Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 837565, 8 pages
http://dx.doi.org/10.1155/2014/837565

Research Article

Spectrums of Solvable Pantograph Differential-Operators for First Order

Z. I. Ismailov[1] and P. Ipek[2]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Karadeniz Technical University, 61080 Trabzon, Turkey*
[2] *Institute of Fundamental Sciences, Karadeniz Technical University, 61080 Trabzon, Turkey*

Correspondence should be addressed to Z. I. Ismailov; zameddin@yahoo.com

Received 23 May 2014; Revised 17 July 2014; Accepted 25 July 2014; Published 14 August 2014

Academic Editor: Abdullah S. Erdoğan

By using the methods of operator theory, all solvable extensions of minimal operator generated by first order pantograph-type delay differential-operator expression in the Hilbert space of vector-functions on finite interval have been considered. As a result, the exact formula for the spectrums of these extensions is presented. Applications of obtained results to the concrete models are illustrated.

1. Introduction

The quantitative and qualitative theory of linear pantograph differential equations, sometimes known as pantograph-type delay differential equations, was first studied in detail by T. Kato and J. B. McLeod [1], L. Fox et al. [2], and A. Iserles [3] in the nineteen seventies.

These equations arose as a mathematical model of an industrial problem involving wave motion in the overhead supply line to an electrified railway system, so they are often called pantograph equations.

In industrial applications in works [2, 4] and in studies on biology and economics, control and electrodynamics in works [5–7] have been researched (for more comprehensive list of features see [3]).

Since an analytical computation of solutions, eigenvalues, and eigenfunctions of corresponding problems is very difficult theoretically and technically, then in this theory methods of numerical analysis play a significant role (for more information see [8–13]).

Let us remember that an operator $S : D(S) \subset H \to H$ in Hilbert space H is called solvable, if S is one-to-one, $SD(S) = H$, and $S^{-1} \in L(H)$.

In this work, by using methods of operator theory all solvable extensions of minimal operator generated by pantograph-type delay differential-operator expression for first order in the Hilbert space of vector-functions at a finite interval have been described in terms of boundary values in Section 2. Consequently, the resolvent operators of these extensions can be written clearly.

The exact formula for the spectrums of these extensions has been given in Section 3. Applications of obtained results to the concrete models have been illustrated in Section 4.

2. Description of Solvable Extensions

In the Hilbert space $L^2(H, (0, 1))$ of vector-functions consider a linear pantograph differential-operator expression for first order in the form

$$l(u) = u'(t) + \sum_{m=1}^{n} A_m(t)\, u(\alpha_m t), \tag{1}$$

where

(1) H is a separable Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$,

(2) operator-function $A_m(\cdot) : [0,1] \to L(H)$, $m = 1, 2, 3, \ldots, n$, is continuous on the uniform operator topology,

(3) $m = 1, 2, 3, \ldots, n-1$, $0 < \alpha_m < 1$, $\alpha_n = 1$.

On the other hand, here the following differential expression will be considered:

$$m(u) = u'(t) \tag{2}$$

in the Hilbert space $L^2(H,(0,1))$ corresponding to (1).

It is clear that the formally adjoint expression of (2) is of the form

$$m^+(v) = -v'(t). \tag{3}$$

Now define operator M_0' on the dense in $L^2(H,(0,1))$ set of vector-functions D_0',

$$\begin{aligned} D_0' &:= \Big\{ u(t) \in L^2(H,(0,1)) : u(t) = \sum_{k=1}^{n} \varphi_k(t) f_k, \\ &\quad \varphi_k(t) \in C_0^\infty(0,1),\ f_k \in H,\ k = 1,2,\ldots,n,\ n \in \mathbb{N} \Big\}, \end{aligned} \tag{4}$$

as $M_0'u = m(u)$.

The closure of M_0' in $L^2(H,(0,1))$ is called the minimal operator generated by differential-operator expression (2) and is denoted by M_0.

In a similar way, the minimal operator M_0^+ in $L^2(H,(0,1))$ corresponding to differential expression (3) can be defined.

The adjoint operator of M_0^+ (M_0) in $L^2(H,(0,1))$ is called the maximal operator generated by (2)((3)) and is denoted by $M(M^+)$.

