Research Article

An Extension of Darbo's Theorem and Its Application

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Here, some extensions of Darbo fixed point theorem associated with measures of noncompactness are proved. Then, as an application, our attention is focused on the existence of solutions of the integral equation $x(t) = F(t, f(t, x(\alpha_1(t)), x(\alpha_2(t))), ((Tx)(t)/\Gamma(\alpha)) \times \int_0^t (u(t, s, \max_{[0,r(s)]}|x(\gamma_1(\tau))|, \max_{[0,r(s)]}|x(\gamma_2(\tau))|) /(t-s)^{1-\alpha})ds, \int_0^\infty v(t, s, x(t))ds), 0 < \alpha \le 1, t \in [0, 1]$ in the space of real functions defined and continuous on the interval $[0, 1]$.

1. Introduction

The concept of measure of noncompactness plays very important role in describing differential and integral equations. It was introduced by Kuratowski [1] as follows:

$$\alpha(S) = \inf \left\{ \delta > 0 : \right.$$
$$\left. S = \bigcup_{i=1}^{n} S_i, \ \text{diam}(S_i) \le \delta, 1 \le i \le n < \infty \right\},$$

$$(1)$$

for bounded subsets S of a metric space X. Darbo [2] used Kuratowski measure of noncompactness to generalize Shauder fixed point theorem to k-set contractive operators, which satisfy the condition $\alpha(T(A)) \le k\alpha(A)$ for some $k \in [0, 1)$. Up to now, other measures of noncompactness have been defined. In recent years many papers have been devoted to the problem of existence of solutions of integral equations, using the technique of measures of noncompactness and Darbo fixed point theorem (cf. [3–6]). Recently, the technique of measure of noncompactness has been used to obtain some extensions of Darbo fixed point theorem and the obtained results have important applications [3, 4]. As some applications of the technique of measures of noncompactness and

Darbo fixed point theorem, the following integral equations have been considered in [5, 6], respectively:

$$x(t) = f(t) + \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, \max_{[0,r(s)]}|x(\tau)|)}{(t-s)^{1-\alpha}} ds,$$

$$0 < \alpha \le 1, \quad t \in [0, 1],$$

$$(2)$$

$$x(t) = F\left(t, f(t, x(\alpha(t))),\right.$$

$$\int_0^{\beta(t)} u(t, s, x(\gamma(s))) d_s(t, s),$$

$$\left. \int_0^\infty v(t, s, x(\delta(s))) ds \right), \quad t \in \mathbb{R}_+.$$

$$(3)$$

Huang and Cao [7] have given a result to find the solution of the integral equation

$$x(t) = q(t) + f_1(t, x(\alpha_1(t)), x(\alpha_2(t)))$$

$$+ \frac{f_2(t, x(\beta_1(t)), x(\beta_2(t)))}{\Gamma(\alpha)}$$

$$\times \int_0^t (t-s)^{\alpha-1} f_3(t, s, x(\gamma_1(s)), x(\gamma_2(s))) ds.$$

$$(4)$$

In this paper, motivated and inspired by the integral equations (2), (3), and (4), we are going to prove a theorem on the existence of solutions of the integral equation

$$
x(t) = F\left(t, f\left(t, x\left(\alpha_1(t)\right), x\left(\alpha_2(t)\right)\right),\right.
$$
$$
\frac{(Tx)(t)}{\Gamma(\alpha)} \times \int_0^t u\left(t, s, \max_{[0,r(s)]}\left|x\left(\gamma_1(\tau)\right)\right|,\right.
$$
$$
\max_{[0,r(s)]}\left|x\left(\gamma_2(\tau)\right)\right|\right)
$$
$$
\times \left((t-s)^{1-\alpha}\right)^{-1} ds,
$$
$$
\left.\int_0^\infty v(t, s, x(t))\, ds\right),
$$
$$
0 < \alpha \le 1, \quad t \in [0, 1],
$$
$$
\tag{5}
$$

in the Banach space $C(I) = C[0, 1]$. Note that (5) has a rather general form and extends the integral equation (2). Our aim will be realized with help of the technique of measure of noncompactness. In Section 2, we present some definitions and preliminary results about the concept of measure of noncompactness. In Section 3, using the existent contractive condition in [8, Theorem 3.1] and the notion of shifting distance functions of [9], some generalizations of Darbo fixed point theorem are proved. In the last section, a result is proved concerning the existence of solutions for the integral equation (5).

2. Preliminaries

In this section, some definitions, notions, and results are presented which will be used in the next sections.

Assume that E is a real Banach space with zero element θ. The closed ball centered at x with radius r and the ball $B(\theta, r)$ are denoted by $B(x, r)$ and B_r, respectively. If X is a nonempty subset of E, then we denote by $\overline{X}$ and conv(X) the closure and closed convex hull of X, respectively. Moreover, let M_E indicate the family of all nonempty bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets of E.

In our considerations, we use the following definition of the concept of measure of noncompactness.

Definition 1 (see [10]). A mapping $\mu : M_E \to [0, \infty)$ is called a measure of noncompactness if it satisfies the following conditions.

   (1) The family Ker$\mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and Ker$\mu \subseteq N_E$.

   (2) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.

   (3) $\mu(\overline{X}) = \mu(X)$.

   (4) $\mu(\text{Conv}(X)) = \mu(X)$.

   (5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

   (6) If $\{X_n\}$ is a sequence of closed sets from M_E such that $X_{n+1} \subseteq X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty}\mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

The family Kerμ described in (1) is said to be the kernel of the measure of noncompactness μ and since $\mu(X_\infty) = \mu(\bigcap_{n=1}^\infty X_n) \le \mu(X_n)$, we infer that $\mu(\bigcap_{n=1}^\infty X_n) = 0$. So, $X_\infty \in \text{Ker}\,\mu$.

