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Research Article

New Exact Solutions for High Dispersive Cubic-Quintic Nonlinear Schrödinger Equation

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We study a class of high dispersive cubic-quintic nonlinear Schrödinger equations, which describes the propagation of femtosecond light pulses in a medium that exhibits a parabolic nonlinearity law. Applying bifurcation theory of dynamical systems and the Fan sub-equations method, more types of exact solutions, particularly solitary wave solutions, are obtained for the first time.

1. Introduction

Propagation of short pulses in optical fibers is governed by the well-known nonlinear Schrödinger equation (NLS) [1]. In recent years, There have been extensive study and application of NLS. The main purpose of this paper is to discuss the traveling wave solutions for a class of high dispersive cubic-Quintic nonlinear Schrödinger equations describing the ultrashort light pulse propagation as in the following:

$$E_z = -i\frac{\beta_2}{2}E_{tt} + i\gamma_1|E|^2E + \frac{\beta_3}{6}E_{ttt} + i\frac{\beta_4}{24}E_{tttt} - i\gamma_2|E|^4E, \tag{1}$$

where $E(z,t)$ is the slowly varying envelope of the electric field, β_2 is the parameter of the group velocity dispersion, β_3 and β_4 are, respectively, the third-order and fourth-order dispersions, and γ_1 and γ_2 are the nonlinearity coefficients. When the higher order terms are ignored, we obtain the NLS. However, for femtosecond light pulses, whose duration is shorter than 10 fs, the last three terms are not ignored. Equation (1) was derived by Palacios and Fernández-Diáz [2]. Azzouzi et al. [3] by using the extended hyperbolic auxiliary equation method in getting the exact explicit solutions to (1). He et al. [4] find the exact bright, dark, and gray analytical nonautonomous soliton solutions of the generalized CQNLSE with spatially inhomogeneous group velocity dispersion (GVD) and amplification or attenuation by the similarity transformation method under certain parametric conditions.

We will study (1) by using the improved Fan subequation method. As a result, more types of exact solutions to (1) are obtained, which include solitons, kink solutions, and Jacobian elliptic function solutions with double periods. The rest of this paper is organized as follows. In Section 2, we give the mathematical framework of the improved method. In Section 3, we apply it to the generalized equation (1) for finding more exact solutions. Finally, some conclusions are given.

2. The Ansatz Solution and Fan Subequation Method

The integrability of a nonlinear equation can be studied by applying the Painleve analysis. It is widely believed that possession of the Painleve property is a sufficient criterion for integrability. Moreover, there exists another technique which basically consists of expressing the solution in terms of an amplitude and a phase function as an approach to find exact solutions of nonlinear evolution equations. We will make use of this formalism looking for exact solution of (1) such as

$$E(z,t) = e^{i(w_0 z - wt)}\varphi(\xi), \tag{2}$$

where $\varphi(\xi)$ is a real fuction and $\xi = v_0 z - vt$. By inserting the expressions (2) into (1), and separating real and imaginary parts, we obtain

$$l_1\varphi'(\xi) + l_3\varphi'''(\xi) = 0, \quad (3a)$$

$$l_0\varphi(\xi) + l_2\varphi''(\xi) + l_4\varphi''''(\xi) + \gamma_1\varphi^3(\xi) - \gamma_2\varphi^5(\xi) = 0, \quad (3b)$$

where

$$\begin{aligned} l_0 &= \frac{1}{2}w^2\left(\beta_2 + \frac{1}{3}\beta_3 w + \frac{1}{12}\beta_4 w^2\right) - w_0, \\ l_1 &= wv\left(\beta_2 + \frac{1}{2}\beta_3 w + \frac{1}{6}\beta_4 w^2\right) - v_0, \\ l_2 &= -v^2\left(\frac{1}{2}\beta_2 + \frac{1}{2}\beta_3 w + \frac{1}{4}\beta_4 w^2\right), \\ l_3 &= -\frac{1}{6}\left(\beta_3 + \beta_4 w\right)v^3, \qquad l_4 = \frac{1}{24}\beta_4 v^4. \end{aligned} \quad (4)$$