Now define an operator P_{α_m} in $L^2(H,(0,1))$ in the form

$$P_{\alpha_m} u(t) = u(\alpha_m t), \quad u \in L^2(H,(0,1)),\ m = 1,2,3,\ldots,n. \tag{5}$$

Then for $u \in L^2(H,(0,1))$ and for $m = 1,2,3,\ldots,n$ it is obtained that

$$\begin{aligned} \left\| P_{\alpha_m} u \right\|^2_{L^2(H,(0,1))} &= \int_0^1 (u(\alpha_m t), u(\alpha_m t))_H dt \\ &= \frac{1}{\alpha_m} \int_0^{\alpha_m} (u(x), u(x))_H dx \\ &\le \frac{1}{\alpha_m} \int_0^1 \|u(x)\|_H^2 dx = \frac{1}{\alpha_m} \|u\|^2_{L^2(H,(0,1))}. \end{aligned} \tag{6}$$

Therefore we have $P_{\alpha_m} \in L(L^2(H,(0,1)))$ and $\|P_{\alpha_m}\| \le 1/\sqrt{\alpha_m}$, $m = 1,2,3,\ldots,n$.

In this situation the following defined operator

$$A_\alpha(t) = \sum_{m=1}^{n} A_m(t) P_{\alpha_m}, \quad 0 < \alpha < 1, \tag{7}$$

is a linear bounded operator in $L^2(H,(0,1))$.

Throughout this work the following defined operators

$$\begin{gathered} L_0 := M_0 + A_\alpha(t), \\ L_0 : \overset{\circ}{W}{}_2^1(H,(0,1)) \subset L^2(H,(0,1)) \longrightarrow L^2(H,(0,1)), \\ L := M + A_\alpha(t), \\ L : W_2^1(H,(0,1)) \subset L^2(H,(0,1)) \longrightarrow L^2(H,(0,1)) \end{gathered} \tag{8}$$

will be called the minimal and maximal operators corresponding to differential expression (1) in $L^2(H,(0,1))$, respectively.

Now let $U(t,s)$, $t,s \in [0,1]$, be the family of evolution operators corresponding to the homogeneous differential equation

$$\begin{gathered} U_t(t,s) f + A_\alpha(t) U(t,s) f = 0, \quad t,s \in [0,1], \\ U(s,s) f = f, \quad f \in H. \end{gathered} \tag{9}$$

The operator $U(t,s)$, $t,s \in [0,1]$, is linear continuous, boundedly invertible in H and

$$U^{-1}(t,s) = U(s,t), \quad s,t \in [0,1] \tag{10}$$

(for more detailed analysis of this concept see [14]).

Let us introduce the operator

$$\begin{gathered} Uz(t) := U(t,0) z(t), \\ U : L^2(H,(0,1)) \longrightarrow L^2(H,(0,1)). \end{gathered} \tag{11}$$

In this case it is easy to see that, for the differentiable vector-function $z \in L^2(H,(0,1))$, $z : [0,1] \rightarrow H$ satisfies the following relation:

$$\begin{aligned} l(Uz) &= (Uz)'(t) + A_\alpha(t) Uz(t) \\ &= Uz'(t) + \left(U_t' + A_\alpha(t) U\right) z(t) = Um(z). \end{aligned} \tag{12}$$

From this $U^{-1} l(Uz) = m(z)$. Hence it is clear that if $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0 \subset \widetilde{L} \subset L$, then

$$\begin{gathered} U^{-1} L_0 U = M_0, \\ M_0 \subset U^{-1} L U = \widetilde{M} \subset M, \\ U^{-1} L U = M. \end{gathered} \tag{13}$$

For example, prove the validity of the last relation. It is known that

$$D(M_0) = \overset{\circ}{W}{}_2^1(H,(0,1)), \qquad D(M) = W_2^1(H,(0,1)). \tag{14}$$

If $u \in D(M)$, then $l(Uz) = Um(z) \in L^2(H,(0,1))$; that is, $Uu \in D(L)$. From the last relation $M \subset U^{-1} L U$. Contrarily, if a vector-function $u \in D(L)$, then

$$m\left(U^{-1} v\right) = U^{-1} l(v) \in L^2(H,(0,1)); \tag{15}$$

that is, $U^{-1} v \in D(M)$. From last relation $U^{-1} L \subset MU$; that is, $U^{-1} L U \subset M$. Hence, $U^{-1} L U = M$.

Theorem 1. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H,(0,1))$ is generated by the pantograph differential-operator expression* (1) *and boundary condition*

$$(K+E)\,u(0)=KU(0,1)\,u(1), \tag{16}$$

where $K\in L(H)$ and E is an identity operator in H. The operator K is determined uniquely by the extension $\widetilde{L}$; that is, $\widetilde{L}=L_K$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfies condition (16) *for some bounded operator $K\in L(H)$ is a solvable extension of the minimal operator L_0 in the $L^2(H,(0,1))$.*

Proof. Firstly, all solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H,(0,1))$ in terms of boundary values are described.