Let $C(I) = C([0, 1])$ denote the Banach space of all real functions defined and continuous on the interval $[0, 1]$ equipped with the norm

$$
\|x\| = \max\{|x(t)|;\ t \in [0, 1]\}. \tag{6}
$$

Fix a nonempty subset X of $C([0, 1])$. For $\varepsilon > 0$ and $x \in X$ define

$$
\omega(x, \varepsilon) = \sup\{|x(t_1) - x(t_2)|;
$$
$$
t_1, t_2 \in [0, 1],\ |t_2 - t_1| \le \varepsilon\},
$$
$$
\omega_0(X, \varepsilon) = \sup\{\omega(x, \varepsilon);\ x \in X\},
$$
$$
\omega_0(X) = \lim_{\varepsilon \to 0}\omega(X, \varepsilon). \tag{7}
$$

Banas and Lecko [11] showed that the function $\omega_0(X)$ is a measure of noncompactness in $C(I)$. Now, we state the following two important theorems which play a key role in the fixed point theory.

Theorem 2 (Schauder [12]). *Let U be a nonempty, bounded, closed, and convex subset of a Banach space E. Then, every continuous and compact map $F : U \to U$ has at least one fixed point in U.*

Theorem 3 (Darbo [10]). *Let Q be a nonempty, closed, bounded, and convex subset of a Banach space E and let $F : Q \to Q$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of Q. Then, F has a fixed point in Q.*

The following definition of the concept of shifting distance functions will be used to generalize Darbo fixed point theorem.

Definition 4 (see [9]). Let $\psi, \phi : [0, \infty) \to \mathbb{R}$ be two functions. The pair of functions (ϕ, ψ) is said to be a pair of shifting distance functions, if the following conditions hold:

   (i) for $u, v \in [0, \infty)$ if $\psi(u) \le \phi(v)$, then $u \le v$;

   (ii) for $\{u_n\}, \{v_n\} \subset [0, \infty)$ with $\lim_{n\to\infty}u_n = \lim_{n\to\infty}v_n = w$, if $\psi(u_n) \le \phi(v_n)$ for all n, then $w = 0$.

Example 5 (see [9]). The conditions (i) and (ii) of the above definition are fulfilled for the functions $\psi, \phi : [0, \infty) \to \mathbb{R}$ defined by $\psi(t) = \ln((1 + 2t)/2)$ and $\phi(t) = \ln((1 + t)/2)$.

3. Some Generalizations of Darbo Fixed Point Theorem

In this section, we prove some generalizations of Darbo fixed point theorem.

Now, using the notion of shifting distance functions of [9], we obtain our generalization of Darbo fixed point theorem as follows.

Theorem 6. *Let U be a nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that $T : U \to U$ is a continuous function such that*

$$\psi\left(\mu\left(TX\right)\right) \le \phi\left(\mu\left(X\right)\right) \tag{8}$$

for any nonempty subset X of U, where μ is an arbitrary measure of noncompactness and $\psi, \phi : [0, \infty) \to R$ are the pair of shifting distance functions. Then, T has a fixed point in U.

Proof. Define the sequence $\{U_n\}$ by $U_0 = U$ and $U_n = \text{Conv}(TU_{n-1})$ for all $n \ge 1$. If there exists an integer $n_0 \ge 0$ such that $\mu(U_{n_0}) = 0$, then U_{n_0} is relatively compact and since $TU_{n_0} \subseteq U_{n_0}$, thus Theorem 2 implies that T has a fixed point. So suppose that $\mu(U_n) > 0$ for all $n \ge 0$. Using (8) we get

$$\begin{aligned}\psi\left(\mu\left(U_{n+1}\right)\right) &= \psi\left(\mu\left(\text{Conv}\left(TU_n\right)\right)\right) \\ &= \psi\left(\mu\left(TU_n\right)\right) \le \phi\left(\mu\left(U_n\right)\right).\end{aligned} \tag{9}$$

Due to condition (i) of Definition 4 and (9) we infer that $\{\mu(U_n)\}$ is a decreasing sequence of positive real numbers. Thus, there exists $r \ge 0$ such that $\mu(U_n) \to r$ as $n \to \infty$. So, in view of (9) and condition (ii) of Definition 4, we get $r = 0$ and hence $\lim_{n \to \infty} \mu(U_n) = 0$. Now, since $U_{n+1} \subseteq U_n$, $TU_n \subseteq U_n$, and $\mu(U_n) \to 0$ as $n \to \infty$, condition (6) of Definition 1 implies that $U_\infty = \bigcap_{n=1}^{\infty} U_n$ is nonempty, closed, convex, and invariant under the operator T and belongs to Ker μ. So, Theorem 2 completes the proof. $\qquad \square$

Taking $\psi = I$ in Theorem 6, we have the following result.

Corollary 7. *Let U be a nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that $T : U \to U$ is a continuous function such that*

$$\mu\left(TX\right) \le \phi\left(\mu\left(X\right)\right) \tag{10}$$

for any nonempty subset X of U, where μ is an arbitrary measure of noncompactness and $\phi : [0, \infty) \to \mathbb{R}$ is a function such that

(a) *for $\{u_n\}, \{v_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = w$, if $u_n \le \phi(v_n)$ for all n, then $w = 0$;*

(b) *for $u, v \in [0, \infty)$ if $u \le \phi(v)$, then $u \le v$.*

Then, T has a fixed point in U.

Using Proposition 9 proved in [9] and Theorem 6, we have the following result.

Corollary 8. *Let U be a nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that $T : U \to U$ is a continuous function such that*

$$\psi\left(\mu\left(TX\right)\right) \le \psi\left(\mu\left(X\right)\right) - \varphi\left(\mu\left(X\right)\right) \tag{11}$$

for any nonempty subset X of U, where μ is an arbitrary measure of noncompactness and $\phi, \psi : [0, \infty) \to [0, \infty)$ are two nondecreasing and continuous functions satisfying $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then, T has a fixed point in U.

