) (1 - 0.5) \\
 &\quad + 5 \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_1(n-s) \\
 &\quad + \sum_{s=0}^{\infty} \frac{e^3-1}{e^3} e^{-3s} x_2(n-s).
 \end{aligned}
 \tag{110}$$

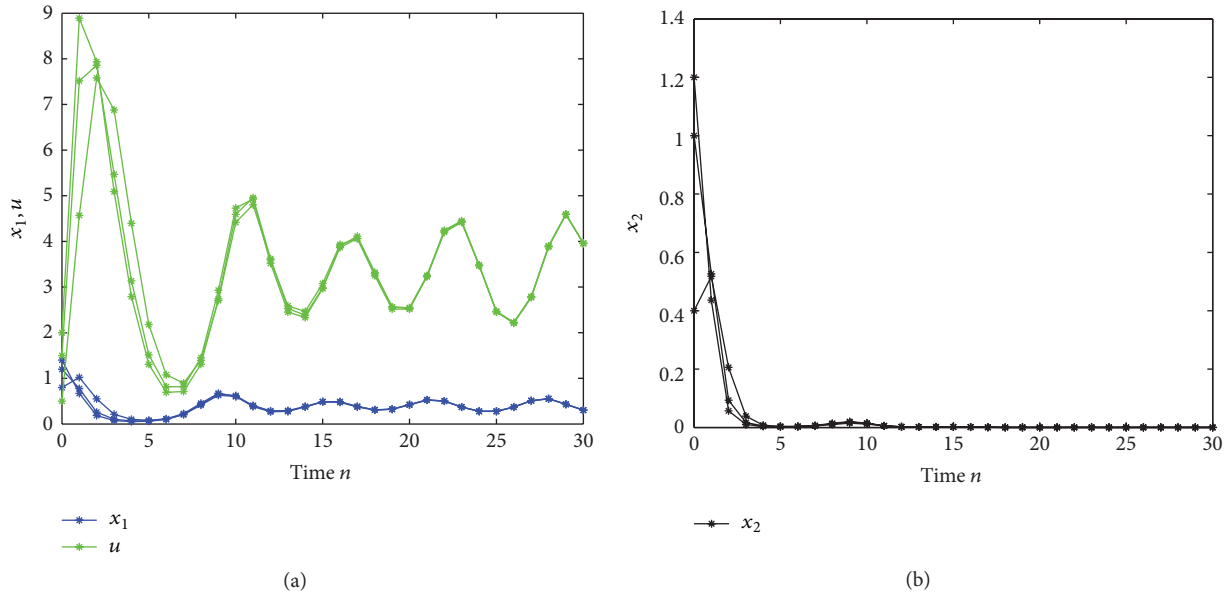


FIGURE 4: Numeric simulations of the solutions  $(x_1(n), x_2(n), u(n))$  of system (110), with the initial conditions  $(x_1(s), x_2(s), u(s)) = (0.8, 0.4, 0.5), (1.4, 1.2, 2),$  and  $(1.2, 1.0, 1.5), s = \dots, -n, -n + 1, \dots, -1, 0,$  respectively.

By calculating, one has

$$\frac{a_1^L}{a_2^M} = \frac{4}{5} > \frac{b_{12}^M e + d_1^M c_2}{b_{22}^L e + d_2^L c_2} = \frac{4}{7},$$

$$\frac{a_1^L}{a_2^M} = \frac{4}{5} > \frac{b_{11}^M e + d_1^M c_1}{b_{21}^L e + d_2^L c_1} = \frac{26}{33}.$$
(111)

Thus, all the conditions of Theorem 9 are satisfied, and so the permanent  $x_2$  species in system (108) become extinct in system (110).

Figure 4 shows the dynamic behaviors of system (110), which supports our assertion.