Consider the following so-called Cauchy extension M_c

$$\begin{gathered} M_c u=u'(t),\quad u(0)=0,\\ M_c: D(M_c)=\left\{u\in W_2^1(H,(0,1)) : u(0)=0\right\}\\ \subset L^2(H,(0,1))\longrightarrow L^2(H,(0,1)) \end{gathered} \tag{17}$$

of the minimal operator M_0. It is clear that M_c is a solvable extension of M_0 and

$$\begin{gathered} M_c^{-1}f(t)=\int_0^t f(x)\,dx,\quad f\in L^2(H,(0,1)),\\ M_c^{-1}: L^2(H,(0,1))\longrightarrow L^2(H,(0,1)). \end{gathered} \tag{18}$$

Now assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H,(0,1))$. In this case it is known that the domain of $\widetilde{M}$ can be written in direct sum in the form

$$D(\widetilde{M})=D(M_0)\oplus\left(M_c^{-1}+K\right)V, \tag{19}$$

where $V=\operatorname{Ker}M=H$, $K\in L(H)$ [15, 16]. Therefore for each $u(t)\in D(\widetilde{M})$ it is true that

$$u(t)=u_0(t)+M_c^{-1}f+Kf,\quad u_0\in D(M_0),\ f\in H. \tag{20}$$

That is, $u(t)=u_0(t)+tf+Kf$, $u_0\in D(M_0)$, $f\in H$. Hence $u(0)=Kf$, $u(1)=f+Kf=(K+E)f$. Hence $u(0)=Kf$, $u(1)=f+Kf=(K+E)f$ and from these relations it is obtained that

$$(K+E)\,u(0)=Ku(1). \tag{21}$$

On the other hand, uniqueness of operator $K\in L(H)$ follows from [15]. Therefore, $\widetilde{M}=M_K$. This completes the necessary part of this assertion.

On the contrary, if M_K is an operator generated by differential expression (2) and boundary condition (21), then M_K is bounded, boundedly invertible, and

$$\begin{gathered} M_K^{-1}: L^2(H,(0,1))\longrightarrow L^2(H,(0,1)),\\ M_K^{-1}f(t)=\int_0^t f(x)\,dx+K\int_0^1 f(x)\,dx,\\ f\in L^2(H,(0,1)). \end{gathered} \tag{22}$$

Consequently, all solvable extensions of the minimal operator M_0 in $L^2(H,(0,1))$ are generated by differential expression (2) and boundary condition (21) with any linear bounded operator K.

Now consider the general case. For this in the $L^2(H,(0,1))$ introduce an operator in the form

$$\begin{gathered} U: L^2(H,(0,1))\longrightarrow L^2(H,(0,1)),\\ (Uz)(t):=U(t,0)\,z(t),\quad z\in L^2(H,(0,1)). \end{gathered} \tag{23}$$

From the properties of the family of evolution operators $U(t,s)$, $t,s\in[0,1]$, it is implied that an operator U is linear bounded and has a bounded inverse and

$$\left(U^{-1}z\right)(t)=U(0,t)\,z(t). \tag{24}$$

On the other hand, from the relations

$$U^{-1}L_0U=M_0,\qquad U^{-1}\widetilde{L}U=\widetilde{M},\qquad U^{-1}LU=M, \tag{25}$$

it is implied that an operator U is one-to-one between sets of solvable extensions of minimal operators L_0 and M_0 in $L^2(H,(0,1))$.

The extension $\widetilde{L}$ of the minimal operator L_0 is solvable in $L^2(H,(0,1))$ if and only if the operator $\widetilde{M}=U^{-1}\widetilde{L}U$ is an extension of the minimal M_0 in $L^2(H,(0,1))$. Then, $u\in D(\widetilde{L})$ if and only if

$$(K+E)\,U(0,0)\,u(0)=KU(0,1)\,u(1); \tag{26}$$

that is, $(K+E)u(0)=KU(0,1)u(1)$. This proves the validity of the claims in the theorem. □

Corollary 2. *In particular the resolvent operator $R_\lambda(L_K)$, $\lambda\in\rho(L_K)$, of any solvable extension L_K of the minimal operator L_0, generated by pantograph-type delay differential expression*

$$l(u)=u'(t)+A(t)\,u(\alpha t),\quad 0<\alpha<1, \tag{27}$$

with boundary condition in $L^2(H,(0,1))$,

$$(K+E)\,u(0)=KU(0,1)\,u(1), \tag{28}$$

is of the form

$$\begin{aligned} &R_\lambda(L_K)\,f(t)\\ &\quad=U(t,0)\left[\left(E+K\left(1-e^{\lambda}\right)\right)^{-1}K\int_0^1 e^{\lambda(1-s)}U(0,s)\,f(s)\,ds\right.\\ &\qquad\left.+\int_0^t e^{\lambda(t-s)}U(0,s)\,f(s)\,ds\right], \end{aligned} \tag{29}$$

$f\in L^2(H,(0,1))$.