Now, motivated and inspired by the contractive condition in [8, Theorem 3.1], we present another generalization of Darbo fixed point theorem as follows.

Theorem 9. *Let U be a nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that $T : U \to U$ is a continuous function such that*

$$\psi\left(\mu\left(TX\right)\right) \le \varphi\left(\mu\left(X\right)\right) - \theta\left(\mu\left(X\right)\right) \tag{12}$$

for any nonempty subset X of U, where μ is an arbitrary measure of noncompactness and $\psi, \phi, \theta : [0, \infty) \to [0, \infty)$ are three functions such that φ and θ are bounded on any bounded interval in $[0, \infty)$ and ψ is continuous. Moreover, assume that

(1) *$\psi(x) \le \varphi(y) \Rightarrow x \le y$;*

(2) *for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \to t > 0$, $\psi(t) - \limsup_{n \to \infty} \varphi(x_n) + \liminf_{n \to \infty} \theta(x_n) > 0$.*

Then, T has a fixed point.

Proof. Similarly as in the proof of Theorem 6, we construct the sequence $\{U_n\}$ by $U_0 = U$ and $U_n = \text{Conv}(TU_{n-1})$ for all $n \ge 1$. If there exists an integer $n_0 \ge 0$ such that $\mu(U_{n_0}) = 0$, then U_{n_0} is relatively compact. Hence, from Theorem 2 we conclude that T has a fixed point in U. Assume that $\mu(U_n) > 0$ for all $n \ge 0$. By applying (12) we get

$$\begin{aligned}\psi\left(\mu\left(U_{n+1}\right)\right) &= \psi\left(\mu\left(\text{Conv}\left(TU_n\right)\right)\right) \\ &= \psi\left(\mu\left(TU_n\right)\right) \le \varphi\left(\mu\left(U_n\right)\right) - \theta\left(\mu\left(U_n\right)\right).\end{aligned} \tag{13}$$

Since $\theta \ge 0$, thus from (13) we get $\psi(\mu(U_{n+1})) \le \varphi(\mu(U_n))$, which by condition (1) implies that $\mu(U_{n+1}) \le \mu(U_n)$. Hence, $\{\mu(U_n)\}$ is a decreasing sequence of positive real numbers. So, there exists $r \ge 0$ such that

$$\lim_{n \to \infty} \mu\left(U_n\right) = r. \tag{14}$$

We will prove that $\mu(U_n) \to 0$ as $n \to \infty$. Taking limit supremum on both sides of (13) and using the properties of the functions ψ, φ, and θ, we have

$$\psi\left(r\right) \le \limsup_{n \to \infty}\varphi\left(\mu U_n\right) + \limsup_{n \to \infty}\left(-\theta\left(\mu U_n\right)\right). \tag{15}$$

Consequently,

$$\psi\left(r\right) \le \limsup_{n \to \infty}\varphi\left(\mu U_n\right) - \liminf_{n \to \infty}\theta\left(\mu U_n\right), \tag{16}$$

which implies that

$$\psi\left(r\right) - \limsup_{n \to \infty}\varphi\left(\mu U_n\right) + \liminf_{n \to \infty}\theta\left(\mu U_n\right) \le 0. \tag{17}$$

So, from condition (2) we conclude that $r = 0$. Thus, $\mu(U_n) \to 0$ as $n \to \infty$ and $\lim_{n \to \infty} \mu(U_n) = 0$. Now, since $U_{n+1} \subseteq U_n$

and $TU_n \subseteq U_n$, thus from condition (6) of Definition 1 we conclude that $U_\infty = \bigcap_{n=1}^{\infty} U_n$ is nonempty, closed, convex, and invariant under the operator T and belongs to $\text{Ker}\,\mu$. Consequently, Theorem 2 implies that T has a fixed point in U. □

Corollary 10 (Darbo [10]). *Let Q be a nonempty, closed, bounded, and convex subset of a Banach space E and let $F : Q \rightarrow Q$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q. Then, F has a fixed point in Q.*

Proof. Take $\psi, \phi = I$ and $\theta(t) = (1 - k)t$ in Theorem 9. □

Taking $\psi = I$ and $\theta = 0$ in Theorem 9, we have the following corollary.

Corollary 11. *Let U be a nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that $T : U \rightarrow U$ is a continuous function such that*

$$\mu(TX) \leq \varphi(\mu(X)) \tag{18}$$

for any nonempty subset X of U, where μ is an arbitrary measure of noncompactness and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is bounded on any bounded interval in $[0, \infty)$. Moreover, assume that $x \leq \varphi(y) \Rightarrow x \leq y$ and, for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \rightarrow t > 0$,

$$\limsup_{n \rightarrow \infty} \varphi(x_n) < t. \tag{19}$$

Then, T has a fixed point.

4. The Solutions of the Integral Equation of Mixed Type

In this section, we consider the integral equation (5) and prove an existence theorem of solutions of that equation. First, we recall the following two Lemmas of [13] which will be used to prove the existence theorem of the integral equation (5).

Lemma 12. *Suppose $x \in C(I)$ and $r : I \rightarrow I$ is a continuous function. Define $(Fx)(t) = \max_{[0,r(t)]} |x(\tau)|$ for $t \in I$. Then, $Fx \in C(I)$.*

Lemma 13. *Let $(x_n) \subset C(I)$ and $x \in C(I)$. Suppose that $x_n \rightarrow x$ in $C(I)$. Then, $Fx_n \rightarrow Fx$ uniformly on I.*

Now, we list the hypotheses which will be used to prove the existence theorem of the integral equation (5).

(H_1) The functions $\alpha_1, \alpha_2, \gamma_1, \gamma_2 : [0, 1] \rightarrow [0, 1]$ are continuous.

(H_2) $r : [0, 1] \rightarrow [0, 1]$ is continuous and nondecreasing on $[0, 1]$.

