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Research Article

On the Paranormed Nörlund Sequence Space of Nonabsolute Type

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Maddox defined the space $\ell(p)$ of the sequences $x=(x_k)$ such that $\sum_{k=0}^{\infty}|x_k|^{p_k}<\infty$, in Maddox, 1967. In the present paper, the Nörlund sequence space $N^t(p)$ of nonabsolute type is introduced and proved that the spaces $N^t(p)$ and $\ell(p)$ are linearly isomorphic. Besides this, the alpha-, beta-, and gamma-duals of the space $N^t(p)$ are computed and the basis of the space $N^t(p)$ is constructed. The classes $(N^t(p):\mu)$ and $(\mu:N^t(p))$ of infinite matrices are characterized. Finally, some geometric properties of the space $N^t(p)$ are investigated.

1. Introduction

We denote the space of all sequences of complex entries by ω. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞, c, and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. Also by bs, cs, ℓ_1, and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous; that is, $|\alpha_n-\alpha| \to 0$ and $g(x_n-x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M=\max\{1,H\}$. Then, the linear spaces $\ell(p)$ and $\ell_\infty(p)$ were defined by Maddox in [1] (see also [2, 3]) as follows:

$$\ell(p)=\left\{x=(x_k)\in\omega:\sum_k |x_k|^{p_k}<\infty\right\}$$
$$\text{with } 0<p_k\le H<\infty,$$
$$\ell_\infty(p)=\left\{x=(x_k)\in\omega:\sup_{k\in\mathbb{N}}|x_k|^{p_k}<\infty\right\} \tag{1}$$

which are the complete spaces paranormed by

$$g_1(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/M},$$
$$g_2(x)=\sup_{k\in\mathbb{N}}|x_k|^{p_k/M} \quad \text{iff } \inf_{k\in\mathbb{N}} p_k>0, \tag{2}$$

respectively, where $\mathbb{N}=\{0,1,2,\ldots\}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We assume throughout that $p_k^{-1}+(p_k')^{-1}=1$, provided $1<\inf p_k\le H<\infty$, and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in\omega: xz=(x_k z_k)\in\mu\ \forall x\in\lambda\}. \tag{3}$$

With the notation of (3), the alpha-, beta-, and gamma-duals of a sequence space λ, which are, respectively, denoted by λ^{α}, λ^{β}, and λ^{γ}, are defined by

$$\lambda^{\alpha} = S(\lambda, \ell_1), \qquad \lambda^{\beta} = S(\lambda, cs), \qquad \lambda^{\gamma} = S(\lambda, bs). \quad (4)$$

If a sequence space λ paranormed by g contains a sequence (b_n) with the property that, for every $x \in \lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0, \quad (5)$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

Let λ, μ be any two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}. \quad (6)$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (6) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$. Also, we write $A_n = (a_{nk})_{k\in\mathbb{N}}$ for the sequence in the nth row of A.

Now, following Peyerimhoff [4, pp. 17–19] and Mears [5], we give short knowledge on the Nörlund means. Let (t_k) be a sequence of nonnegative real numbers with $t_0 > 0$ and write $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then, the Nörlund means with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a^t_{nk})$ which is given by

$$a^t_{nk} = \begin{cases} \dfrac{t_{n-k}}{T_n}, & 0 \le k \le n, \\ 0, & k > n \end{cases} \quad (7)$$

for all $k, n \in \mathbb{N}$. It is known that the Nörlund matrix N^t is a Toeplitz matrix if and only if $t_n/T_n \to 0$, as $n \to \infty$, and is reduced in the case $t = e = (1, 1, 1, \ldots)$ to the matrix C_1 of arithmetic means. Additionally, for $t_n = A^{r-1}_n$ for all $n \in \mathbb{N}$, the method N^t is reduced to the Cesàro method C_r of order $r > -1$, where

$$A^r_n = \begin{cases} \dfrac{(r+1)(r+2)\cdots(r+n)}{n!}, & n = 1, 2, 3, \ldots, \\ 1, & n = 0. \end{cases} \quad (8)$$

Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1, 2, 3, \ldots\}$ by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}. \quad (9)$$

The inverse matrix $U^t = (u^t_{nk})$ of the matrix $N^t = (a^t_{nk})$ is given by Mears in [5] as follows:

$$u^t_{nk} = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \le k \le n, \\ 0, & k > n \end{cases} \quad (10)$$

for all $k, n \in \mathbb{N}$. Also, one can derive by straightforward calculation for all $k \in \{1, 2, 3, \ldots\}$ that

$$D_k = \sum_{j=1}^{k-1} (-1)^{j-1} t_j D_{k-j} + (-1)^{k-1} t_k. \quad (11)$$

The rest of this paper is organized as follows.