Let $l_1 = l_3 = 0$, and we get

$$\begin{aligned} w &= -\frac{\beta_3}{\beta_4}, \qquad v_0 = \frac{v\beta_3\left(\beta_3^2 - 3\beta_2\beta_4\right)}{3\beta_4^2}, \\ l_0 &= -w_0 - \frac{\beta_3^2\left(\beta_3^2 - 4\beta_2\beta_4\right)}{8\beta_4^3}, \qquad l_2 = \frac{v^2\left(\beta_3^2 - 2\beta_2\beta_4\right)}{4\beta_4}, \\ l_4 &= \frac{1}{24}\beta_4 v^4. \end{aligned} \quad (5)$$

Then, (3a) and (3b) become

$$l_0\varphi(\xi) + l_2\varphi''(\xi) + l_4\varphi''''(\xi) + \gamma_1\varphi^3(\xi) - \gamma_2\varphi^5(\xi) = 0. \quad (6)$$

We introduce auxiliary equation:

$$\varphi'(\xi) = \epsilon\sqrt{c_0 + c_1\varphi(\xi) + c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi)}, \quad (7)$$

where $\epsilon = \pm 1$, which is known as Fan subequation method and proposed by Fan in [5]. This method is proposed to seek more types of exact solutions of nonlinear partial differential equations. Obviously, (7) is equivalent to the two-dimensional systems as follows:

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{1}{2}\left(c_1 + 2c_2\varphi + 3c_3\varphi^2 + 4c_4\varphi^3\right), \quad (8)$$

which has the Hamiltonian function:

$$H(\varphi, y) = y^2 - \left(c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4\right) = c_0. \quad (9)$$

One can easily find that c_0 corresponds to the Hamiltonian constant and (7) is equivalent to the Hamiltonian system (8). Thus, in order to search the exact solutions of (7) we need only to discuss (8). For a fixed c_0, (9) determines a set of orbits of (8). As c_0 varies, (9) defines different families of orbits of (8) which have different dynamical behavior. Below we will first study the bifurcation of phase portraits of (8) by making use of bifurcation method of dynamical systems and with the aid of the computer symbolic system Mathematica. Then according to the obtained bifurcation and the Hamiltonian function (9), we will gain many new exact solutions of (7) for all possible parameters c_j [6].

Substituting (7) into (6), we have

$$\begin{aligned} &\left(-\gamma_2 + 24l_4c_4^2\right)\varphi^5 + 30l_4c_3c_4\varphi^4 \\ &+ \left(2l_2c_4 + l_4\left(20c_2c_4 + \frac{15}{2}c_3^2\right) + \gamma_1\right)\varphi^3 \\ &+ \left(\frac{3}{2}l_2c_3 + l_4\left(\frac{15}{2}c_2c_3 + 1_5c_1c_4\right)\right)\varphi^2 \\ &+ \left(l_0 + l_2c_2 + l_4\left(\frac{9}{2}c_1c_3 + 1_2c_0c_4 + c_2^2\right)\right)\varphi \\ &+ \frac{1}{2}l_2c_1 + l_4\left(3c_0c_3 + \frac{1}{2}c_1c_2\right) = 0. \end{aligned} \quad (10)$$

Setting all the coefficients of φ^i ($i = 0, 1, \ldots, 5$) to zero, and solving the obtained algebraic equations, we find the following sets of solutions (I)

$$\begin{aligned} c_1 &= c_3 = 0, \\ c_2 &= \frac{1}{v^2}\left(-\frac{3\left(\beta_3^2 - 2\beta_2\beta_4\right)}{5\beta_4^2} - \frac{6\gamma_1\sqrt{\gamma_2/\beta_4}}{5\gamma_2}\right), \\ c_4 &= \frac{1}{v^2}\sqrt{\frac{\gamma_2}{\beta_4}}, \\ c_0 &= \Bigg(\Big(50w_0\gamma_2\beta_4^3 - 3\gamma_1^2\beta_4^3 - 52\gamma_2\beta_2\beta_3^2\beta_4 \\ &\qquad + 27\gamma_2\beta_2^2\beta_4^2 + 13\gamma_2\beta_3^4\Big) \\ &\qquad \times \sqrt{\frac{\gamma_2}{\beta_4}} + 12\gamma_1\gamma_2\beta_4\left(\beta_3^2 - 2\beta_2\beta_4\right)\Bigg) \\ &\quad \times \left(25\gamma_2^2\beta_4^3v^2\right)^{-1} \end{aligned} \quad (11)$$