### 7. Discussion

(1) Li et al. [3] consider a continuous and autonomous Lotka-Volterra competitive system with infinite delays and feedback controls; if the Lotka-Volterra competitive system is globally stable, then they showed that the feedback controls only change the position of the unique positive equilibrium and retain the stable property. As a consequence of this result, feedback control variables have no influence on the persistent property of the system they considered; Xu et al. [5] proposed and studied the discrete nonautonomous Lotka-Volterra competitive system with delays and feedback controls (model (4)). Theorem 3.2 in [5] showed that feedback controls do not affect the persistent property of system (4). In this paper, we propose and study a discrete Lotka-Volterra competition system with single feedback control variable. Remarks 12 and 13 show that the feedback control variable plays an important role

in the persistent property of the system (2). If the Lotka-Volterra competitive system is permanent, then we show that, by choosing suitable feedback control variable, one of the species will be driven to extinction; that is, feedback control variable, which represents the biological control or some harvesting procedure, is an unstable factor of the system. Such a finding overturns the previous recognition on feedback control variables ([3, 5]) and shows that it is better for us to take a single policy to protect the rare species through reducing the species' competitions than to take two different policies.

(2) In [2], Chen proposed system (2) and investigated the persistent property of the system (see Theorem A in introduction section); however, he did not investigate the stability property and extinction property of the system. As a direct corollary of Theorem 7, concerned with the stability property of system (2), we have the following.

**Theorem B.** Assume that there exist positive constants  $\alpha_i, i = 1, 2,$  such that

$$\alpha_1 A_{11} - \alpha_2 b_{21}^M > 0, \quad \alpha_2 A_{22} - \alpha_1 b_{12}^M > 0 \tag{112}$$

hold; then, for any two positive solutions  $(x_1(n), x_2(n))$  and  $(x_1^*(n), x_2^*(n))$  of system (2), one has

$$\lim_{n \rightarrow +\infty} (x_i(n) - x_i^*(n)) = 0, \tag{113}$$

where

$$A_{ii} = \min \left\{ b_{ii}^L, \frac{2}{M_i} - b_{ii}^M \right\}, \quad i = 1, 2. \tag{114}$$



Now let us consider the autonomous case of system (2). Assume that  $a_i > 1, i = 1, 2$ ; then

$$A_{ii} = \min \left\{ b_{ii}, \frac{2b_{ii}}{\exp \{a_i - 1\}} - b_{ii} \right\} = \left( \frac{2}{\exp \{a_i - 1\}} - 1 \right) b_{ii}. \tag{115}$$

As a corollary of Theorem B, we have the following.

**Theorem B'.** Assume that  $a_i > 1, i = 1, 2$ , and

$$\begin{aligned} \left( \frac{2}{\exp \{a_1 - 1\}} - 1 \right) b_{11} - b_{21} &> 0, \\ \left( \frac{2}{\exp \{a_2 - 1\}} - 1 \right) b_{22} - b_{12} &> 0 \end{aligned} \tag{116}$$

hold; then, for any two positive solutions  $(x_1(n), x_2(n))$  and  $(x_1^*(n), x_2^*(n))$  of the system (2), one has

$$\lim_{n \rightarrow +\infty} (x_i(n) - x_i^*(n)) = 0. \tag{117}$$

*Remark 15.* Wu and Zhang [19] obtained a set of sufficient conditions which ensure the global attractivity of a discrete autonomous Lotka-Volterra competition system with infinite delays and feedback controls; as a direct corollary of their main result, one could also obtain a set of sufficient conditions which ensure the global stability of the autonomous case of system (2). However, their result required the assumption  $a_i \leq 1, i = 1, 2$ . While Theorem B' solves the case  $a_i > 1, i = 1, 2$ . It is in this sense that our result Theorem B' supplements and complements the main result of Wu and Zhang [19].

Now, concerned with the extinction property of the system (2), with some minor revision of the proof of Theorems 8 and 9, we could obtain the following results.

