

Research Article

The F -Analogue of Riordan Representation of Pascal Matrices via Fibonomial Coefficients

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We study an analogue of Riordan representation of Pascal matrices via Fibonomial coefficients. In particular, we establish a relationship between the Riordan array and Fibonomial coefficients, and we show that such Pascal matrices can be represented by an F -Riordan pair.

1. Introduction

Pascal matrices are infinite matrices whose entries are formed by binomial coefficients. Fibonomial coefficients are a certain class of generalized binomial coefficients, and its theory is now well understood. The Fibonomial coefficients are used to define Pascal matrices, called Pascal matrices via Fibonomial coefficients.

The Riordan group is quite easily developed but unifies many themes in enumeration. In the recent literature, special attention has been given to the concept of Riordan arrays, which is a generalization of the well-known Pascal triangle. Riordan arrays are infinite lower-triangular matrices defined by the generating function of their columns. They form a group, called the Riordan group (see [1]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli (see [2, 3]).

Setting infinite dimensional Pascal matrix via Fibonomial coefficients requires very long applications and operations, whereas it was seen that these long applications and operations can be culminated more short and in a practical way through the aid of Riordan representation. Therefore, one can

obtain Pascal matrix via Fibonomial coefficients through the aid of Riordan representation.

The aim of this study is to establish the Riordan representation of Pascal matrices via Fibonomial coefficients. It is confronted by a problem while obtaining Riordan representation. Using the usual operation can not be a solution for this problem. For overcoming this problem, a new binary operation is required to define. By using this operation, F -analogue of Riordan representation is obtained. In particular, we show that Pascal matrices via Fibonomial coefficients of the first and the second kinds can be represented with an F -analogue of Riordan pair.

2. Preliminaries

The Fibonacci sequence is the starting point of our discussion. Thus, we briefly review some basic concepts and properties of Fibonomial coefficients. The Fibonacci numbers F_n are defined by the initial conditions $F_0 = 0$, $F_1 = 1$ and the recurrence

$$F_n = F_{n-1} + F_{n-2} \quad (1)$$

for $n \geq 2$. Let n and k be integers with $n \geq k \geq 0$. Then, the Fibonomial coefficients are defined by

$$\binom{n}{k}_F = \frac{[n]_F!}{[k]_F! [n-k]_F!}, \tag{2}$$

where $[n]_F! = F_n F_{n-1} \cdots F_1$ and $[0]_F! = 1$. It can be shown that Fibonomial coefficients satisfy the following recursion relation:

$$\binom{n}{k}_F = F_{k+1} \binom{n-1}{k}_F + F_{n-k-1} \binom{n-1}{k-1}_F \tag{3}$$

(see [4–6]). Let $0 \leq i, j \leq n-1$. The $n \times n$ Pascal matrix, $P_n = (p_{ij})$, is defined by

$$p_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The Pascal matrix via Fibonomial coefficients is denoted by $\mathcal{P}(k, m) = (p_{ij})$ and is defined by

$$p_{ij} = \begin{cases} \binom{k+jm-1+i-j}{k+jm-1}_F & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $k, m \in \mathbb{N}$ and $\binom{k+jm-1+i-j}{k+jm-1}_F$ is the Fibonomial coefficient. Specially, $\mathcal{P}(1, 1) = \mathcal{P} = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Similarly, the generalized Pascal matrix via Fibonomial coefficients of the first kind, $U_n[x] = (U_n(x, i, j))$, is defined by

$$U_n(x, i, j) = \begin{cases} x^{i-j} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and the generalized Pascal matrix via Fibonomial coefficients of the second kind, $M_n[x] = (M_n(x, i, j))$, is defined by

$$M_n(x, i, j) = \begin{cases} x^{i+j-2} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Moreover, the extended generalized Pascal matrix via Fibonomial coefficients, $\Phi_n[x, y]_F = (\varphi_n(x, y; i, j)_F)$, is defined by

$$\varphi_n(x, y; i, j)_F = \begin{cases} x^{i-j} y^{i+j-2} \binom{i}{j}_F & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

See [7–11] for details. Note that all of \mathcal{P} , U_n , M_n , and Φ_n are $n \times n$ matrices.

The Riordan group is a set of infinite lower-triangular matrices each of which is defined by two generating functions, called a Riordan pair. Any infinite matrix of this group is called a Riordan array, and Riordan arrays are generalizations of Pascal’s triangle. In fact, Pascal matrices via Fibonomial coefficients are Riordan arrays, and we will show that they can be represented by a Riordan pair. To this purpose we briefly review the Riordan group and we refer to [1–3, 12, 13] for a detailed treatment of the subject.