In Section 2, the complete paranormed Nörlund sequence space $N^t(p)$ is introduced and proved that $N^t(p)$ is linearly isomorphic to the space $\ell(p)$ and the basis for the space $N^t(p)$ is determined. Section 3 is devoted to the alpha-, beta-, and gamma-duals of the space $N^t(p)$. In Section 4, the classes $(N^t(p) : \mu)$ and $(\mu : N^t(p))$ of infinite matrices are characterized, where μ denotes any given sequence space. In Section 5, the rotundity of the space $N^t(p)$ is characterized and some results related to this concept are given. In the final section of the paper, the significance of the space is mentioned and further suggestions are recorded.

2. The Nörlund Sequence Space $N^t(p)$ of Nonabsolute Type

In this section, we define the Nörlund sequence space $N^t(p)$ and prove that $N^t(p)$ is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Finally, we give the basis for the space $N^t(p)$.

Let λ be any sequence space. Then, the matrix domain λ_A of an infinite matrix A in λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}. \quad (12)$$

In [6], Choudhary and Mishra defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that B-transforms of them are in $\ell(p)$, where $B = (b_{nk})$ is defined by

$$b_{nk} = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n. \end{cases} \quad (13)$$

Başar and Altay [7] examined the space $bs(p)$ which was formerly defined by Başar [8] as the set of all series whose sequences of partial sums are in the space $\ell_\infty(p)$. With the notation of (12), the spaces $\overline{\ell(p)}$ and $bs(p)$ can be redefined by

$$\overline{\ell(p)} = [\ell(p)]_B, \qquad bs(p) = [\ell_\infty(p)]_B. \quad (14)$$

In [9], Başar and Altay defined the sequence space $r^q(p)$ which consists of all sequences such that R^q-transforms of them are in $\ell(p)$, where $R^q = (r^q_{nk})$ is the matrix of Riesz mean; that is,

$$r^q(p) = \{\ell(p)\}_{R^q}, \qquad r^q_p = (\ell_p)_{R^q}. \tag{15}$$

In [10], Wang defined the sequence space $X_{a(p)}$ consisting of all sequences whose N^t-transforms are in ℓ_p which is a Banach space with the norm

$$\|x\|_p = \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^p\right)^{1/p} \quad \text{with } 1 \le p < \infty. \tag{16}$$

Now, we introduce the Nörlund sequence space $N^t(p)$ defined by

$$N^t(p) := \left\{x = (x_k) \in \omega : \sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} < \infty\right\} \quad \text{with } 0 < p_k \le H < \infty. \tag{17}$$

It is natural that the space $N^t(p)$ can also be defined with the notation of (12) that $N^t(p) = \{\ell(p)\}_{N^t}$.

Define the sequence $y = (y_k)$ by the N^t-transform of a sequence $x = (x_k)$; that is,

$$y_k = \frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j \quad \forall k \in \mathbb{N}. \tag{18}$$

Theorem 1. *$N^t(p)$ is a complete linear metric space paranormed by g defined by*

$$g(x) = \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k}\right)^{1/M} \quad \textit{with } 0 < p_k \le H < \infty. \tag{19}$$

Proof. Since this can be shown by a routine verification, we omit the detail. □

Remark 2. One can easily see that the absolute property does not hold on the space $N^t(p)$; that is, $g(x) \ne g(|x|)$ for at least one sequence in the space $N^t(p)$, and this says that $N^t(p)$ is a sequence space of nonabsolute type, where $|x| = (|x_k|)$.

Theorem 3. *The Nörlund sequence space $N^t(p)$ of nonabsolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.*

Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $N^t(p)$ and $\ell(p)$ for $0 < p_k \le H < \infty$. Consider the transformation T defined, with the notation of (18), from $N^t(p)$ to $\ell(p)$ by $x \mapsto y = Tx = N^t x$. The linearity of T is clear. Further, it is trivial that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let us take any $y \in \ell(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^k (-1)^{k-i} D_{k-i} T_i y_i \quad \forall k \in \mathbb{N}. \tag{20}$$

Therefore, we see from (19) that

$$\begin{aligned} g(x) &= \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j} \sum_{i=0}^j (-1)^{j-i} D_{j-i} T_i y_i\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_k |y_k|^{p_k}\right)^{1/M} = g_1(y) < \infty. \end{aligned} \tag{21}$$

This means that $x \in N^t(p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is linear bijection and this says us that the spaces $N^t(p)$ and $\ell(p)$ are linearly isomorphic. Therefore, the proof is completed. □

We determine the basis for the paranormed space $N^t(p)$.