and (II)

$$\begin{aligned} c_1 &= c_3 = 0, \\ c_2 &= \frac{1}{5v^2}\left(-\frac{3\left(\beta_3^2 - 2\beta_2\beta_4\right)}{\beta_4^2} + \frac{6\gamma_1\sqrt{\gamma_2/\beta_4}}{\gamma_2}\right), \\ c_4 &= -\frac{1}{v^2}\sqrt{\frac{\gamma_2}{\beta_4}}, \end{aligned}$$

$$c_0 = \Bigg(-\left(50w_0\gamma_2\beta_4^3 - 3\gamma_1^2\beta_4^3 - 52\gamma_2\beta_2\beta_3^2\beta_4 + 27\gamma_2\beta_2^2\beta_4^2 + 13\gamma_2\beta_3^4\right) \times \sqrt{\frac{\gamma_2}{\beta_4}} + 12\gamma_1\gamma_2\beta_4\left(\beta_3^2 - 2\beta_2\beta_4\right)\Bigg) \times \left(25\gamma_2^2\beta_4^3 v^2\right)^{-1} \tag{12}$$

and (III), $\gamma_2 = 0$:

$$\begin{aligned} c_0 &= -\left(\left(-\beta_3^2 + 2\beta_2\beta_4\right) \times \left(200\beta_4^3 w_0 + 73\beta_3^4 - 292\beta_2\beta_3^2\beta_4 + 192\beta_2^2\beta_4^2\right)\right) \times \left(150\beta_4^5\gamma_1 v^2\right)^{-1}, \\ c_1 &= \mp \frac{200\beta_4^3 w_0 + 73\beta_3^4 - 292\beta_2\beta_3^2\beta_4 + 192\beta_2^2\beta_4^2}{150\beta_4^3\gamma_1 v^2} \times \sqrt{\frac{-5\gamma_1}{\beta_4}}, \\ c_2 &= \frac{6\left(-\beta_3^2 + 2\beta_2\beta_4\right)}{5\beta_4^2 v^2}, \qquad c_3 = \pm\frac{4}{5v^2}\sqrt{\frac{-5\gamma_1}{\beta_4}}, \\ c_4 &= 0, \end{aligned} \tag{13}$$

where w_0, v are any real number and $v \neq 0$.

3. Exact Solutions of High Dispersive Cubic-Quintic Nonlinear Schrödinger Equation

3.1. Case (I) and Case (II). In this case, the Hamiltonian system (8) becomes

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = c_2\varphi + 2c_4\varphi^3, \tag{14}$$

and the Hamiltonian function (9) reduces to

$$H_1(\varphi, y) = y^2 - c_2\varphi^2 - c_4\varphi^4 = c_0. \tag{15}$$

Now we discuss the bifurcations of phase portraits of (14). Obviously all the equilibrium points of (14) lie in the φ-axis and their abscissas are the real zeros of $f(\varphi) = c_2\varphi + 2c_4\phi^3$. Thus, we have the following proposition on the distribution of the equilibrium points of (14).

Proposition 1.

(1) *For* $c_2c_4 < 0$, (14) *has three equilibria points at* $E_{10}(0,0)$, $E_{11}(\psi_{11},0)$, *and* $E_{12}(\psi_{12},0)$, *where* $\psi_{11} = -\sqrt{-c_2/(2c_4)}$, $\psi_{12} = \sqrt{-c_2/(2c_4)}$.

(2) *For* $c_2c_4 \geq 0$, (14) *has a unique equilibrium at* $E_{10}(0,0)$.

Using the bifurcation theory of dynamical systems [7–9], the (c_2, c_4)-plane was divided into four subregions:

$$\begin{aligned} &D1: \{c_2 > 0, c_4 > 0\}; \qquad D2: \{c_2 < 0, c_4 > 0\}; \\ &D3: \{c_2 < 0, c_4 < 0\}; \qquad D4: \{c_2 > 0, c_4 < 0\}. \end{aligned} \tag{16}$$

The phase portraits of (14) are shown in Figure 1.