(H_3) $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and there exist positive real numbers $L_1, L_2, L_3 < 1$ such that

$$\left| F(t, x_1, x_2, x_3) - F(t, y_1, y_2, y_3) \right|$$
$$\leq L_1 |x_1 - y_1| + L_2 |x_2 - y_2| + L_3 |x_3 - y_3|, \tag{20}$$

for $t \in [0, 1]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Moreover, assume that $\overline{F} = \sup_{t \in [0,1]} |F(t, 0, 0, 0)|$.

(H_4) $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $f_0 = \sup\{|f(t, 0, 0)|; \ t \in [0, 1]\}$. Further, there exists continuous and nondecreasing function $\psi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\psi_1(0) = 0$ so that

(a) $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \psi_1((|x_1 - x_2| + |y_1 - y_2|)/2)$, for $t \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H_5) $T : C(I) \rightarrow C(I)$ is a continuous operator such that

$$|(Tx)(t_1) - (Tx)(t_2)| \leq \psi_1(|x(t_2) - x(t_1)|),$$
$$|(Tx)(t)| \leq a + b\|x\|, \tag{21}$$

for $x \in C(I)$ and $t, t_1, t_2 \in [0, 1]$, where ψ_1 is the existent function in the assumption (H_4) and a, b are positive real numbers.

(H_6) $u : [0, 1] \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function and there exists a function $n : [0, 1] \rightarrow [0, 1]$ being continuous on $[0, 1]$ and a function $\phi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\phi_1(0) = 0$ such that

$$|u(t, s, x_1, y_1)| \leq n(t)\phi_1\left(\frac{|x_1 - y_1|}{2}\right),$$
$$|u(t, s, x_1, y_1) - u(t, s, x_2, y_2)| \tag{22}$$
$$\leq n(t)\phi_1\left(\frac{|x_1 - x_2| + |y_1 - y_2|}{2}\right),$$

for $t, s \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}_+$.

(H_7) $v : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and there exists a continuous function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a continuous function $k : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that m and $s \rightarrow k(t, s)$ are integrable over $\mathbb{R}_+$. Moreover, there exists the function $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that ϕ_2 is continuous and nondecreasing with $\phi_2(0) = 0$ and the following conditions hold.

(a) $|v(t_1, s, x) - v(t_2, s, x)| \leq m(s)\phi_2(|t_2 - t_1|)$, $|v(t, s, x)| \leq \phi_2(|x|)k(t, s)$, $|v(t, s, x) - v(t, s, y)| \leq \phi_2(|x - y|)k(t, s)$, for all $t, t_1, t_2 \in [0, 1]$, $s \in \mathbb{R}_+$, and $x, y \in \mathbb{R}$.

(b) $\phi_2(t) + \psi_1(t) \leq \phi(t)$ for all $t \geq 0$, where ψ_1 and ϕ are the existent functions in the assumption (H_4) and Corollary 7, respectively. Moreover, assume that $M = \sup_{t \in [0,1]} \int_0^\infty k(t, s)ds$.

(H_8) There exists a positive solution r_0 such that

$$L_1\psi_1(r_0) + L_2\overline{n}\frac{(a+br_0)}{\Gamma(\alpha+1)}\phi_1(r_0)$$

$$+ L_3M\phi_2(r_0) + L_1f_0 + \overline{F} \le r_0, \qquad (23)$$

$$L_3M \le L_1 + L_2\phi_1(r_0) \times \frac{1}{\Gamma(\alpha+1)} \le 1,$$

where $f_0 = \sup\{f(t,0,0); t \in [0,1]\}$ and $\overline{n} = \sup\{n(t)t^\alpha; t \in [0,1]\}$.

Now, we can present and prove the main result of this section.

Theorem 14. *Under the assumptions (H_1)–(H_8), (5) has at least one solution $x = x(t)$ belonging to the Banach space $C(I)$.*

Proof. Define the operator G on the Banach space $C(I)$ by the formula

$$(Gx)(t) = F\left(t, f(t,x(\alpha_1(t)),x(\alpha_2(t))), \frac{(Tx)(t)}{\Gamma(\alpha)}\right.$$

$$\times \int_0^t u(t,s,\max_{[0,r(s)]}|x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]}|x(\gamma_2(\tau))|)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1}ds,$$

$$\left.\int_0^\infty v(t,s,x(t))ds\right),$$

$$0 < \alpha \le 1, \quad t \in [0,1].$$

Using the imposed assumptions we infer that Gx is continuous on I for each $x \in C(I)$. Moreover, by using our assumptions, for any $t \in I$, we get

$$|(Gx)(t)|$$

$$= \left|F\left(t, f(t,x(\alpha_1(t)),x(\alpha_2(t))),\right.\right.$$

$$\frac{(Tx)(t)}{\Gamma(\alpha)}\int_0^t u(t,s,\max_{[0,r(s)]}|x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]}|x(\gamma_2(\tau))|)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1}ds,$$

$$\left.\left.\int_0^\infty v(t,s,x(t))ds\right)\right|$$

$$\le \left|F\left(t, f(t,x(\alpha_1(t)),x(\alpha_2(t))),\right.\right.$$

$$\frac{(Tx)(t)}{\Gamma(\alpha)}\int_0^t u(t,s,\max_{[0,r(s)]}|x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]}|x(\gamma_2(\tau))|)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1}ds,$$

$$\left.\int_0^\infty v(t,s,x(t))ds\right) - F(t,0,0,0)\bigg|$$

$$+ |F(t,0,0,0)|$$

$$\le L_1\left|f(t,x(\alpha_1(t)),x(\alpha_2(t))) - f(t,0,0)\right|$$

$$+ L_1\left|f(t,0,0)\right|$$

$$+ L_2\left|\frac{(Tx)(t)}{\Gamma(\alpha)}\int_0^t u(t,s,\max_{[0,r(s)]}|x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]}|x(\gamma_2(\tau))|)$$