Remark 3. Note that in the general case $AP_\alpha\neq P_\alpha A$, for any $A\in L(H)$.

Indeed, if

$$(Af)(t) = tf(t), \quad f \in L^2(H,(0,1)), \quad A : L^2(0,1) \longrightarrow L^2(0,1), \tag{30}$$

then for $0 < \alpha < 1$ and $f \in L^2(0,1)$

$$\begin{aligned}(AP_\alpha) f(t) &= A(P_\alpha f(t)) = A(f(\alpha t)) \\ &= tf(\alpha t), \quad 0 < t < 1, \\ (P_\alpha A) f(t) &= P_\alpha (Af(t)) = P_\alpha (tf(t)) \\ &= (\alpha t) f(\alpha t), \quad 0 < t < 1.\end{aligned} \tag{31}$$

Corollary 4. *Assume that for any $t \in (0,1)$ and any $u \in W_2^1(H,(0,1))$*

$$(A_\alpha u)(\alpha t) = A_\alpha u(\alpha t). \tag{32}$$

In this case, all solvable extensions of minimal operator L_0 are generated by the following differential expression

$$l(u) = u'(t) + A(t) u(\alpha t), \quad 0 < \alpha < 1, \tag{33}$$

and boundary condition

$$(K+E) u(0) = K \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A^n u(\alpha^n), \quad K \in B(H) \tag{34}$$

in the Hilbert $L^2(H,(0,1))$ and vice versa.

Note that the series in the right side of the last equality is convergent, because for any $u \in W_2^1(H,(0,1))$

$$\sum_{n=0}^{\infty} \left\| \frac{(-1)^n}{n!} A^n u(\alpha^n) \right\|_H \le \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} \max_{[0,1]} \|u\|_H < +\infty. \tag{35}$$

Corollary 5. *All solvable extensions L_K of the minimal operator L_0 generated by pantograph differential expression $l(u) = u'(t) + u(\alpha t)$, $0 < \alpha < 1$, are described with boundary conditions*

$$\begin{aligned}(K+E) u(0) &= K \left[u(1) - \frac{u(\alpha)}{1!} - \frac{u(\alpha^2)}{2!} + \cdots \right] \\ &= K \sum_{n=0}^{\infty} \frac{1}{n!} u(\alpha^n)\end{aligned} \tag{36}$$

in the Hilbert space $L^2(H,(0,1))$.

Corollary 6. *It can be proved that all the solvable extensions of the minimal operator are generated by pantograph-type differential-operator expressions for first order*

$$l(u) = u'(t) + u(\alpha_1 t) + u(\alpha_2 t) \tag{37}$$

in $L^2(H,(0,1))$ generated by $l(\cdot)$ and boundary condition

$$\begin{aligned}(K+E) u(0) = K \Big[& u(1) - (u(\alpha_1) + u(\alpha_2)) \\ & + \frac{1}{2!} \left(u(\alpha_1^2) + 2u(\alpha_1\alpha_2) + u(\alpha_2^2) \right) \\ & - \frac{1}{3!} \left(u(\alpha_1^3) + u(\alpha_1\alpha_2^2) + 2u(\alpha_1^2\alpha_2) \right. \\ & \left. + 2u(\alpha_1\alpha_2^2) + u(\alpha_1^2\alpha_2) + u(\alpha_2^3) \right) + \cdots \Big]\end{aligned} \tag{38}$$

in $L^2(H,(0,1))$.

Remark 7. Theorem 1 can be generalized in the differential expression

$$l_\varphi(u) := u'(t) + \sum_{m=1}^{n} A_m(t) u(\varphi_m(t)), \tag{39}$$

where $\varphi_m : [0,1] \to [0,1]$, $\varphi_m > 0$ (< 0), $\varphi_m \in C^1[0,1]$, $P_{\varphi_m} : L^2(H,(0,1)) \to L^2(H,(0,1))$, $P_{\varphi_m} u(t) = u(\varphi_m(t))$, $m = 1, 2, \ldots, n$, $A_\varphi(t) := \sum_{m=1}^{n} A_m(t) P_{\varphi_m}$.

Theorem 8. *All solvable extensions of minimal operator corresponding to pantograph-type delay differential-operator expression $l_\varphi(\cdot)$ in Hilbert space $L^2(H,(0,1))$ are described by $l_\varphi(\cdot)$ and boundary condition*

$$(K+E) u(0) = K U_\varphi(0,1) u(1), \tag{40}$$

where $K \in B(H)$ and $U_\varphi(t,s)$, $t,s \in [0,1]$, is a family of evolution operators corresponding to the homogeneous differential equation

$$(U_\varphi)'_t(t,s) + A_\varphi(t) U(t,s) = 0, \quad t,s \in [0,1], \tag{41}$$

with boundary condition $U_\varphi(s,s)f = f$, $f \in H$ and vice versa.