For the function defined by (15), we have

$$\begin{aligned} h_{10} &= H_1(\psi_{10}, 0) = 0, \\ h_{11} &= h_{12} = H_1(\psi_{11}, 0) = H_1(\psi_{12}, 0) = \frac{c_2^2}{4c_4}. \end{aligned} \tag{17}$$

Let $M(w_e, z_e)$ be the coefficient matrix of the system (14) at an equilibrium point (w_e, z_e). Then, we have

$$J(w_e, 0) = \det\left(M(w_e, 0)\right) = -\left(c_2 + 6c_4w_e^2\right). \tag{18}$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and Trace $(M(w_e)) = 0$, then it is a center point; if $J > 0$ and $(\text{Trace}(M(w_e, z_e)))^2 - 4J(w_e, z_e) > 0$, then it is a node; if $J = 0$ and the index of the equilibrium point is 0, then it is a cusp; otherwise, it is a high order equilibrium point.

Below, we will give explicit and exact solutions of (7) (also (6)). We always let $\Delta = c_2^2 - 4c_0c_4$ and only pay attention to the bounded solutions of (6).

(1) Suppose that $(c_2, c_4) \in D4$.

(a) For $c_0 \in (c_2^2/(4c_4), 0)$, (7) has periodic solutions:

$$\varphi_{\pm 1} = \pm\sqrt{\frac{c_2 + \sqrt{\Delta}}{-2c_4} + \frac{\sqrt{\Delta}}{c_4}\text{sn}^2\left(\sqrt{\frac{c_2 + \sqrt{\Delta}}{2}}\xi, \sqrt{\frac{2\sqrt{\Delta}}{c_2 + \sqrt{\Delta}}}\right)}, \tag{19}$$

where $\text{sn}(x, k)$ and below $\text{cn}(x, k)$ are Jacobian elliptic functions with modulus k [10]. The profiles of periodic solutions are shown in Figure 2.

Thus, we obtain the following solutions of (1):

$$E(z,t) = \pm e^{i(w_0 z + \beta_3 t/\beta_4)} \times \sqrt{\frac{c_2 + \sqrt{\Delta}}{-2c_4} + \frac{\sqrt{\Delta}}{c_4}\text{sn}^2\left(\sqrt{\frac{c_2 + \sqrt{\Delta}}{2}}\xi, \sqrt{\frac{2\sqrt{\Delta}}{c_2 + \sqrt{\Delta}}}\right)}. \tag{20}$$