Definition 1 (see [1]). Let g and f be two functions defined by

$$\begin{aligned} g(x) &= g_0 + g_1 x + g_2 x^2 + \cdots, \\ f(x) &= f_1 x + f_2 x^2 + f_3 x^3 + \cdots \end{aligned} \tag{10}$$

with $g_0 \neq 0$. Let us denote by (g, f) the infinite lower-triangular matrix whose j th column is formed by the coefficients of the power series

$$g(x) f(x)^j, \quad j = 0, 1, 2, \dots \tag{11}$$

The first column of this matrix is called the 0th column. Let \mathcal{R} be the set of all infinite lower-triangular matrices defined by (11), and let (g, f) and $(u, v) \in \mathcal{R}$. Then \mathcal{R} becomes a group under the operation

$$(g, f) * (u, v) := (g(u \circ f), v \circ f). \tag{12}$$

In particular, \mathcal{R} is called the Riordan group and any element (g, f) of \mathcal{R} is called a Riordan pair. The identity element of \mathcal{R} is

$$I = (1, x). \tag{13}$$

And the inverse of any (g, f) is

$$(g, f)^{-1} = \left(\frac{1}{g \circ \bar{f}}, \bar{f} \right). \tag{14}$$

Here, \bar{f} is the compositional inverse of f ; that is, $f(\bar{f}(x)) = \bar{f}(f(x)) = x$.

3. F-Analogue of FTRA

In this section, firstly $*_F$ operation is defined and then using this operation a new theorem which is called F -analogue of FTRA is obtained.

The generating functions of the Fibonomial coefficients [14–18] are

$$\prod_{i=1}^n (1 - \alpha^{(n-i)} \beta^{(i-1)} x) = \sum_{k \geq 0} (-1)^{k(k+1)/2} \binom{n}{k}_F x^k, \tag{15}$$

$$\prod_{i=1}^n \frac{1}{(1 - \alpha^{(i-1)} \beta^{(n-i)} x)} = \sum_{k \geq 0} \binom{n+k-1}{k}_F x^k,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. For simplicity of notation, we denote these two generating functions as follows.

Definition 2. Let $n \geq 0$; then

$$(1-x)_F^n \stackrel{\text{def}}{=} \prod_{i=1}^n (1 - \alpha^{(n-i)} \beta^{(i-1)} x), \tag{16}$$

$$\frac{1}{(1-x)_F^n} \stackrel{\text{def}}{=} \prod_{i=1}^n \frac{1}{(1 - \alpha^{i-1} \beta^{n-i} x)}.$$

Definition 3. Let \mathcal{F} denote the set of elements

$$\frac{tx^m}{(1-x)_F^n} \stackrel{\text{def}}{=} tx^m \prod_{i=1}^n \frac{1}{(1 - \alpha^{i-1} \beta^{n-i} x)} \tag{17}$$

$$= t \sum_{k \geq 0} \binom{n+k-m-1}{n-1}_F x^k$$

for all integers $m, n \geq 0$ and $t \in \mathbb{R}$. Let $*_F : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ be a binary operation defined as follows:

$$\frac{tx^a}{(1-x)_F^A} *_F \frac{ux^b}{(1-x)_F^B} \stackrel{\text{def}}{=} \frac{tx^a}{(1 - (\alpha^B x))_F^A} \cdot \frac{ux^b}{(1 - (\beta^A x))_F^B} \tag{18}$$

$$= \frac{tux^{a+b}}{(1-x)_F^{A+B}}.$$

Lemma 4. The pair $(\mathcal{F}, *_F)$ is a monoid.

Proof. (1) Closure. Indeed, for any two elements from \mathcal{F} we obtain an element from \mathcal{F} . That is,

$$\frac{tx^a}{(1-x)_F^A} *_F \frac{ux^b}{(1-x)_F^B} = \frac{tux^{a+b}}{(1-x)_F^{A+B}} \in \mathcal{F}. \tag{19}$$

- (2) Associativity is satisfied straightforwardly.
- (3) An identity element is

$$\frac{t^0 x^0}{(1-x)_F^0}. \tag{20}$$

□

Let g and f belong to \mathcal{F} with $g(0) \neq 0$ and $f(0) = 0$. The infinite lower-triangular matrix whose j th column is formed by the coefficients of the power series is

$$g(x) *_F f(x)^{[j]}, \quad j = 0, 1, 2, \dots, \tag{21}$$

where $f(x)^{[j]} := \underbrace{f(x) *_F f(x) *_F \dots *_F f(x)}_{j \text{ times}}$. By using new

$*_F$ binary operation and (21), we obtain a representation which is the analogue of the Riordan representation. We call the representation F -analogue of Riordan representation and denote it by $(g, f)_F$.