$$\left.\times\left((t-s)^{1-\alpha}\right)^{-1}ds\right|$$

$$+ L_3\left|\int_0^\infty v(t,s,x(t))ds\right| + \overline{F}$$

$$\le L_1\psi_1\left(\frac{|x(\alpha_1(t))| + |x(\alpha_2(t))|}{2}\right) + L_2\frac{(a+b\|x\|)}{\Gamma(\alpha)}n(t)$$

$$\times \int_0^t \phi_1\left(|\max_{[0,r(s)]}|x(\gamma_2(\tau))|\right.$$

$$\left.-\max_{[0,r(s)]}|x(\gamma_1(\tau))|\|\frac{1}{2}\right)$$

$$\times (t-s)^{\alpha-1}ds$$

$$+ L_3\int_0^\infty \phi_2(|x(t)|)k(t,s)ds + \overline{F} + L_1f_0.$$

$$(25)$$

Since ψ_1, ϕ_1, and ϕ_2 are nondecreasing, then from (25) we conclude that

$$\|(Gx)(t)\| \le L_1\psi_1(\|x\|) + L_2\overline{n}\frac{(a+b\|x\|)}{\Gamma(\alpha+1)}\phi_1(\|x\|)$$

$$(26)$$

$$+ L_3M\phi_2(\|x\|) + \overline{F} + L_1f_0.$$

So, in view of assumption (H_8), the operator G transforms B_{r_0} into itself. Next, we show that G is continuous on the ball B_{r_0}. To do this, assume that $\{x_n\}$ is a sequence in B_{r_0} such that $x_n \to x$ and show that $Gx_n \to Gx$. Indeed, for each $t \in I$, we have

$$|(Gx_n)(t) - (Gx)(t)|$$

$$= \left|F\left(t, f(t,x_n(\alpha_1(t)),x_n(\alpha_2(t))),\right.\right.$$

$$\frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x_n(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x_n(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds,$$

$$\int_0^\infty v(t,s,x_n(t)) ds\right)$$

$$- F\left(t, f(t, x(\alpha_1(t)), x(\alpha_2(t))),\right.$$

$$\frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds,$$

$$\int_0^\infty v(t,s,x(t)) ds\right)\right|$$

$$\leq L_1 \left| f(t, x_n(\alpha_1(t)), x_n(\alpha_2(t)))\right.$$

$$- f(t, x(\alpha_1(t)), x(\alpha_2(t)))\right|$$

$$+ L_2 \left| \frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x_n(\gamma_1(\tau))|,\right.\right.$$

$$\max_{[0,r(s)]} |x_n(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds$$

$$- \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds\right|$$

$$+ L_3 \left| \int_0^\infty v(t,s,x_n(t)) ds - \int_0^\infty v(t,s,x(t)) ds\right|.$$

$$(27)$$

On the other hand, we have

$$\left| \frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x_n(\gamma_1(\tau))|,\right.\right.$$

$$\max_{[0,r(s)]} |x_n(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds$$

$$- \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds\right|$$

$$\leq \left| \frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x_n(\gamma_1(\tau))|,\right.\right.$$

$$\max_{[0,r(s)]} |x_n(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds$$

$$- \frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds\right|$$

$$+ \left| \frac{(Tx_n)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds$$

$$- \frac{(Tx)(t)}{\Gamma(\alpha)} \int_0^t u\left(t, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\right)$$

$$\times \left((t-s)^{1-\alpha}\right)^{-1} ds\right|$$

$$\leq \frac{(a+b\|x_n\|)}{\Gamma(\alpha+1)} t^\alpha n(t) \phi_1$$

$$\times \left(\frac{1}{2} \left[\left| \max_{[0,r(s)]} |x_n(\gamma_1(\tau))|\right.\right.\right.$$

$$- \max_{[0,r(s)]} |x(\gamma_1(\tau))|\right|$$

$$+ \left| \max_{[0,r(s)]} |x_n(\gamma_2(\tau))|\right.$$

$$- \max_{[0,r(s)]} |x(\gamma_2(\tau))|\right| \right] \right)$$

$$+ \frac{\|Tx_n - Tx\|}{\Gamma(\alpha+1)} n(t) t^\alpha \phi_1 \left(\left| \max_{[0,r(s)]} |x(\gamma_1(\tau))|\right.\right.$$

$$- \max_{[0,r(s)]} |x(\gamma_2(\tau))|\right| \frac{1}{2} \right).$$

$$(28)$$

Due to (27), (28), and our assumptions, we derive that

$$|(Gx_n)(t) - (Gx)(t)|$$

$$\leq L_1 \psi_1 \left(\left(|x_n(\alpha_1(t)) - x(\alpha_1(t))|\right.\right.$$

$$+ |x_n(\alpha_2(t)) - x(\alpha_2(t))| \right) \frac{1}{2} \right)$$

$$+ L_2 \frac{(a + b\|x_n\|)}{\Gamma(\alpha + 1)} t^\alpha n(t) \phi_1$$

$$\times \left(\left| \max_{[0,r(s)]} |x_n(\gamma_1(\tau))| \right. \right.$$

$$\left. - \max_{[0,r(s)]} |x(\gamma_1(\tau))| \right|$$

$$+ \left| \max_{[0,r(s)]} |x_n(\gamma_2(\tau))| \right.$$

$$\left. \left. - \max_{[0,r(s)]} |x(\gamma_2(\tau))| \right| \right)$$

$$+ \frac{\|Tx_n - Tx\|}{\Gamma(\alpha + 1)} n(t) t^\alpha \phi_1$$

$$\times \left(\left| \max_{[0,r(s)]} |x(\gamma_2(\tau))| \right. \right.$$

$$\left. - \max_{[0,r(s)]} |x(\gamma_1(\tau))| \right|$$

$$\left. - \max_{[0,r(s)]} |x(\gamma_1(\tau))| \right| \right)$$

$$+ L_3 \phi_2 \left(|x_n(t) - x(t)| \right) \int_0^\infty k(t, s) \, ds.$$

$$(29)$$

Using (29), Lemma 13, and the imposed assumptions, we have $\|Gx_n - Gx\| \to 0$ as $n \to \infty$.