Theorem 4. *Define the sequence $b^{(k)}(t) = \{b^{(k)}_n(t)\}_{n\in\mathbb{N}}$ of the elements of the space $N^t(p)$ for every fixed $k \in \mathbb{N}$ by*

$$b^{(k)}_n(t) = \begin{cases} (-1)^{n-k} D_{n-k} T_k, & 0 \le k \le n, \\ 0, & k > n. \end{cases} \tag{22}$$

Then, the sequence $\{b^{(k)}(t)\}_{k\in\mathbb{N}}$ is a basis for the space $N^t(p)$ and any $x \in N^t(p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(t) b^{(k)}(t), \tag{23}$$

where $\lambda_k(t) = (N^t x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$.

Proof. It is clear that $\{b^{(k)}(t)\} \subset N^t(p)$, since

$$N^t b^{(k)}(t) = e^{(k)} \in \ell(p) \quad \forall k \in \mathbb{N}, \tag{24}$$

where $e^{(k)}$ is the sequence whose only nonzero term is a 1 in the kth place for each $k \in \mathbb{N}$ and $0 < p_k \le H < \infty$.

Let $x \in N^t(p)$ be given. For every nonnegative integer m, we put

$$x^{[m]} = \sum_{k=0}^m \lambda_k(t) b^{(k)}(t). \tag{25}$$

Then, we obtain by applying N^t to (25) with (24) that

$$\begin{gathered} N^t x^{[m]} = \sum_{k=0}^m \lambda_k(t) N^t b^{(k)}(t) = \sum_{k=0}^m (N^t x)_k e^{(k)}, \\ \{N^t(x - x^{[m]})\}_i = \begin{cases} 0, & 0 \le i \le m, \\ (N^t x)_i, & i > m, \end{cases} \end{gathered} \tag{26}$$

where $i, m \in \mathbb{N}$. Given $\epsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m+1}^{\infty} \left|\left(N^t x\right)_i\right|^{p_k}\right]^{1/M} < \epsilon \tag{27}$$

for all $(m+1) \geq m_0$. Hence,

$$\begin{aligned} g\left[N^t\left(x - x^{[m]}\right)\right] &= \left[\sum_{i=m+1}^{\infty} \left|\left(N^t x\right)_i\right|^{p_k}\right]^{1/M} \\ &\leq \left[\sum_{i=m_0}^{\infty} \left|\left(N^t x\right)_i\right|^{p_k}\right]^{1/M} < \epsilon \end{aligned} \tag{28}$$

for all $(m+1) \geq m_0$ which proves that $x \in N^t(p)$ is represented as in (23).

Let us show the uniqueness of the representation for $x \in N^t(p)$ given by (23). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k(t) b^{(k)}(t)$. Since the linear transformation T, from $N^t(p)$ to $\ell(p)$, used in the proof of Theorem 3 is continuous, we have at this stage that

$$\left(N^t x\right)_n = \sum_k \mu_k(t) \left\{N^t b^{(k)}(t)\right\}_n = \sum_k \mu_k(t) e_n^{(k)} = \mu_n(t) \tag{29}$$

for all $n \in \mathbb{N}$ which contradicts the fact that $(N^t x)_n = \lambda_n(t)$ for all $n \in \mathbb{N}$. Hence, the representation (23) of $x \in N^t(p)$ is unique. This completes the proof. □

3. The Alpha-, Beta-, and Gamma-Duals of the Space $N^t(p)$

In this section, we determine the alpha-, beta-, and gamma-duals of the space $N^t(p)$. We will quote some lemmas which are needed in proving our theorems.