3. Spectrum of Solvable Extensions

In this section, the spectrum structure of solvable extensions of minimal operator L_0 in $L^2(H,(0,1))$ will be investigated.

Firstly, prove the following fact.

Theorem 9. *If $\widetilde{L}$ is a solvable extension of a minimal operator L_0 and $\widetilde{M} = U^{-1}\widetilde{L}U$ corresponds to the solvable extension of a minimal operator M_0, then the spectrum of these extensions is true $\sigma(\widetilde{L}) = \sigma(\widetilde{M})$.*

Proof. Consider a problem to the spectrum for a solvable extension L_K of a minimal operator L_0 generated by pantograph differential-operator expression (1); that is,

$$L_K u = \lambda u + f, \quad \lambda \in \mathbb{C}, \quad f \in L^2(H,(0,1)). \tag{42}$$

From this it is obtained that

$$(L_K - \lambda E) u = f \tag{43}$$

or $(UM_K U^{-1} - \lambda E) u = f$. Hence $U(M_K - \lambda)(U^{-1}u) = f$.

Therefore, the validity of the theorem is clear. □

Now prove the following result for the spectrum of solvable extension.

Theorem 10. *If L_K is a solvable extension of the minimal operator L_0 in the space $L^2(H,(0,1))$, then spectrum of L_K has the form*

$$\sigma(L_K) = \left\{\lambda \in \mathbb{C} : \lambda = \ln\left|\frac{\mu+1}{\mu}\right| + i\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi i, \mu \in \sigma(K)\setminus\{0,-1\}, n \in \mathbb{Z}\right\}. \tag{44}$$

Proof. Firstly, the spectrum of the solvable extension $M_K = U^{-1}L_K U$ of the minimal operator M_0 in $L^2(H,(0,1))$ will be investigated.

Consequently, consider the following problem for the spectrum; that is, $M_K u = \lambda u + f$, $\lambda \in \mathbb{C}$, $f \in L^2(H,(0,1))$. Then

$$u' = \lambda u + f,$$
$$(K+E)u(0) = Ku(1), \quad \lambda \in \mathbb{C}, \tag{45}$$
$$f \in L^2(H,(0,1)), \quad K \in L(H).$$

It is clear that a general solution of the above differential equation in $L^2(H,(0,1))$ has the form

$$u_\lambda(t) = e^{\lambda t} f_0 + \int_0^t e^{\lambda(t-s)} f(s)\,ds, \quad f_0 \in H. \tag{46}$$

Therefore, from the boundary condition $(K+E)u_\lambda(0) = Ku_\lambda(1)$ it is obtained that

$$\left(E + K\left(1 - e^\lambda\right)\right) f_0 = K\int_0^1 e^{\lambda(1-s)} f(s)\,ds. \tag{47}$$

For $\lambda_m = 2m\pi$, $m \in \mathbb{Z}$, from the last relation it is established that

$$f_0^{(m)} = K\int_0^1 e^{\lambda_m(1-s)} f(s)\,ds, \quad m \in \mathbb{Z}. \tag{48}$$

Consequently, in this case the resolvent operator of M_K is in the form

$$R_{\lambda_m}(M_K) f(t) = Ke^{\lambda_m t}\int_0^1 e^{\lambda_m(1-s)} f(s)\,ds + \int_0^t e^{\lambda(t-s)} f(s)\,ds, \quad f \in L^2(H,(0,1)), \quad m \in \mathbb{Z}. \tag{49}$$

On the other hand, it is clear that $R_{\lambda_m}(M_K) \in B(L^2(H,(0,1)))$, $m \in \mathbb{Z}$.

Now assume that $\lambda \neq 2m\pi$, $m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$. Then using (47) we have

$$\left(K - \frac{1}{e^\lambda - 1}E\right) f_0 = \frac{1}{1-e^\lambda} K\int_0^1 e^{\lambda(1-s)} f(s)\,ds, \quad f_0 \in H, \quad f \in L^2(H,(0,1)). \tag{50}$$

Therefore, $\lambda \in \sigma(M_K)$ if and only if

$$\mu = \frac{1}{e^\lambda - 1} \in \sigma(K). \tag{51}$$

In this case since $\mu \neq 0$,

$$e^\lambda = \frac{\mu+1}{\mu}, \quad \mu \in \sigma(K), \ \mu \neq -1. \tag{52}$$

Then

$$\lambda_n = \ln\left|\frac{\mu+1}{\mu}\right| + i\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{53}$$

Later on, using the last relation and Theorem 9 the validity of the claim in the theorem is proved. □

Now one result on the asymptotic behavior of eigenvalues of solvable extensions in special cases will be proved.