Therefore, we can write

$$\left(\frac{t}{(1-x)_F^n} \right)_F *_F \left(\frac{ux}{(1-x)_F^m} \right)^{[j]} = \frac{t u^j x^j}{(1-x)_F^{n+mj}} \tag{22}$$

for any $j \geq 0$.

The following theorem is analogous to the fundamental theorem of Riordan arrays.

Theorem 5 (F -analogue of FTTRA). Let $g(x), f(x) \in \mathcal{F}$ with $g(0) \neq 0$ and $f(0) = 0$. The F -analogue of the fundamental theorem of Riordan arrays is

$$(g(x), f(x))_F \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{bmatrix}, \tag{23}$$

where the generating functions of the column vectors are given, respectively, by $A(x)$ and $B(x)$. Then, equation of (23) is true if and only if the following equation holds:

$$g(x) *_F A(f(x)) = B(x). \tag{24}$$

Proof. Let $g(x), f(x) \in \mathcal{F}$ with $g(0) \neq 0$ and $f(0) = 0$. Then, it can be written as

$$g(x) = \frac{t}{(1-x)_F^n}, \quad f(x) = \frac{ux}{(1-x)_F^m}, \tag{25}$$

where $m, n \geq 0$ and $t, u \in \mathbb{R}$. In this case, the matrix turns to

$$\begin{bmatrix} | & & & \dots & | \\ | & | & & & | \\ g & g *_F f^{[1]} & g *_F f^{[2]} & & | \\ | & | & | & | & | \\ | & | & | & | & | \\ \vdots & & & \ddots & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{bmatrix} \tag{26}$$

and then

$$\begin{aligned}
 & \begin{bmatrix} t \binom{n-1}{n-1}_F & 0 & 0 & 0 & \dots \\ t \binom{n}{n-1}_F & tu \binom{n+m-2}{n-1}_F & 0 & 0 & \\ t \binom{n+1}{n-1}_F & tu \binom{n+m-1}{n-1}_F & tu^2 \binom{n+2m-3}{n-1}_F & 0 & \\ t \binom{n+2}{n-1}_F & tu \binom{n+m}{n-1}_F & tu^2 \binom{n+2m-2}{n-1}_F & tu^3 \binom{n+3m-4}{n-1}_F & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \\
 & = \begin{bmatrix} a_0 t \binom{n-1}{n-1}_F \\ a_0 t \binom{n}{n-1}_F + a_1 tu \binom{n+m-2}{n-1}_F \\ a_0 t \binom{n+1}{n-1}_F + a_1 tu \binom{n+m-1}{n-1}_F + a_2 tu^2 \binom{n+2m-3}{n-1}_F \\ a_0 t \binom{n+2}{n-1}_F + a_1 tu \binom{n+m}{n-1}_F + a_2 tu^2 \binom{n+2m-2}{n-1}_F + a_3 tu^3 \binom{n+3m-4}{n-1}_F \\ \vdots \end{bmatrix} \tag{27}
 \end{aligned}$$

and this yields

$$\begin{aligned}
 & a_0 t \binom{n-1}{n-1}_F \\
 & + \left[a_0 t \binom{n}{n-1}_F + a_1 tu \binom{n+m-2}{n-1}_F \right] x \\
 & + \left[a_0 t \binom{n+1}{n-1}_F + a_1 tu \binom{n+m-1}{n-1}_F \right. \\
 & \quad \left. + a_2 tu^2 \binom{n+2m-3}{n-1}_F \right] x^2 + \dots \\
 & = a_0 \left[t \binom{n-1}{n-1}_F + t \binom{n}{n-1}_F \right. \\
 & \quad \left. + t \binom{n+1}{n-1}_F x^2 + \dots \right] \\
 & + a_1 \left[tu \binom{n+m-2}{n-1}_F x \right. \\
 & \quad \left. + tu \binom{n+m-1}{n-1}_F x^2 + \dots \right] \\
 & + a_2 \left[tu^2 \binom{n+2m-3}{n-1}_F x^2 \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. + tu^2 \binom{n+2m-2}{n-1}_F x^3 + \dots \right] + \dots \\
 & = a_0 g(x) + a_1 [g(x) *_F f(x)] \\
 & \quad + a_2 [g(x) *_F f(x)^{[2]}] + a_3 [g(x) *_F f(x)^{[3]}] + \dots \\
 & = g(x) *_F [a_0 + a_1 f(x) \\
 & \quad + a_2 f(x)^{[2]} + a_3 f(x)^{[3]} + \dots] \\
 & = g(x) *_F A(f(x)) = B(x) \tag{28}
 \end{aligned}$$

and we have our result. □

4. The *F*-Analogue of Riordan Representation of Pascal Matrices via Fibonomial Coefficients

Obtaining the entries of infinite dimensional Pascal matrix via Fibonomial coefficients requires cumbersome calculations. However, there is an alternative method using the Riordan group which appears to be more convenient. To this

purpose, we start with the following theorem in which we obtain the F -analogue of Riordan representation of \mathcal{P} .