Lemma 5 (see [11], Theorem 5.1.0). *The following statements hold.*

(i) *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{N \in \mathcal{F}} \sum_k \left|\sum_{n \in N} a_{nk} B^{-1}\right|^{p'_k} < \infty. \tag{30}$$

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left|\sum_{n \in N} a_{nk}\right|^{p_k} < \infty. \tag{31}$$

Lemma 6 (see [12], Theorem 1). *The following statements hold.*

(i) *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left|a_{nk} B^{-1}\right|^{p'_k} < \infty. \tag{32}$$

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k \in \mathbb{N}} \left|a_{nk}\right|^{p_k} < \infty. \tag{33}$$

Lemma 7 (see [12], Theorem 1). *Let $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (32), (33) hold and there is $\beta_k \in \mathbb{C}$ such that $a_{nk} \to \beta_k$ for each $k \in \mathbb{N}$.*

Theorem 8. *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Define the sets $D_1(p)$, $D_2(p)$, and $D_3(p)$ as follows:*

$$\begin{aligned} D_1(p) : &= \left\{a = (a_k) \in \omega \right. \\ &\left. : \sup_{N \in \mathcal{F}} \sum_k \left|\sum_{n \in N} (-1)^{n-k} a_n D_{n-k} T_k B^{-1}\right|^{p'_k} < \infty\right\}, \\ D_2(p) : &= \left\{a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k \left|\sum_{i=k}^{n} (-1)^{i-k} a_i D_{i-k} T_k B^{-1}\right|^{p'_k} \right. \\ &\left. < \infty, \left\{\left(a_n T_n B^{-1}\right)^{p'_k}\right\} \in \ell_\infty\right\}, \\ D_3(p) &= cs. \end{aligned} \tag{34}$$

Then, the following statements hold:

(i) $\{N^t(p)\}^\alpha = D_1(p)$;

(ii) $\{N^t(p)\}^\gamma = D_2(p)$;

(iii) $\{N^t(p)\}^\beta = D_2(p) \cap D_3(p)$.

Proof. (i) Let us take $a = (a_k) \in \omega$. We easily derive with (20) that

$$a_n x_n = \sum_{k=0}^{n} (-1)^{n-k} a_n D_{n-k} T_k y_k = (Cy)_n \quad \forall n \in \mathbb{N}, \tag{35}$$

where $C = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} (-1)^{n-k} a_n D_{n-k} T_k, & 0 \le k \le n, \\ 0, & k > n \end{cases} \quad (36)$$

for all $k, n \in \mathbb{N}$. Thus, we observe by combining (35) with Part (i) of Lemma 5 that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in N^t(p)$ if and only if $Cy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This gives the result that $\{N^t(p)\}^\alpha = D_1(p)$.

(ii) Consider the equality

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n-1} \sum_{i=k}^{n} (-1)^{i-k} a_i D_{i-k} T_k y_k + a_n T_n y_n \\ &= (Ey)_n \quad \forall n \in \mathbb{N}, \end{aligned} \quad (37)$$

where $E = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} \sum_{i=k}^{n} (-1)^{i-k} a_i D_{i-k} T_k, & 0 \le k \le n-1, \\ a_n T_n, & k = n, \\ 0, & k > n \end{cases} \quad (38)$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from Part (i) of Lemma 6 with (37) that $ax = (a_n x_n) \in bs$ whenever $x = (x_k) \in N^t(p)$ if and only if $Ey \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$. Therefore, we obtain from Part (i) of Lemma 6 that $\{N^t(p)\}^\gamma = D_2(p)$.

(iii) We see from Lemma 7 that $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in N^t(p)$ if and only if $Ey \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive from Lemma 7 that $\{N^t(p)\}^\beta = D_2(p) \cap D_3(p)$.

Therefore, the proof is completed. □

Theorem 9. *Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Define the sets $D_4(p)$ and $D_5(p)$ by*

$$\begin{aligned} D_4(p) :&= \Bigg\{ a = (a_k) \in \omega \\ &\quad : \sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} (-1)^{n-k} a_n D_{n-k} T_k \right|^{p_k} < \infty \Bigg\}, \\ D_5(p) :&= \Bigg\{ a = (a_k) \in \omega : \sup_{n,k \in \mathbb{N}} \left| \sum_{i=k}^{n} (-1)^{i-k} a_i D_{i-k} T_k \right|^{p_k} \\ &\quad < \infty, \left\{ (a_n T_n)^{p_k} \right\} \in \ell_\infty \Bigg\}. \end{aligned} \quad (39)$$

Then, the following statements hold:

(i) $\{N^t(p)\}^\alpha = D_4(p)$;

(ii) $\{N^t(p)\}^\gamma = D_5(p)$;

(iii) $\{N^t(p)\}^\beta = D_3(p) \cap D_5(p)$.