Theorem 11. *Let $K \in L(H)$, $K \neq 0$, and $\sigma(K) = \sigma_P(K)$. In addition, assume that there exists $\alpha, \beta > 0$ such that for any $\mu \epsilon \sigma_P(K)$ are true $|\mu| \geq \alpha > 0$ and $|\mu+1| \geq \beta > 0$. Then $\lambda_n(L_K) \sim 2n\pi$ as $n \to \infty$. (i.e., there exist $\lim_{n\to\infty}(|\lambda_n(L_k)|/2n\pi)$ and $0 < \lim_{n\to\infty}(|\lambda_n(L_k)|/2n\pi) < \infty$).*

Proof. In this case for $n \geq 1$

$$|\lambda_n(M_K)|^2 = \ln^2\left|\frac{\mu+1}{\mu}\right| + \left|\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi\right|^2. \tag{54}$$

Since for any $\mu \in \sigma_p(K)$

$$\left|\frac{\mu+1}{\mu}\right| \geq \frac{\beta}{|\mu|} \geq \frac{\beta}{\|K\|} > 0, \qquad \left|\frac{\mu+1}{\mu}\right| \leq 1 + \frac{1}{|\mu|} \leq 1 + \frac{1}{\alpha}, \tag{55}$$

then

$$\ln\frac{\beta}{\|K\|} \leq \ln\left|\frac{\mu+1}{\mu}\right| \leq \ln\left(1+\frac{1}{\alpha}\right). \tag{56}$$

Therefore, for any $\mu \in \sigma_p(K)$ is true

$$\min\left\{\left|\ln\left(\frac{\beta}{\|K\|}\right)\right|, \left|\ln\left(1+\frac{1}{\alpha}\right)\right|\right\} \leq \left|\ln\left|\frac{\mu+1}{\mu}\right|\right| \leq \max\left\{\left|\ln\left(\frac{\beta}{\|K\|}\right)\right|, \left|\ln\left(1+\frac{1}{\alpha}\right)\right|\right\}. \tag{57}$$

On the other hand, for any $n \in \mathbb{Z}$

$$(2n\pi)^2 \leq \left|\arg\left(\frac{\mu+1}{\mu}\right) + 2n\pi\right|^2 \leq (2(n+1)\pi)^2. \tag{58}$$

Consequently, for any $n \in \mathbb{N}$

$$(2n\pi)^2\left(1+\frac{1}{4n^2\pi^2}\min^2\left\{\left|\ln\left(\frac{\beta}{\|K\|}\right)\right|,\left|\ln\left(1+\frac{1}{\alpha}\right)\right|\right\}\right)$$
$$\leq\left|\lambda_n\left(M_K\right)\right|^2\leq(2n\pi)^2\left(\left(\frac{2(n+1)\pi}{2n\pi}\right)^2\right.$$
$$\left.+\frac{1}{(2n\pi)^2}\max^2\left\{\left|\ln\left(\frac{\beta}{\|K\|}\right)\right|,\left|\ln\left(1+\frac{1}{\alpha}\right)\right|\right\}\right). \tag{59}$$

This means that $\lambda_n(M_K) \sim 2n\pi$ as $n \to \infty$. □

4. Applications

Example 12. Let

$$(H,\|\cdot\|_H)=(\mathbb{C},|\cdot|),\qquad A(t)=a(t)\in C(\mathbb{R}). \tag{60}$$

By Theorem 1, all solvable extensions L_k of minimal operator L_0 generated by $l(u) = u'(t) + a(t)u(\alpha t)$, $0 < \alpha < t$, in $L^2(0,1)$ are described with $l(\cdot)$ and boundary condition

$$(k+1)u(0)=k\exp\left(-\int_0^1 a(t)\,dt\right)u(1),\quad k\in\mathbb{C}. \tag{61}$$

In addition, the resolvent operator of these extensions is in the form

$$\begin{aligned}R_\lambda(L_k)f(t)&=\exp\left(-\int_0^t a(x)P_\alpha dx\right)\\&\times\left[\left(1+k\left(1-e^\lambda\right)^{-1}\right)k\right.\\&\quad\times\int_0^1\exp\left(\lambda(1-s)+\int_0^s a(x)P_\alpha dx\right)f(s)\,ds\\&\quad\left.+\int_0^t\exp\left(\lambda(t-s)+\int_0^s a(x)P_\alpha dx\right)f(s)\,ds\right],\end{aligned} \tag{62}$$