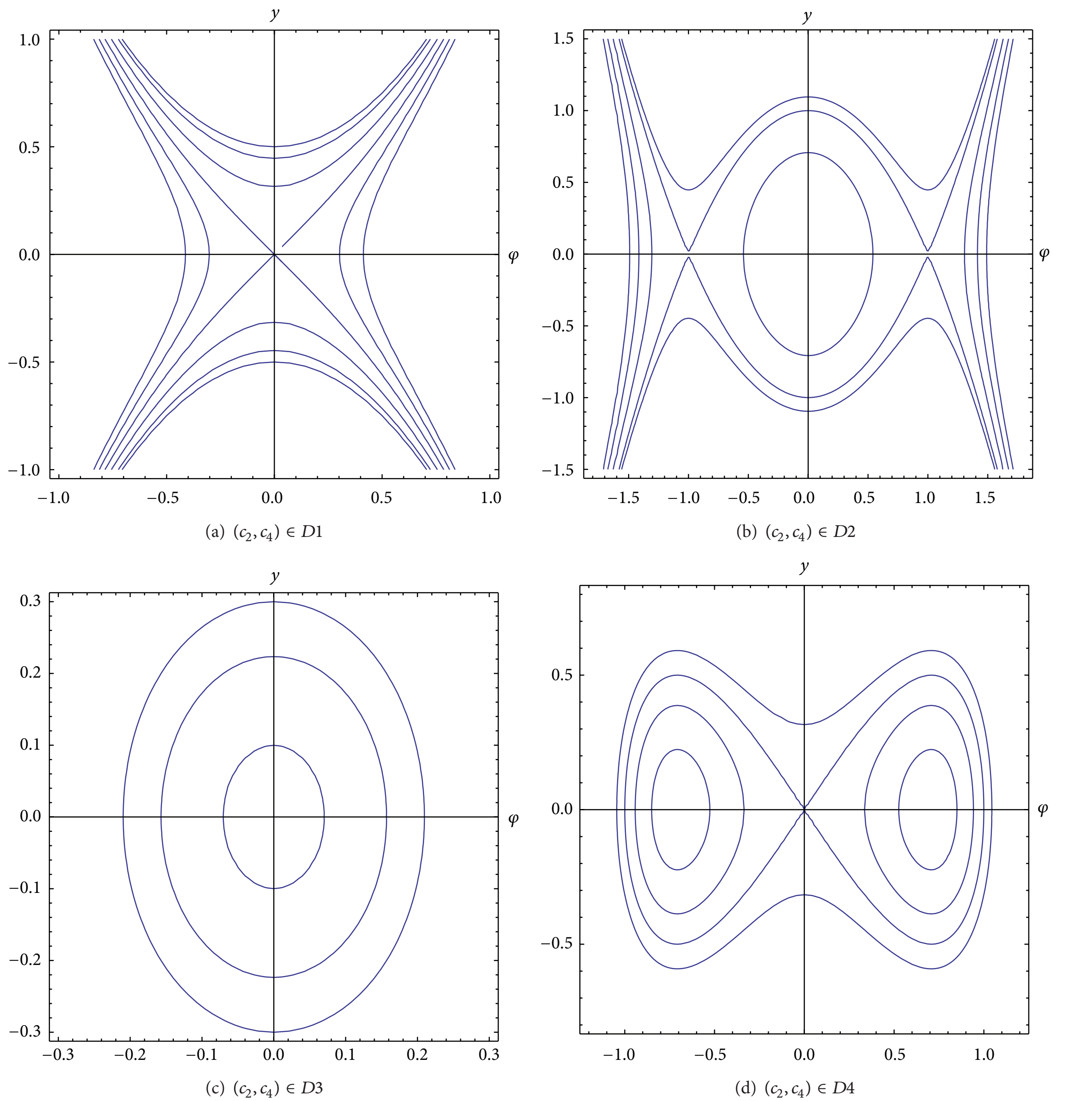
(b) For $c_0 = 0$, we have solitary wave solutions of (7):

$$\varphi_{\pm 2} = \pm\sqrt{-\frac{c_2}{c_4}}\,\text{sech}\left(\sqrt{c_2}\xi\right). \tag{21}$$

Thus, we obtain the following solutions of (1):

$$E(z,t) = \pm e^{i(w_0 z + \beta_3 t/\beta_4)}\sqrt{-\frac{c_2}{c_4}}\,\text{sech}\left(\sqrt{c_2}\xi\right). \tag{22}$$

The profiles of solutions of (22) are shown in Figure 3.



(a) $(c_2, c_4) \in D1$

(b) $(c_2, c_4) \in D2$

(c) $(c_2, c_4) \in D3$

(d) $(c_2, c_4) \in D4$

Figure 1: The phase portraits of system (14).

Remark 2. By the expression of c_0, there always exists a w_0 such that $c_0 = 0$ if $(c_2, c_4) \in D4$.

Because of the limitation of length, we omit the expression of $E(z,t)$, beginning from here.

(c) For $c_0 > 0$, (6) has periodic solution:

$$\varphi_3 = \sqrt{\frac{c_2 + \sqrt{\Delta}}{-2c_4}} \operatorname{cn}\left(\sqrt[4]{\Delta}\xi, \sqrt{\frac{c_2 + \sqrt{\Delta}}{2\sqrt{\Delta}}}\right). \tag{23}$$

