Research Article

A Class of Volterra-Fredholm Type Weakly Singular Difference Inequalities with Power Functions and Their Applications

Yange Huang,[1] Wu-Sheng Wang,[2] and Yong Huang[1]

[1] Department of Mathematics and Computer Information Engineering, Baise University, Baise 533000, China
[2] School of Mathematics and Statistics, Hechi University, Yizhou, Guangxi 546300, China

Correspondence should be addressed to Wu-Sheng Wang; wang4896@126.com

We discuss a class of Volterra-Fredholm type difference inequalities with weakly singular. The upper bounds of the embedded unknown functions are estimated explicitly by analysis techniques. An application of the obtained inequalities to the estimation of Volterra-Fredholm type difference equations is given.

1. Introduction

Being an important tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds, and other qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities [1, 2] and their applications have attracted great interests of many mathematicians [3–5]. Some recent works can be found in [6–28].

In 1981, Henry [12] discussed the following linear singular integral inequality:

$$u(t) \leq a + b \int_0^t (t-s)^{\beta-1} u(s)\, ds. \tag{1}$$

In 2007, Ye et al. [18] discussed linear singular integral inequality

$$u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds. \tag{2}$$

In 2014, Cheng et al. [28] discussed the following inequalities:

$$u^m(t) \leq a(t) + b(t) \int_0^t f(s) u^n(s)\, ds$$
$$+ c(t) \int_0^T g(s) u^r(s)\, ds,$$

$$u^m(t) \leq a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1-1} s^{\gamma_1-1} f(s) u^n(s)\, ds$$
$$+ c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2-1} s^{\gamma_2-1} g(s) u^r(s)\, ds. \tag{3}$$

On the other hand, difference inequalities which give explicit bounds on unknown functions provide a very useful and important tool in the study of many qualitative as well as quantitative properties of solutions of nonlinear difference equations. More attentions are paid to some discrete versions of Gronwall-Bellman type inequalities (such as [29–50]).

In 2002, Pachpatte [36] discussed the following difference inequality:

$$u(n) \leq c + \sum_{s=\alpha}^{n-1} f(n,s) u(s)\, ds + \sum_{s=\alpha}^{\beta} g(n,s) u(s),$$
$$n \in \mathbb{N} \cap [\alpha, \beta]. \tag{4}$$

In 2010, Ma [45] discussed the following difference inequality with two variables:

$$u^i(m,n) \leq a(m,n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) u^j(s,t)$$
$$+ \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u^r(s,t). \tag{5}$$

In 2014, Huang at el. [50] discussed the following linear singular difference inequality:

$$u(n) \le a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{\beta-1} \tau_s w_1(u(s))$$
$$\times \left[u(s) + h(s) + \sum_{\sigma=0}^{s-1} (t_s - t_\sigma)^{\beta-1} \tau_\sigma w_2(u(\sigma)) \right]. \tag{6}$$

Motivated by the results given in [6, 11, 28, 36, 45, 49, 50], in this paper, we discuss the following inequalities:

$$u(m,n) \le a(m,n) + b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) u(s,t)$$
$$+ c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t), \tag{7}$$

$$u^i(m,n) \le a(m,n) + b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) u^j(s,t)$$
$$+ c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u^r(s,t), \tag{8}$$

$$u^i(n) \le a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{\beta-1} t_s^{\gamma-1} \tau_s f(s) u^j(s)$$
$$+ c(n) \sum_{s=0}^{N-1} (t_N - t_s)^{\beta-1} t_s^{\gamma-1} \tau_s g(s) u^r(s). \tag{9}$$

2. Difference Inequalities with Two Variables

Throughout this paper, let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, \ldots\}$, and $\Omega_{X,Y} = \{(m,n) : m_0 \le m \le X, n_0 \le n \le Y, m, n, X, Y \in \mathbb{N}\}$. For a function $z(m,n)$, its first-order difference is defined by $\Delta_1 z(m,n) = z(m+1, n) - z(m,n)$. Obviously, the linear difference equation $\Delta z(n) = b(n)$ with the initial condition $z(n_0) = 0$ has the solution $z(n) = \sum_{s=n_0}^{n-1} b(s)$. For convenience, in the sequel, we complementarily define that $\sum_{s=n_0}^{n_0-1} b(s) = 0$.