Now, let X be a nonempty subset of the ball B_{r_0}, $x \in X$, and fix arbitrarily $\varepsilon > 0$. Choose $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \varepsilon$. Then, taking into account our assumptions, we have

$$|(Gx)(t_2) - (Gx)(t_1)|$$

$$= \left| F\left(t_2, f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2))), \right. \right.$$

$$\frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds,$$

$$\int_0^\infty v(t_2, s, x(t_2)) \, ds \right)$$

$$- F\left(t_1, f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))), \right.$$

$$\frac{(Tx)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds,$$

$$\left. \left. \int_0^\infty v(t_1, s, x(t_1)) \, ds \right) \right|$$

$$\leq L_1 \left| f(t_2, x(\alpha_1(t_2)), x(\alpha_2(t_2))) \right.$$

$$\left. - f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1))) \right|$$

$$+ L_1 \left| f(t_2, x(\alpha_1(t_1)), x(\alpha_2(t_1))) \right.$$

$$\left. - f(t_1, x(\alpha_1(t_1)), x(\alpha_2(t_1))) \right|$$

$$+ L_2 \left| \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \frac{(Tx)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\left. \times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \right|$$

$$+ L_3 \int_0^\infty \left| v(t_2, s, x(t_2)) - v(t_1, s, x(t_2)) \right| ds$$

$$+ L_3 \int_0^\infty \left| v(t_1, s, x(t_2)) - v(t_1, s, x(t_1)) \right| ds.$$

$$(30)$$

Moreover, we have

$$\left| \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \frac{(Tx)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\left. \times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \right|$$

$$\leq \left| \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} u(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_1} u(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|,$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \bigg|$$

$$+ \left| \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \frac{(Tx)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \bigg|$$

$$\leq \frac{(Tx)(t_2)}{\Gamma(\alpha)} \left[\left| \int_0^{t_2} u\left(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \int_0^{t_1} u\left(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds \bigg|$$

$$+ \left| \int_0^{t_1} u\left(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds \bigg|$$

$$+ \left| \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \bigg| \bigg]$$

$$+ \frac{|(Tx)(t_2) - (Tx)(t_1)|}{\Gamma(\alpha)}$$

$$\times \left| \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \bigg|.$$

$$(31)$$

Thus,

$$\left| \frac{(Tx)(t_2)}{\Gamma(\alpha)} \int_0^{t_2} u\left(t_2, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.\right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_2 - s)^{1-\alpha} \right)^{-1} ds$$

$$- \frac{(Tx)(t_1)}{\Gamma(\alpha)} \int_0^{t_1} u\left(t_1, s, \max_{[0,r(s)]} |x(\gamma_1(\tau))|, \right.$$

$$\max_{[0,r(s)]} |x(\gamma_2(\tau))|\Big)$$

$$\times \left((t_1 - s)^{1-\alpha} \right)^{-1} ds \bigg|$$

$$\leq \frac{(a + b \|x\|)}{\Gamma(\alpha)} n(t_2) \phi_1(\|x\|)$$

$$\times \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \omega_{r_0}(u, \varepsilon) \int_0^{t_1} (t_2 - s)^{\alpha-1} ds$$

$$+ n(t_1) \phi_1(\|x\|) \int_0^{t_1} \left\{ (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right\} ds$$

$$+ \frac{\psi_1(\omega(x, \varepsilon))}{\Gamma(\alpha)} n(t_1) \phi_1(\|x\|) \int_0^{t_1} (t_1 - s)^{\alpha-1} ds$$

$$= \frac{(a + b \|x\|)}{\Gamma(\alpha + 1)} n(t_2) \phi_1(\|x\|) (t_2 - t_1)^{\alpha}$$

$$+ \omega_{r_0}(u, \varepsilon) \times \frac{1}{\alpha} \left(-(t_2 - t_1)^{\alpha} + t_2^{\alpha} \right)$$

$$+ n(t_1) \phi_1(\|x\|) \left((t_2 - t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha} \right) \times \frac{1}{\alpha}$$

$$+ \frac{1}{\Gamma(\alpha + 1)} \times \psi_1(\omega(x, \varepsilon)) n(t_1) \phi_1(\|x\|) \times t_1^{\alpha}$$

$$\leq \frac{(a + br_0)}{\Gamma(\alpha + 1)} \phi_1(r_0) \times \varepsilon^{\alpha} + \omega_{r_0}(u, \varepsilon) \times \frac{1}{\alpha} \times -\varepsilon^{\alpha}$$

$$+ \phi_1(r_0)\varepsilon^\alpha \times \frac{1}{\alpha}$$

$$+ \frac{1}{\Gamma(\alpha+1)}\psi_1(\omega(x,\varepsilon))\phi_1(r_0).$$

(32)

Using (30), (32), and the assumptions of Theorem 14, we earn

$$|(Gx)(t_2) - (Gx)(t_1)|$$

$$\leq L_1\left[\psi_1\left(\frac{\omega(x,\omega(\alpha_1,\varepsilon)) + \omega(x,\omega(\alpha_2,\varepsilon))}{2}\right)\right.$$

$$\left. + \omega_{r_0}(f,\varepsilon)\right]$$

$$+ L_2\left[\frac{(a+br_0)}{\Gamma(\alpha)}\phi_1(r_0) \times \frac{1}{\alpha}\varepsilon^\alpha\right.$$

$$\left. + \omega_{r_0}(u,\varepsilon) \times \frac{-1}{\alpha}\varepsilon^\alpha + \phi_1(r_0)\varepsilon^\alpha \times \frac{1}{\alpha}\right]$$