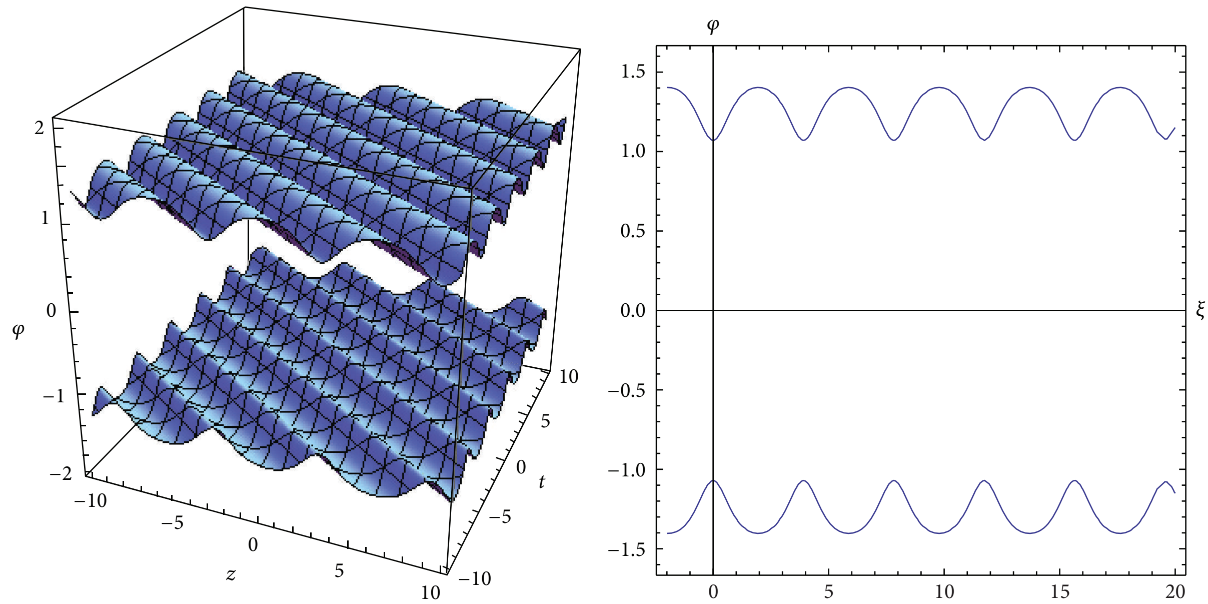
(2) Suppose that $(c_2, c_4) \in D2$.

(a) For $c_0 \in (0, c_2^2/(4c_4))$, system (14) has periodic solution:

$$\varphi_{\pm 4} = \pm\sqrt{\frac{-c_2 - \sqrt{\Delta}}{-c_2 + \sqrt{\Delta}}} \operatorname{sn}\left(\sqrt{\frac{-c_2 + \sqrt{\Delta}}{2}}\xi, \sqrt{\frac{-c_2 - \sqrt{\Delta}}{-c_2 + \sqrt{\Delta}}}\right). \tag{24}$$

(b) For $c_0 = c_2^2/(4c_4)$, there exists smooth kink wave solution of (14) as follows:

$$\varphi_{\pm 5} = \pm\sqrt{\frac{-c_2}{2c_4}} \tanh\left(\sqrt{\frac{-c_2}{2}}\xi\right). \tag{25}$$



(a) The 3D wave profiles of solutions $\varphi_{\pm 1}$

(b) The 2D wave profiles of solutions $\varphi_{\pm 1}$ for $t = 0$

FIGURE 2: The wave profiles of solutions $\varphi_{\pm 1}$ with the parameters values: $\gamma_1 = \gamma_2 = \beta_1 = \beta_2 = \beta_3 = \nu = 1$, $w_0 = 0.38$.

(3) If $(c_2, c_4) \in D_3$, for $c_0 > 0$, we have the following periodic solution of (14):

$$\varphi_{\pm 6} = \pm\sqrt{\frac{-c_2 - \sqrt{\Delta}}{2c_4}}\,\mathrm{cn}\left(\sqrt[4]{\Delta}\xi, \sqrt{\frac{c_2 + \sqrt{\Delta}}{2\sqrt{\Delta}}}\right). \tag{26}$$

(4) If $(c_2, c_4) \in D_1$, it is observed from Figure 1(a) that there is no bounded solution of system (14).