Proof. This is easily obtained by proceeding as in the proof of Theorem 8 by using Lemma 7 and the second parts of Lemmas 5 and 6 instead of the first parts. So, we omit the detail. □

4. Some Matrix Transformations Related to the Sequence Space $N^t(p)$

In the present section, we characterize the matrix transformations from the space $N^t(p)$ into any given sequence space μ and from a given sequence space μ into the space $N^t(p)$. Since $\mu_A \cong \mu$ for any triangle A and any sequence space μ, it is trivial that the equivalence "$x \in \mu_A$ if and only if $y = Ax \in \mu$" holds.

Now, we can give the following theorem.

Theorem 10. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $F = (f_{nk})$ are connected with the relation*

$$f_{nk} := \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k} T_k a_{nj} \quad (40)$$

for all $k, n \in \mathbb{N}$ and μ is any given sequence space. Then, $A \in (N^t(p) : \mu)$ if and only if $A_n \in \{N^t(p)\}^\beta$ for all $n \in \mathbb{N}$ and $F \in (\ell(p) : \mu)$.

Proof. Let μ be any given sequence space. Suppose that (40) holds between the elements of the matrices $A = (a_{nk})$ and $F = (f_{nk})$, and take into account that the spaces $N^t(p)$ and $\ell(p)$ are linearly isomorphic.

Let $A \in (N^t(p) : \mu)$ and take any $y \in \ell(p)$. Then

$$\left(FN^t \right)_{nk} = \sum_{j=k}^{\infty} f_{nj} a_{jk}^t = \sum_{j=k}^{\infty} \sum_{i=j}^{\infty} (-1)^{i-j} D_{i-j} a_{ni} T_j \frac{t_{j-k}}{T_j} = a_{nk}. \quad (41)$$

That is, FN^t exists and $A_n \in \{N^t(p)\}^\beta$ which yields that $F_n \in \ell_1$ for each $n \in \mathbb{N}$. Hence, Fy exists and thus

$$\begin{aligned} \sum_k f_{nk} y_k &= \sum_k \sum_{i=k}^{\infty} (-1)^{i-k} D_{i-k} a_{ni} T_k \\ &\quad \times \left(\frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \right) = \sum_k a_{nk} x_k \end{aligned} \quad (42)$$

for all $n \in \mathbb{N}$. So, we have that $Fy = Ax$, which leads us to the consequence $F \in (\ell(p) : \mu)$.

Conversely, let $A_n \in \{N^t(p)\}^\beta$ for each $n \in \mathbb{N}$ and $F \in (\ell(p) : \mu)$, and take $x = (x_k) \in N^t(p)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_k a_{nk} x_k = \sum_k a_{nk} \left[\sum_{i=0}^{k} (-1)^{k-i} D_{k-i} T_i y_i \right] = \sum_k f_{nk} y_k \quad \forall n \in \mathbb{N} \quad (43)$$

that $Ax = Fy$ and this shows that $A \in (N^t(p) : \mu)$. This completes the proof. □

By changing the roles of the spaces $N^t(p)$ with μ in Theorem 10, we have the following.

Theorem 11. *Suppose that μ is any given sequence space and the elements of the infinite matrices $A = (a_{nk})$ and $G = (g_{nk})$ are connected with the relation $g_{nk} = \sum_{j=0}^{n}(t_{n-j}/T_n)a_{jk}$ for all $k, n \in \mathbb{N}$. Then, $A \in (\mu : N^t(p))$ if and only if $G \in (\mu : \ell(p))$.*

Proof. Let $x = (x_k) \in \mu$ and consider the following equality:

$$\sum_{j=0}^{n}\frac{t_{n-j}}{T_n}\sum_{k=0}^{m}a_{jk}x_k = \sum_{k=0}^{m}g_{nk}x_k \quad \forall n \in \mathbb{N}. \tag{44}$$

Then, by letting $m \to \infty$ in (44), we have $\{N^t(Ax)\}_n = (Gx)_n$ for all $n \in \mathbb{N}$. Since $Ax \in N^t(p)$, $N^t(Ax) = Gx \in \ell(p)$. This completes the proof. □

5. The Rotundity of the Space $N^t(p)$

In functional analysis, the rotundity of Banach spaces is one of the most important geometric properties. For details, the reader may refer to [13–15]. In this section, we give the necessary and sufficient condition in order to the space $N^t(p)$ be rotund and present some results related to this concept.