Theorem 6. Let $\mathcal{P}(n, m)$ be the infinite Pascal matrix via Fibonomial coefficients as in (5). Then, the F -analogue of Riordan representation of $\mathcal{P}(n, m)$ is given by

$$\mathcal{P}(n, m) = \left(\frac{t}{(1-x)_F^n}, \frac{ux}{(1-x)_F^m} \right)_F. \quad (29)$$

Proof. We consider the infinite Pascal matrix via Fibonomial coefficients $\mathcal{P}(n, m)$. The entries of the j th column are given by

$$p_{ij} = [x^i] tu^j \binom{n+jm-1+i-j}{n+jm-1}_F \quad i = 0, 1, 2, \dots \quad (30)$$

Let

$$g(x) = \frac{t}{(1-x)_F^n}, \quad f(x) = \frac{ux}{(1-x)_F^m}; \quad (31)$$

then

$$g(x) *_F (f(x))^{[j]} = \frac{t}{(1-x)_F^n} *_F \left(\frac{ux}{(1-x)_F^m} \right)^{[j]}. \quad (32)$$

Using (22), we obtain

$$g(x) *_F (f(x))^{[j]} = \frac{tu^j x^j}{(1-x)_F^{n+mj}}. \quad (33)$$

Taking (17) into account, we have

$$\begin{aligned} g(x) *_F (f(x))^{[j]} &= \frac{tu^j x^j}{(1-x)_F^{n+mj}} \\ &= \sum_{i \geq 0} tu^j \binom{n+jm-1+i-j}{n+jm-1}_F x^i. \end{aligned} \quad (34)$$

This proves that the generating function of the j th column of $\mathcal{P}(n, m)$ is

$$\frac{tu^j x^j}{(1-x)_F^{n+mj}}. \quad (35)$$

In conclusion, the F -analogue of Riordan representation of $\mathcal{P}(n, m)$ is

$$\left(\frac{t}{(1-x)_F^n}, \frac{ux}{(1-x)_F^m} \right)_F. \quad (36)$$

□

Corollary 7. Let \mathcal{P} be in (6). Then, the F -analogue of Riordan representation of \mathcal{P} is given by

$$\mathcal{P} = \left(\frac{1}{(1-x)_F}, \frac{x}{(1-x)_F} \right)_F. \quad (37)$$

Corollary 8. Let U be the Pascal matrix via Fibonomial coefficients of the first kind. Then, the F -analogue of Riordan representation of U is given by

$$\begin{aligned} &\left(\frac{1}{(1-xt)_F}, \frac{t}{(1-xt)_F} \right)_F \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ x & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ x^2 & x & 1 & 0 & 0 & 0 & 0 & \dots \\ x^3 & 2x^2 & 2x & 1 & 0 & 0 & 0 & \dots \\ x^4 & 3x^3 & 6x^2 & 3x & 1 & 0 & 0 & \dots \\ x^5 & 5x^4 & 15x^3 & 15x^2 & 5x & 1 & 0 & \dots \\ x^6 & 8x^5 & 40x^4 & 60x^3 & 40x^2 & 8x & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \quad (38)$$

Corollary 9. Let M be the Pascal matrix via Fibonomial coefficients of the second kind. Then, the F -analogue of Riordan representation of M is given by

$$\begin{aligned} &\left(\frac{1}{(1-xt)_F}, \frac{x^2 t}{(1-xt)_F} \right)_F \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ x & x^2 & 0 & 0 & 0 & 0 & \dots \\ x^2 & x^3 & x^4 & 0 & 0 & 0 & \dots \\ x^3 & 2x^4 & 2x^5 & x^6 & 0 & 0 & \dots \\ x^4 & 3x^5 & 6x^6 & 3x^7 & x^8 & 0 & \dots \\ x^5 & 5x^6 & 15x^7 & 15x^8 & 5x^9 & x^{10} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \quad (39)$$

Corollary 10. Let Φ be the extended generalized Pascal matrix via Fibonomial coefficients. Then, F -analogue of the Riordan representation of Φ is

$$\begin{aligned} &\left(\frac{1}{(1-xyt)_F}, \frac{y^2 t}{(1-xyt)_F} \right)_F \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ xy & y^2 & 0 & 0 & 0 & \dots \\ x^2 y^2 & xy^3 & y^4 & 0 & 0 & \dots \\ x^3 y^3 & 2x^2 y^4 & 2xy^5 & y^6 & 0 & \dots \\ x^4 y^4 & 3x^3 y^5 & 6x^2 y^6 & 3xy^7 & y^8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \quad (40)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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