3.2. The Case (III). For this case, the Hamiltonian system (8) becomes

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = f(\varphi) = \frac{1}{2}\left(c_1 + 2c_2\varphi + 3c_3\varphi^2\right) \tag{27}$$

and the Hamiltonian function (9) reduces to

$$H_1(\varphi, y) = y^2 - c_1\varphi - c_2\varphi^2 - c_3\varphi^3 = c_0. \tag{28}$$

Similar to the previous discussion, we have the following proposition.

Proposition 3.

(1) *For $c_2^2 - 3c_1c_3 > 0$, (27) has two equilibria at $E_{21}(\psi_{21}, 0)$ and $E_{22}(\psi_{22}, 0)$, where*

$$\psi_{21} = \frac{-c_2 - \sqrt{c_2^2 - 3c_1c_3}}{3c_3}, \qquad \psi_{22} = \frac{-c_2 + \sqrt{c_2^2 - 3c_1c_3}}{3c_3}. \tag{29}$$

(2) *For $c_2^2 - 3c_1c_3 = 0$, (27) has a unique equilibrium at $E_{20}(\psi_{20}, 0)$, where $\psi_{20} = -c_2/(3c_3)$.*

(3) *For $c_2^2 - 3c_1c_3 < 0$, (27) has no equilibrium.*

Let $h_{2i} = H_1(\psi_{2i}, 0)$, $i = 0, 1, 2$, and notice that we need only to consider the case $c_3 \geq 0$ because of the invariance of (27) under the transformations $\phi \to -\phi$, $y \to -y$, and $c_3 \to -c_3$.

(1) $c_2^2 - 3c_1c_3 > 0$ and $c_0 \in (h_{21}, h_{22})$. In this case,

$$c_3\varphi^3 + c_2\varphi^2 + c_1\varphi + c_0 = 0 \tag{30}$$

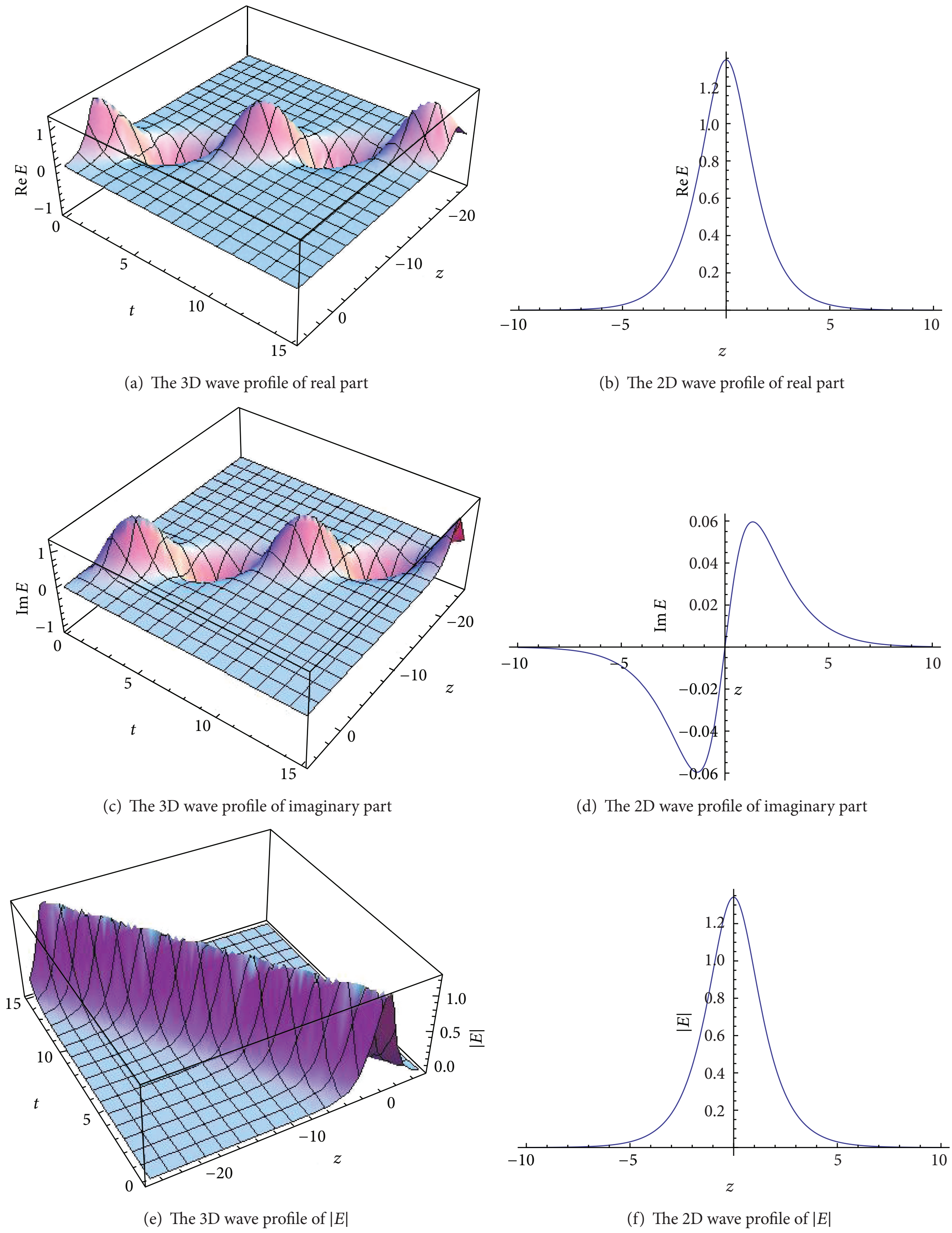
has three mutually different real roots $\varphi_m < \varphi_l < \varphi_M$; thus,