Definition 12. Let $S(X)$ be the unit sphere of a Banach space X. Then, a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Definition 13. A Banach space X is said to have Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 14. A Banach space X is said to have

(i) the Opial property if every sequence (x_n) weakly convergent to $x_0 \in X$ satisfies

$$\liminf_{n\to\infty}\|x_n - x_0\| < \liminf_{n\to\infty}\|x_n + x\| \tag{45}$$

for every $x \in X$ with $x \neq x_0$;

(ii) the uniform Opial property if for each $\epsilon > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty}\|x_n + x\| \tag{46}$$

for each $x \in X$ with $\|x\| \geq \epsilon$ and each sequence (x_n) in X such that $x_n \xrightarrow{w} 0$ and $\liminf_{n\to\infty}\|x_n\| \geq 1$.

Definition 15. Let X be a real vector space. A functional $\sigma : X \to [0,\infty)$ is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;

(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;

(iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(iv) the modular σ is called convex if $\sigma(\alpha x+\beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular σ on X is called

(a) right continuous if $\lim_{\alpha\to 1^+}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$;

(b) left continuous if $\lim_{\alpha\to 1^-}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$;

(c) continuous if it is both right and left continuous, where

$$X_\sigma = \left\{x \in X : \lim_{\alpha\to 0^+}\sigma(\alpha x) = 0\right\}. \tag{47}$$

We define σ_p on $N^t(p)$ by $\sigma_p(x) = \sum_k |(1/T_k)\sum_{j=0}^{k} t_{k-j}x_j|^{p_k}$. If $p_k \geq 1$ for all $k \in \mathbb{N}_1 = \{1, 2, \ldots\}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, σ_p is a convex modular on $N^t(p)$. We consider $N^t(p)$ equipped with Luxemburg norm given by

$$\|x\| = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\}. \tag{48}$$

$N^t(p)$ is a Banach space with this norm. This can be shown by the similar way used in the proof of Theorem 7 in [16].

We establish some basic properties for the modular σ_p.

Proposition 16. *The modular σ_p on $N^t(p)$ satisfies the following properties with $p_k \geq 1$ for all $k \in \mathbb{N}$.*

(i) *If $0 < \alpha \leq 1$, then $\alpha^M\sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.*

(ii) *If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M\sigma_p(x/\alpha)$.*

(iii) *If $\alpha \geq 1$, then $\alpha\sigma_p(x/\alpha) \leq \sigma_p(x)$.*

(iv) *The modular σ_p is continuous.*

Proof. (i) Let $0 < \alpha \leq 1$. Then $\alpha^M/\alpha^{p_k} \leq 1$ for all $p_k \geq 1$. So, we have

$$\begin{aligned} \alpha^M\sigma_p\left(\frac{x}{\alpha}\right) &= \sum_k \frac{\alpha^M}{\alpha^{p_k}}\left|\frac{1}{T_k}\sum_{j=0}^{k}t_{k-j}x_j\right|^{p_k} \\ &\leq \sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}t_{k-j}x_j\right|^{p_k} = \sigma_p(x), \\ \sigma_p(\alpha x) &= \sum_k \alpha^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^{k}t_{k-j}x_j\right|^{p_k} \\ &\leq \alpha\sum_k\left|\frac{1}{T_k}\sum_{j=0}^{k}t_{k-j}x_j\right|^{p_k} = \alpha\sigma_p(x). \end{aligned} \tag{49}$$

(ii) Let $\alpha \geq 1$. Then $1 \leq \alpha^M/\alpha^{p_k}$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) \leq \frac{\alpha^M}{\alpha^{p_k}}\sigma_p(x) = \alpha^M\sigma_p\left(\frac{x}{\alpha}\right). \tag{50}$$

(iii) Let $\alpha \geq 1$. Then $\alpha/\alpha^{p_k} \leq 1$ for all $p_k \geq 1$. Therefore, one can easily see that

$$\begin{aligned} \alpha\sigma_p\left(\frac{x}{\alpha}\right) &= \sum_k \frac{\alpha}{\alpha^{p_k}}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} \\ &\leq \sum_k \left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} = \sigma_p(x). \end{aligned} \tag{51}$$

(iv) If $\alpha > 1$, then we have

$$\begin{aligned} \sum_k \alpha\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} &\leq \sum_k \alpha^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} \\ &\leq \sum_k \alpha^{M}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k}; \end{aligned} \tag{52}$$

that is,

$$\alpha\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M \sigma_p(x). \tag{53}$$

By passing to limit as $\alpha \to 1^+$ in (53), we have $\sigma_p(\alpha x) \to \sigma_p(x)$. Hence, σ_p is right continuous.