$\lambda \in \sigma(L_k)$, $f \in L^2(0,1)$ and for $k \neq 0,-1$ spectrum of this extension L_k is in the form

$$\sigma(L_k)=\left\{\lambda\in\mathbb{C}:\lambda=\ln\left|\frac{k+1}{k}\right|+i\arg\left(\frac{k+1}{k}\right)+2n\pi i,n\in\mathbb{Z}\right\}. \tag{63}$$

Example 13. Let

$$(H,\|\cdot\|_H)=(\mathbb{C},|\cdot|),\quad a,b\in C(\mathbb{C}). \tag{64}$$

Consider the following pantograph functional-differential expression in the form

$$l(u)=u'(t)+a(t)u(t)+b(t)u(\alpha t),\quad 0<\alpha<1, \tag{65}$$

in $L^2(0,1)$. Then by Theorem 1, all solvable extensions L_k of minimal operator L_0 are generated by $l(\cdot)$ and boundary condition

$$(k+1)u(0)=k\exp\left(\int_0^1\left(a(s)+b(s)P_\alpha\right)ds\right)u(1),\quad k\in\mathbb{C} \tag{66}$$

and vice versa.

Moreover, the resolvent operator of these extensions is

$$\begin{aligned}R_\lambda(L_k)f(t)&=\exp\left(-\int_0^t\left(a(s)+b(s)P_\alpha\right)ds\right)\\&\times\left[\left(1+k\left(1-e^\lambda\right)^{-1}\right)k\int_0^1\exp(\lambda(1-s)\right.\\&\quad+\int_0^s\left(a(x)+b(x)P_\alpha\right)dx)f(s)\,ds\\&\quad+\int_0^t\exp(\lambda(t-s)\\&\quad\left.+\int_0^s\left(a(x)+b(x)P_\alpha\right)dx)f(s)\,ds\right],\\&\qquad\lambda\in\mathbb{C},\quad f\in L^2(0,1).\end{aligned} \tag{67}$$

On the other hand, by Theorem 9 for $k \in \mathbb{C} \setminus \{(0,0),(-1,0)\}$ the spectrum of solvable extension L_k is in the form

$$\sigma(L_k)=\left\{\lambda\in\mathbb{C}:\lambda=\ln\left|\frac{\tau+1}{\tau}\right|+i\arg\left(\frac{\tau+1}{\tau}\right)+2n\pi i,n\in\mathbb{Z}\right\}. \tag{68}$$

Now consider the following differential equation

$$u'(t)=a(t)u(\alpha t)+b(t)u(t)+f(t) \tag{69}$$

with initial-boundary value problem

$$u(0)=u_0 \tag{70}$$

in the Hilbert space $L^2(0,1)$, where $a(\cdot), b(\cdot) \in C[0,1]$, $f \in L^2(0,1)$, and $u_0 \in \mathbb{C}$.

In order to solve this problem change the function $u(t)$ by

$$y(t)=u(t)-u_0,\quad 0<t<1. \tag{71}$$

Then the considered problem transforms the following problem:

$$y'(t)=a(t)y(\alpha t)+b(t)y(t)+g(t),\quad y(0)=0, \tag{72}$$

where $g(t) = f(t) - (a(t) + b(t))u_0$.

The last problem can be written in the form

$$y'(t)+A_\alpha(t)y(t)=g(t),\quad y(0)=0, \tag{73}$$

where $A_\alpha(t) = -a(t)P_\alpha - b(t)E$.

Then solution of the above Cauchy problem by Corollary 2 can be analytically expressed in the form ($K = 0$)

$$\begin{aligned} y(t) &= R_0(L_c)\, g(t) = L_c^{-1} g(t) \\ &= U(t,0)\int_0^t U(0,x)\, g(x)\, dx. \end{aligned} \quad (74)$$

Another approach to this problem has been investigated in [17].

Example 14. Consider the following integrodifferential equation for first order in the form

$$u'(t) + \int_0^t u(\alpha x)\, dx = f(t), \quad 0 < \alpha < 1,\ u(0) = u_0 \quad (75)$$

in Hilbert space $L^2(0,1)$. Changing the unknown function $u(t)$ by

$$y(t) = u(t) - u_0, \quad 0 < t < 1, \quad (76)$$

the following initial-value problem for integrodifferential equation is obtained:

$$y'(t) + \int_0^t y(\alpha x)\, dx = g(t), \quad y(0) = 0, \quad (77)$$

where $g(t) = f(t) - u_0 t$, in $L^2(0,1)$.