Lemma 1. *Assume that $u(m,n)$, $a(m,n)$, $c(m,n)$, and $g(m,n)$ are nonnegative functions on $\Omega_{M,N} = \{(m,n) : m_0 \le m \le M, n_0 \le n \le N, m, n, M, N \in \mathbb{N}\}$. If $\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} g(s,t) c(s,t) < 1$ and $u(m,n)$ satisfies the following difference inequality:*

$$u(m,n) \le a(m,n) + c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t),$$
$$\forall (m,n) \in \Omega_{M,N}, \tag{10}$$

then

$$u(m,n) \le a(m,n) + \frac{c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t)}{1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t)},$$
$$\forall (m,n) \in \Omega_{M,N}. \tag{11}$$

Proof. Since $\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t)$ is a constant. Let $\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t) = K$. From (10), we have

$$u(m,n) \le a(m,n) + c(m,n) K, \quad \forall (m,n) \in \Omega_{M,N}. \tag{12}$$

Since $g(m,n)$ is nonnegative, we have

$$g(m,n) u(m,n) \le g(m,n) a(m,n) + c(m,n) g(m,n) K. \tag{13}$$

Let $s = m$ and $t = n$ in (13) and substituting $s = m_0, m_1, m_2, \ldots, M-1$ and $t = n_0, n_1, n_2, \ldots, N-1$, successively, we obtain

$$K = \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t)$$
$$\le \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t)$$
$$+ \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t) K. \tag{14}$$

From (14), we have

$$K \le \frac{\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) a(s,t)}{1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t)}, \tag{15}$$

where $\sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} g(s,t) c(s,t) < 1$. Substituting inequality (15) into (13), we get the explicit estimation (11) for $u(m,n)$. □

Theorem 2. *Assume that $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $f(m,n)$, and $g(m,n)$ are nonnegative functions on $\Omega_{M,N}$ and $a(m,n)$, $b(m,n)$, and $c(m,n)$ are nondecreasing in both m and n. If*

$$\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) c(s,t)$$
$$\times \exp\left(b(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f(\tau,\xi) \right) < 1, \tag{16}$$
$$\forall (m,n) \in \Omega_{M,N},$$

and $u(m,n)$ satisfies the difference inequality (7), then

$$u(m,n)$$
$$\le \exp\left(b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right)$$

$$\times \left[a\,(m,n) + c\,(m,n) \right.$$

$$\times \left(\left(\left(\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g\,(s,t)\,a\,(s,t) \right. \right. \right.$$

$$\left. \times \exp\left(b\,(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f\,(\tau,\xi) \right) \right)$$

$$\times \left(1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g\,(s,t)\,c\,(s,t) \right.$$

$$\times \exp\left(b\,(s,t) \right.$$

$$\left. \left. \left. \times \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} f\,(\tau,\xi) \right) \right)^{-1} \right) \right],$$

(17)

for all $(m,n) \in \Omega_{M,N}$.

Proof. Fixing any arbitrary $(X,Y) \in \Omega_{M,N}$, from (7), we have

$$u\,(m,n) \le a\,(X,Y) + b\,(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f\,(s,t)\,u\,(s,t)$$

$$+ c\,(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g\,(s,t)\,u\,(s,t),$$

(18)

for all $(m,n) \in \Omega_{X,Y}$, where $a(m,n)$, $b(m,n)$, and $c(m,n)$ are nondecreasing in both m and n.