$$c_3\varphi^3 + c_2\varphi^2 + c_1\varphi + c_0 = c_3(\varphi - \varphi_m)(\varphi - \varphi_l)(\varphi - \varphi_M). \tag{31}$$

Equation (7) has periodic wave solutions as follows:

$$\begin{aligned}\varphi_8 &= \varphi_M - ((\varphi_M - \varphi_l)(\varphi_M - \varphi_m)) \\ &\quad\times\left(\varphi_M - \varphi_m - (\varphi_l - \varphi_m)\,\mathrm{sn}^2\right. \\ &\quad\left.\times\left(\frac{\sqrt{c_3(\varphi_M - \varphi_m)}}{2}\xi, \sqrt{\frac{\varphi_l - \varphi_m}{\varphi_M - \varphi_m}}\right)\right)^{-1}, \\ \varphi_9 &= \varphi_m + (\varphi_l - \varphi_m)\,\mathrm{sn}^2\left(\frac{\sqrt{c_3(\varphi_M - \varphi_m)}}{2}\xi, \sqrt{\frac{\varphi_l - \varphi_m}{\varphi_M - \varphi_m}}\right).\end{aligned} \tag{32}$$

(2) $c_2^2 - 3c_1c_3 > 0$ and $c_0 = h_{22}$. In this case, φ_{22} is double root of (30); suppose that φ_t is other root of the equation,



(a) The 3D wave profile of real part
(b) The 2D wave profile of real part
(c) The 3D wave profile of imaginary part
(d) The 2D wave profile of imaginary part
(e) The 3D wave profile of $|E|$
(f) The 2D wave profile of $|E|$

FIGURE 3: The wave profile of solutions (22), with the parameters values: $\gamma_1 = \gamma_2 = \beta_1 = \beta_2 = \beta_3 = \nu = 1$, $w_0 = 0.06$.

obviously $\varphi_t < \varphi_{22}$, and we have a solitary wave solution of peak type of (7) as follows:

$$\varphi_{10} = \varphi_t + (\varphi_{22} - \varphi_t)\tanh^2\left(\frac{1}{2}\sqrt{c_3(\varphi_{22} - \varphi_t)}\xi\right). \quad (33)$$

4. Conclusions

In this study, we apply bifurcation theory of dynamical systems and the Fan subequation method to investigate (1), and many new exact solutions have been obtained; most importantly, under more general conditions than [3], to the best of our knowledge, these solutions have not been reported in the literature. This method can help us find exact solutions of other types of nonlinear dispersion partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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