**Theorem C.** Assume that

$$\frac{a_1^M}{a_2^L} < \frac{b_{11}^L}{b_{21}^M}, \quad \frac{a_1^M}{a_2^L} < \frac{b_{12}^L}{b_{22}^M} \tag{118}$$

hold; let  $(x_1(n), x_2(n))$  be any positive solution of system (2); then

$$\lim_{n \rightarrow +\infty} x_1(n) = 0. \tag{119}$$

**Theorem D.** Assume that

$$\frac{a_1^L}{a_2^M} > \frac{b_{11}^M}{b_{21}^L}, \quad \frac{a_1^L}{a_2^M} > \frac{b_{12}^M}{b_{22}^L} \tag{120}$$

hold; let  $(x_1(n), x_2(n))$  be any positive solution of system (2); then

$$\lim_{n \rightarrow +\infty} x_2(n) = 0. \tag{121}$$

It is interesting to investigate the stability property of the rest of the species when one of the species in system (2) is driven to extinction. In this case, as a direct corollary of Theorems 10 and 11, by simple computation, we have the following.

**Theorem E.** Assume that (118) holds and also

$$\frac{b_{22}^M}{b_{22}^L} \exp \{a_2^M - 1\} < 2 \tag{122}$$

holds; then, for any positive solution  $(x_1(n), x_2(n))$  of system (2), we have

$$\lim_{n \rightarrow +\infty} x_1(n) = 0, \quad \lim_{n \rightarrow +\infty} (x_2(n) - x_2^*(n)) = 0, \tag{123}$$

where  $x_2^*(n)$  is any positive solution of the system  $x_2(n+1) = x_2(n) \exp(a_2(n) - b_{22}(n)x_2(n))$ .

**Theorem F.** Assume that (120) holds and also

$$\frac{b_{11}^M}{b_{11}^L} \exp \{a_1^M - 1\} < 2 \tag{124}$$

holds; then, for any positive solution  $(x_1(n), x_2(n))$  of system (2), we have

$$\lim_{n \rightarrow +\infty} x_2(n) = 0, \quad \lim_{n \rightarrow +\infty} (x_1(n) - x_1^*(n)) = 0, \tag{125}$$

where  $x_1^*(n)$  is any positive solution of the system  $x_1(n+1) = x_1(n) \exp(a_1(n) - b_{11}(n)x_1(n))$ .

Obviously, Theorems B–F complement and supplement the main results of Chen [2].

### Appendix

The proof of the solution of system (5) which satisfies the initial conditions (8) is positive.

*Proof.* For convenience, we introduce the following notations:

$$\begin{aligned} F_i(n) &= a_i(n) - b_{ii}(n)x_i(n) - b_{ij}(n) \\ &\times \sum_{s=0}^{\infty} k_{ij}(s)x_j(n-s) - d_i(n) \sum_{s=0}^{\infty} h_i(s)u(n-s), \\ & \qquad \qquad \qquad i, j = 1, 2, i \neq j, \end{aligned}$$

$$\begin{aligned} b(n) &= c_1(n) \sum_{s=0}^{\infty} l_1(s)x_1(n-s) \\ &+ c_2(n) \sum_{s=0}^{\infty} l_2(s)x_2(n-s). \end{aligned} \tag{A.1}$$

And so, from the first and second equations of system (5), we have

$$\frac{x_i(n+1)}{x_i(n)} = \exp \{F_i(n)\}. \tag{A.2}$$

Therefore

$$x_i(n) = x_i(0) \exp \left\{ \sum_{p=0}^{n-1} F_i(p) \right\} > 0. \tag{A.3}$$

From the third equation of system (5), we have

$$u(n+1) \geq (1 - e^u)u(n) + b(n). \quad (\text{A.4})$$

Therefore

$$u(n) \geq (1 - e^u)^n u(0) + \sum_{p=0}^{n-1} (1 - e^u)^p b(n-p-1). \quad (\text{A.5})$$

From conditions (H<sub>1</sub>) and (8), we can know

$$1 - e^u > 0, \quad b(n-p-1) > 0. \quad (\text{A.6})$$

So

$$u(n) > 0. \quad (\text{A.7})$$

□

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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