The last equation can be rewritten in the form

$$y'(t) + P_\alpha y(t) = g(t), \quad y(0) = 0. \quad (78)$$

It is easy to see that the analytical solution of this problem is in the form

$$y(t) = \int_0^t e^{-P_\alpha(t-s)} g(s)\, ds. \quad (79)$$

Consequently, for $0 < t < 1$

$$\begin{aligned} u(t) &= u_0 + \int_0^t e^{-P_\alpha(t-s)} \left(f(s) - u_0 s\right) ds \\ &= u_0 - e^{-P_\alpha t}\int_0^t s e^{P_\alpha s} ds u_0 \\ &\quad + e^{-P_\alpha t}\int_0^t e^{P_\alpha s} f(s)\, ds. \end{aligned} \quad (80)$$

Example 15. All solvable extensions of minimal operator generated by differential expression

$$\begin{aligned} l(u) = \frac{\partial u(t,x)}{\partial t} + x u(\alpha t, x), \quad x \in (-1,1), \\ 0 < t < 1, \quad 0 < \alpha < 1, \end{aligned} \quad (81)$$

in the Hilbert space $L^2((-1,1)\times(0,1))$ are described by this $l(\cdot)$ and boundary condition

$$(K + E)\, u(0,x) = KU(0,1)\, u(1,x), \quad (82)$$

where $K \in B(L^2(-1,1))$ and $U(t,s)$, $t,s \in [0,1]$, is a solution of operator equation

$$\begin{aligned} U_t'(t,s) + x P_\alpha U(t,s) = 0, \quad t,s \in [0,1], \\ U(s,s) f = f, \quad f \in L^2(-1,1), \end{aligned} \quad (83)$$

where $P_\alpha u(t) = u(\alpha t)$, $P_\alpha : L^2(0,1) \to L^2(0,1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank Professor A. Ashyralyev (Fatih University, Istanbul, Turkey) for his various comments and suggestions.

References

[1] T. Kato and J. B. McLeod, "The functional differential equation $y'(x) = ay(\lambda x) + by(x)$," *Bulletin of the American Mathematical Society*, vol. 77, pp. 891–937, 1971.

[2] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, "On a functional differential equation," *Journal of the Institute of Mathematics and Its Applications*, vol. 8, pp. 271–307, 1971.

[3] A. Iserles, "On the generalized pantograph functional-differential equation," *European Journal of Applied Mathematics*, vol. 4, no. 1, pp. 1–38, 1993.

[4] J. R. Ockendon and A. B. Tayler, "The dynamics of a current collection system for an electric locomotive," *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences*, vol. 322, pp. 447–468, 1971.

[5] W. G. Aiello, H. I. Freedman, and J. Wu, "Analysis of a model representing stage-structured population growth with state-dependent time delay," *SIAM Journal on Applied Mathematics*, vol. 52, no. 3, pp. 855–869, 1992.

[6] M. Buhmann and A. Iserles, "Stability of the discretized pantograph differential equation," *Mathematics of Computation*, vol. 60, no. 202, pp. 575–589, 1993.

[7] V. P. Spiridonov, "The factorization method, self-similar potentials and quantum algebras," in *Special Functions 2000: Current Perspective and Future Directions*, vol. 30 of *NATO Science Series II: Mathematics, Physics and Chemistry*, pp. 335–364, Springer, Amsterdam, The Netherlands, 2001.

[8] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford, UK, Oxford University Press, 2003.

[9] A. Bellen, "Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay," *IMA Journal of Numerical Analysis*, vol. 22, no. 4, pp. 529–536, 2002.

[10] A. Bellen, H. Brunner, S. Maset, and L. Torelli, "Superconvergence in collocation methods on quasi-graded meshes for functional differential equations with vanishing delays," *BIT: Numerical Mathematics*, vol. 46, no. 2, pp. 229–247, 2006.

[11] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, Cambridge, UK, 2004.

[12] H. Brunner and Q. Hu, "Optimal superconvergence results for delay integro-differential equations of pantograph type," *SIAM Journal on Numerical Analysis*, vol. 45, no. 3, pp. 986–1004, 2007.

[13] I. Ali, H. Brunner, and T. Tang, "A spectral method for pantograph-type delay differential equations and its convergence analysis," *Journal of Computational Mathematics*, vol. 27, no. 2-3, pp. 254–265, 2009.

[14] S. G. Krein, *Linear Differential Equations in Banach Spaces*, vol. 29 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1971.

[15] M. I. Višik, "Linear extensions of operators and boundary conditions," *Doklady Akad Nauk SSSR*, vol. 65, pp. 433–436, 1949 (Russian).

[16] M. I. Višik, "On general boundary problems for elliptic differential equations," *Trudy Matematicheskogo Obshestva*, vol. 1, pp. 187–246, 1952 (Russian).

[17] E. Ishiwata, Y. Muroya, and H. Brunner, "A super-attainable order in collocation methods for differential equations with proportional delay," *Applied Mathematics and Computation*, vol. 198, no. 1, pp. 227–236, 2008.