(33)

$$+ L_2 \times \frac{1}{\Gamma(\alpha+1)}\psi_1(\omega(x,\varepsilon))\phi_1(r_0)$$

$$+ L_3\phi_2(\varepsilon)\int_0^\infty m(s)\,ds$$

$$+ L_3\phi_2(|x(t_2) - x(t_1)|)\int_0^\infty k(t_1,s)\,ds,$$

where

$$\omega_{r_0}(u,\varepsilon) = \sup\{|u(t,s,x_1,x_2) - u(\tau,s,x_1,x_2)|;$$

$$t,s,\tau \in [0,1],$$

$$x_1,x_2 \in [0,r_0], \ |t-\tau| \leq \varepsilon\},$$

$$\omega_{r_0}(f,\varepsilon) = \sup\{|f(t_2,x,y) - f(t_1,x,y)| :$$

$$t_1,t_2 \in [0,1],$$

$$|t_2 - t_1| < \varepsilon, \ x,y \in [-r_0,r_0]\}.$$

(34)

The estimate (33) implies that

$$\omega(Gx,\varepsilon)$$

$$\leq L_1\psi_1\left(\frac{\omega(x,\omega(\alpha_1,\varepsilon)) + \omega(x,\omega(\alpha_2,\varepsilon))}{2}\right)$$

$$+ L_1\omega_{r_0}(f,\varepsilon)$$

$$+ L_2\left[\frac{(a+br_0)}{\Gamma(\alpha+1)}\phi_1(r_0) \times \frac{1}{\alpha}\varepsilon^\alpha\right.$$

$$\left. + \omega_{r_0}(u,\varepsilon) \times \frac{-1}{\alpha}\varepsilon^\alpha + \phi_1(r_0)\varepsilon^\alpha \times \frac{1}{\alpha}\right]$$

$$+ L_2 \times \frac{1}{\Gamma(\alpha+1)}\psi_1(\omega(x,\varepsilon))\phi_1(r_0)$$

$$+ L_3\int_0^\infty \phi_2(\varepsilon)m(s)\,ds + L_3\phi_2(\omega(x,\varepsilon))M.$$

(35)

It follows from assumptions (H_1), (H_4), and (H_6) that the functions α_1, α_2, f, and u are uniformly continuous on the sets $[0,1]$, $[0,1]$, $[0,1]\times[-r_0,r_0]\times[-r_0,r_0]$, and $[0,1]\times[0,1]\times[0,r_0]\times[0,r_0]$, respectively. Consequently, we infer that

$$\lim_{\varepsilon\to0}\omega(\alpha_1,\varepsilon) = \lim_{\varepsilon\to0}\omega(\alpha_2,\varepsilon)$$

$$= \lim_{\varepsilon\to0}\omega_{r_0}(f,\varepsilon) = \lim_{\varepsilon\to0}\omega_{r_0}(u,\varepsilon) = 0.$$

(36)

Hence, using assumptions (H_4) and (H_7) and estimate (35), we get

$$\omega_0(GX)$$

$$\leq L_1\lim_{\varepsilon\to0}\psi_1\left(\frac{\omega(X,\omega(\alpha_1,\varepsilon)) + \omega(X,\omega(\alpha_2,\varepsilon))}{2}\right)$$

$$+ \lim_{\varepsilon\to0}\left(L_2 \times \frac{1}{\Gamma(\alpha+1)}\psi_1(\omega(X,\varepsilon))\phi_1(r_0)\right.$$

$$\left. + L_3M\phi_2(\omega(X,\varepsilon))\right).$$

(37)

Since the functions ψ_1 and ϕ_2 are continuous and $\psi_1(t) + \phi_2(t) \leq \phi(t)$, then from (36), (37), and the assumption (H_8), we conclude that

$$\omega_0(GX) \leq L_1\psi_1(\omega_0(X))$$

$$+ L_2 \times \frac{1}{\Gamma(\alpha+1)}\phi_1(r_0) \times \psi_1(\omega_0(X))$$

$$+ L_3M\phi_2(\omega_0(X))$$

$$\leq \left(L_1 + L_2\phi_1(r_0) \times \frac{1}{\Gamma(\alpha+1)}\right)$$

$$\times (\psi_1(\omega_0(X)) + \phi_2(\omega_0(X)))$$

$$\leq \phi(\omega_0(X)).$$

(38)

Thus, Corollary 7 completes the proof. □

Now, we present the following example which shows that Theorem 14 can be applied to obtain solutions of the integral equation (39) but the existent results in [5, 6] are inapplicable.

Example 15. Now, we investigate the following integral equation:

$$
x(t)
$$

$$
= \frac{e^{-t}}{100} \cos \left(\frac{e^{-t}}{800 (1+t^2)(1+|x(t)|+|x(2t)|)} \right.
$$

$$
+ \frac{1}{100} \times \cos \left(\frac{1}{(1+(1/8)|x(t)|)} \right) \frac{1}{\Gamma(1/2)}
$$

$$
\times \int_0^t \left(\arctan \left(e^{-ts} \left| \max_{[0,s]} |x(\tau)| \right. \right. \right.
$$

$$
\left. \left. \left. - \max_{[0,s]} \left| x(2\tau) \right| \right| \right) \frac{1}{8} \right) \right)
$$

$$
\times \left((t-s)^{1/2} \right)^{-1} ds
$$

$$
\left. + \frac{1}{100} \int_0^\infty \frac{e^{-s}}{1+t/8} \sin \left(\frac{|x(t)|}{8} \right) ds \right),
$$