If $0 < \alpha < 1$, we have

$$\begin{aligned} \sum_k \alpha^{M}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} &\leq \sum_k \alpha^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} \\ &\leq \sum_k \alpha\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k}; \end{aligned} \tag{54}$$

that is,

$$\alpha^M\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha\sigma_p(x). \tag{55}$$

By letting $\alpha \to 1^-$ in (55), we have $\sigma_p(\alpha x) \to \sigma_p(x)$. Hence, σ_p is left continuous. Since σ_p is both right and left continuous, it is continuous. □

Now, we give some relationships between the modular σ_p and the Luxemburg norm on $N^t(p)$.

Proposition 17. *For any $x \in N^t(p)$, the following statements hold.*

(i) *If $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$.*

(ii) *If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$.*

(iii) *$\|x\| = 1$ if and only if $\sigma_p(x) = 1$.*

(iv) *$\|x\| < 1$ if and only if $\sigma_p(x) < 1$.*

(v) *$\|x\| > 1$ if and only if $\sigma_p(x) > 1$.*

(vi) *If $0 < \alpha < 1$ and $\|x\| > \alpha$, then $\sigma_p(x) > \alpha^M$.*

(vii) *If $\alpha \geq 1$ and $\|x\| < \alpha$, then $\sigma_p(x) < \alpha^M$.*

Proof. Let $x \in N^t(p)$.

(i) Let $\epsilon > 0$ such that $0 < \epsilon < 1-\|x\|$. By the definition of $\|\cdot\|$ in (48), there exists an $\alpha > 0$ such that $\|x\|+\epsilon > \alpha$ and $\sigma_p(x/\alpha) \leq 1$. So, we have

$$\begin{aligned} \sigma_p(x) &\leq \sum_k \left(\frac{\|x\|+\epsilon}{\alpha}\right)^{p_k}\left|\frac{1}{T_k}\sum_{j=0}^k t_{k-j}x_j\right|^{p_k} \\ &\leq (\|x\|+\epsilon)\,\sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\|+\epsilon. \end{aligned} \tag{56}$$

Since ϵ is arbitrary, we have $\sigma_p(x) \leq \|x\|$ from (56).

(ii) If we choose $\epsilon > 0$ such that $0 < \epsilon < 1 - 1/\|x\|$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By the definition of $\|\cdot\|$ in (48) and Part (iii) of Proposition 16, we have

$$1 < \sigma_p\left[\frac{x}{(1-\epsilon)\|x\|}\right] \leq \frac{1}{(1-\epsilon)\|x\|}\sigma_p(x). \tag{57}$$

So, $(1-\epsilon)\|x\| < \|x\|$ for all $\epsilon \in (0, 1-(1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$.

(iii) Since σ_p is continuous, by Theorem 1.4 of [15] we directly have (iii).

(iv) This follows from Parts (i) and (iii).

(v) This follows from Parts (ii) and (iii).

(vi) This follows from Part (ii) and Part (i) of Proposition 16.

(vii) This follows from Part (i) and Part (ii) of Proposition 16.

□

Theorem 18. *The space $N^t(p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Let $N^t(p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$ for all $k < 3$. Consider the following sequences given by

$$\begin{aligned} x &= (1, -D_1, D_2, -D_3, D_4, \ldots), \\ y &= (0, T_1, -T_1D_1, T_1D_2, -T_1D_3, \ldots). \end{aligned} \tag{58}$$

Then, obviously $x \neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1. \tag{59}$$

By Part (iii) of Proposition 17, $x, y, (x+y)/2 \in S[N^t(p)]$ which leads us to the contradiction that the sequence space $N^t(p)$ is not rotund. Hence, $p_k > 1$ for all $k \in \mathbb{N}$.

Conversely, let $x \in S[N^t(p)]$ and $v, z \in S[N^t(p)]$ with $x = (v+z)/2$. By convexity of σ_p and Part (iii) of Proposition 17, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} = 1, \tag{60}$$

which gives that

$$\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{61}$$

Also, since $x = (v + z)/2$ and from (61), we obtain that

$$\begin{aligned}\sum_k \left| \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} \frac{(v_j + z_j)}{2} \right|^{p_k} \\ = \frac{1}{2} \left(\sum_k \left| \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} v_j \right|^{p_k} + \sum_k \left| \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} z_j \right|^{p_k} \right).\end{aligned} \tag{62}$$

This implies that

$$\left| \frac{v_j + z_j}{2} \right|^{p_k} = \frac{\left|v_j\right|^{p_k} + \left|z_j\right|^{p_k}}{2} \tag{63}$$

for all $k \in \mathbb{N}$. Since the function $t \to |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (63) that $v_k = z_k$ for all $k \in \mathbb{N}$. Hence, $v = z$. That is, $N^t(p)$ is rotund. □

Theorem 19. *Let (x_n) be a sequence in $N^t(p)$. Then, the following statements hold:*

(i) $\lim_{n\to\infty}\|x_n\| = 1$ *implies* $\lim_{n\to\infty}\sigma_p(x_n) = 1$;

(ii) $\lim_{n\to\infty}\sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty}\|x_n\| = 0$.