(39)

for $t \in [0,1]$. Equation (39) is a special case of the integral equation (5), where

$$
F(t,x,y,z) = \frac{e^{-t}}{100} \cos \left(\frac{x+y+z}{100} \right),
$$

$$
f(t,x,y) = \frac{e^{-t}}{8(1+t^2)(1+|x|+|y|)},
$$

$$
\alpha_1(t) = \gamma_1(t) = r(t) = t,
$$

$$
\alpha_2(t) = \gamma_2(t) = 2t,
$$

$$
(Tx)(t) = \cos \left(\frac{1}{(1+(1/8)|x(t)|)} \right),
$$

$$
u(t,s,x,y) = \arctan \left(\frac{e^{-ts}|x-y|}{8} \right),
$$

$$
v(t,s,x) = \frac{e^{-s}}{1+t/8} \sin \left(\frac{|x|}{8} \right).
$$

(40)

Then, it is easily seen that α_1, α_2, γ_1, γ_2, and r satisfy the assumptions (H_1) and (H_2) of the Theorem 14. Further, the function F satisfies assumption (H_3) with $L_1 = L_2 = L_3 = 1/10000$ and $\overline{F} = 1/100$. Since $f(t,x,y) = e^{-t}/8(1+t^2)(1+|x|+|y|)$, then, for all $t \in [0,1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$
\left| f(t,x_1,y_1) - f(t,x_2,y_2) \right|
$$

$$
= \left| \frac{e^{-t}}{8(1+t^2)(1+|x_1|+|y_1|)} \right.
$$

$$
\left. - \frac{e^{-t}}{8(1+t^2)(1+|x_2|+|y_2|)} \right|
$$

$$
= \frac{e^{-t}}{8(1+t^2)} \left| \frac{|x_2|+|y_2|-|x_1|-|y_1|}{(1+|x_1|+|y_1|)(1+|x_2|+|y_2|)} \right|
$$

$$
\leq \frac{1}{8} \left(|x_2 - x_1| + |y_2 - y_1| \right).
$$

(41)

So, the assumption (H_4) is satisfied with $\psi_1(t) = (1/4)t$ and $f_0 = 1/8$. In this example, we have $(Tx)(t) = \cos(1/(1+(1/8)|x(t)|))$ and this operator satisfies assumption (H_5) with $\psi_1(t) = t/8$, $a = 1$, and $b = 0$. On the other hand, for all $t, s \in [0,1]$ and $x, y, x_1, x_2, y_1, y_2 \in \mathbb{R}_+$, we get

$$
|u(t,s,x,y)| = \left| \arctan \left(e^{-ts} \frac{|x-y|}{8} \right) \right|
$$

$$
\leq e^{-t} \left| \frac{|x-y|}{8} \right|,
$$

$$
|u(t,s,x_1,y_1) - u(t,s,x_2,y_2)| = \left| \arctan \left(\frac{e^{-ts}|x_1-y_1|}{8} \right) \right.
$$

$$
\left. - \arctan \left(\frac{e^{-ts}|x_2-y_2|}{8} \right) \right|
$$

$$
\leq e^{-t} \left| \frac{|x_1-y_1|}{8} - \frac{|x_2-y_2|}{8} \right|
$$

$$
= e^{-t} \left| \frac{|x_1-y_1|-|x_2-y_2|}{8} \right|
$$

$$
\leq e^{-t} \frac{|x_1-x_2|+|y_1-y_2|}{8}.
$$

(42)

So, $\phi_1(t) = (1/4)t$ and $n(t) = e^{-t}$. Moreover, $\overline{n} = \sup\{\sqrt{t}e^{-t}; t \in [0,1]\} = 1/e$. Thus, assumption (H_6) holds. Now, notice that the function $v(t,s,x) = (e^{-s}/(1+t/8)) \sin(|x|/8)$ is continuous and satisfies the conditions (a) and (b) of the assumption (H_7). Indeed, for all $t, t_1, t_2 \in [0,1]$, $s \in \mathbb{R}_+$, and $x, y \in \mathbb{R}$, we have

$$
|v(t_2,s,x) - v(t_1,s,x)|
$$

$$
= \left| \frac{e^{-s}}{1+t_2/8} \sin \left(\frac{|x|}{8} \right) - \frac{e^{-s}}{1+t_1/8} \sin \left(\frac{|x|}{8} \right) \right|
$$

$$
\leq e^{-s} \frac{|t_2 - t_1|}{8},
$$

$$
|v(t,s,x)| \leq \frac{e^{-s}}{1+t/8} \times \frac{|x|}{8},
$$

(43)

$$
|v(t,s,x) - v(t,s,y)|
$$

$$
= \left| \frac{e^{-s}}{1+t/8} \sin \left(\frac{|x|}{8} \right) - \frac{e^{-s}}{1+t/8} \sin \left(\frac{|y|}{8} \right) \right|
$$

$$
\leq \frac{e^{-s}}{1+t/8} \left(\frac{|x-y|}{8} \right).
$$

So, $\phi_2(t) = t/8$, $m(s) = e^{-s}$, and $k(t, s) = e^{-s}/(1 + t/8)$. Moreover, $M = \sup_{t \in [0,1]} \int_0^{\infty} k(t, s)ds = 1$, $\psi_1(t) + \phi_2(t) = t/4 = \phi(t)$, and the condition (b) of the assumption (H_7) holds with $\phi(t) = t/4$. Thus, the existent inequalities in assumption (H_8) have the forms

$$\frac{1}{40000}r_0 + \frac{1}{40000e} \times \frac{1}{\Gamma(3/2)}r_0 + \frac{1}{10000} \times \frac{r_0}{8}$$

$$+ \frac{1}{8} \times \frac{1}{10000} + \frac{1}{100} \leq r_0, \qquad (44)$$

$$\frac{1}{10000} \leq \frac{1}{10000} + \frac{1}{10000} \times \frac{r_0}{4} \times \frac{1}{\Gamma(3/2)} \leq 1.$$

It is easily seen that the last inequalities have a positive solution. For example, $r_0 = 1/2$. We see that all assumptions of Theorem 14 are satisfied. Consequently, from Theorem 14 the integral equation (39) has a solution in the space $C(I)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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