Proof. The proof is similar to that of Theorem 10 in [16]. □

Theorem 20. *Let $x \in N^t(p)$ and $(x^{(n)}) \subset N^t(p)$. If $\sigma_p(x^{(n)}) \to \sigma_p(x)$ as $n \to \infty$ and $x_k^{(n)} \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$, then $x^{(n)} \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $x \in N^t(p)$ and $(x^{(n)}) \subset N^t(p)$, $\sigma_p(x^{(n)} - x) = \sum_k |\{N^t(x^{(n)} - x)\}_k|^{p_k} < \infty$. So, there exists an $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left|\left\{N^t\left(x^{(n)} - x\right)\right\}_k\right|^{p_k} < \frac{\epsilon}{2}. \tag{64}$$

Also, since $x_k^{(n)} \to x_k$ as $n \to \infty$, we have

$$\sum_{k=1}^{k_0} \left|\left\{N^t\left(x^{(n)} - x\right)\right\}_k\right|^{p_k} < \frac{\epsilon}{2}. \tag{65}$$

Therefore, we obtain from (64) and (65) that $\sigma_p(x^{(n)} - x) < \epsilon$. This means that $\sigma_p(x^{(n)} - x) \to 0$ as $n \to \infty$. This result implies $\|x^{(n)} - x\| \to 0$ as $n \to \infty$ from Part (ii) of Theorem 19. Hence, $x_n \to x$ as $n \to \infty$. □

Theorem 21. *The sequence space $N^t(p)$ has the Kadec-Klee property.*

Proof. Let $x \in S[N^t(p)]$ and $(x^{(n)}) \subset N^t(p)$ such that $\|x^{(n)}\| \to 1$ and $x^{(n)} \xrightarrow{w} x$ are given. By Part (i) of Theorem 19, we have $\sigma_p(x^{(n)}) \to 1$ as $n \to \infty$. Also, $x \in S[N^t(p)]$ implies $\|x\| = 1$. By Part (iii) of Proposition 17, we obtain $\sigma_p(x) = 1$. Therefore, we have $\sigma_p(x^{(n)}) \to \sigma_p(x)$ as $n \to \infty$.

Since $x^{(n)} \xrightarrow{w} x$ and $q_k : N^t(p) \to \mathbb{R}$ (or $\mathbb{C}$) defined by $q_k(x) = x_k$ is continuous, $x_k^{(n)} \to x_k$ as $n \to \infty$. Therefore, $x^{(n)} \to x$ as $n \to \infty$. This completes the proof. □

Theorem 22. *For any $1 < p < \infty$, the space $X_{a(p)}$ has the uniform Opial property.*

Proof. Since the proof can be given by the similar way used in proving Theorem 13 of Nergiz and Başar [16], we omit the detail. □

6. Conclusion

Wang introduced the sequence space $X_{a(p)}$, in [10]. Although the domain of several triangle matrices in the classical sequence spaces ℓ_p, c_0, c, and ℓ_∞ and in the Maddox spaces $\ell(p)$, $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ was investigated by researchers, the domain of Nörlund mean neither in a normed sequence space nor in a paranormed sequence space was not studied and is still as an open problem. So, we have worked on the domain of Nörlund mean in the Maddox space $\ell(p)$. Additionally, we emphasize on some geometric properties of the new space $N^t(p)$. It is obvious that the matrix N^t is not comparable with the matrices E^r, A^r, or $B(r,s)$. So, the present results are new.

It is clear that by depending on the choice of the sequence space μ, the characterization of several classes of matrix transformations from the space $N^t(p)$ and into the space $N^t(p)$ can be obtained from Theorems 10 and 11, respectively. As a natural continuation of this paper, we will study the domain of the Nörlund mean in Maddox's spaces $\ell_\infty(p)$, $c(p)$, and $c_0(p)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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