Define a function $z(m,n)$ by the right side of (18); that is,

$$z\,(m,n) := a\,(X,Y) + b\,(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f\,(s,t)\,u\,(s,t)$$

$$+ c\,(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g\,(s,t)\,u\,(s,t),$$

(19)

for all $(m,n) \in \Omega_{X,Y}$. Obviously, we have

$$u\,(m,n) \le z\,(m,n), \quad \forall (m,n) \in \Omega_{X,Y}, \qquad (20)$$

$$z\,(m_0,n) = a\,(X,Y) + c\,(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g\,(s,t)\,u\,(s,t). \quad (21)$$

Using the difference formula $\Delta_1 z(m,n) = z(m+1,n) - z(m,n)$ and relation (20), from (21), we have

$$\Delta_1 z\,(m,n) = b\,(X,Y) \sum_{t=n_0}^{n-1} f\,(m,t)\,u\,(m,t)$$

$$\le b\,(X,Y) \sum_{t=n_0}^{n-1} f\,(m,t)\,z\,(m,t) \qquad (22)$$

$$\le b\,(X,Y)\,z\,(m,n) \sum_{t=n_0}^{n-1} f\,(m,t),$$

where we have used the monotonicity of z in n. From (22), we observe that

$$\frac{\Delta_1 z\,(m,n)}{z\,(m,n)} \le b\,(X,Y) \sum_{t=n_0}^{n-1} f\,(m,t), \quad \forall (m,n) \in \Omega_{X,Y}. \quad (23)$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given integers m, n with $(m+1,n)$, $(m,n) \in \Omega_{X,Y}$, there exists ξ in the open interval $(z(m,n), z(m,n+1))$ such that

$$\ln z\,(m+1,n) - \ln z\,(m,n) = \int_{z(m,n)}^{z(m+1,n)} \frac{ds}{s} = \frac{\Delta_1 z\,(m,n)}{\xi}$$

$$\le \frac{\Delta_1 z\,(m,n)}{z\,(m,n)}.$$

(24)

From (23) and (24), we have

$$\ln z\,(m+1,n) - \ln z\,(m,n) \le b\,(X,Y) \sum_{t=n_0}^{n-1} f\,(m,t),$$

(25)

$$\forall (m,n) \in \Omega_{X,Y}.$$

Let $s = m$ and $t = n$ in (25), and substituting $s = m_0, m_1, m_2, \ldots, m-1$ and $t = n_0, n_1, n_2, \ldots, n-1$, successively, we obtain

$$\ln z\,(m,n) - \ln z\,(m_0,n) \le b\,(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f\,(s,t),$$

(26)

$$\forall (m,n) \in \Omega_{X,Y}.$$

It implies that

$$z\,(m,n) \le z\,(m_0,n) \exp\left(b\,(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f\,(s,t) \right),$$

(27)

$$\forall (m,n) \in \Omega_{X,Y}.$$

Using (20) and (21), from (27), we have

$$u(m,n)$$

$$\leq \left(a(X,Y) + c(X,Y) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t) \right)$$

$$\times \exp\left(b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right)$$

$$= a(X,Y) \exp\left(b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right) \tag{28}$$

$$+ c(X,Y) \exp\left(b(X,Y) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right)$$

$$\times \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t),$$

for all $(m,n) \in \Omega_{X,Y}$. Taking $m = X$ and $n = Y$ in (28), we have

$$u(X,Y)$$

$$\leq a(X,Y) \exp\left(b(X,Y) \sum_{s=m_0}^{X-1} \sum_{t=n_0}^{Y-1} f(s,t) \right)$$

$$+ c(X,Y) \exp\left(b(X,Y) \sum_{s=m_0}^{X-1} \sum_{t=n_0}^{Y-1} f(s,t) \right) \tag{29}$$

$$\times \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t).$$

Since X, Y are chosen arbitrarily, we replace X and Y in (29) with m and n, respectively, and obtain that

$$u(m,n)$$

$$\leq a(m,n) \exp\left(b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right)$$

$$+ c(m,n) \exp\left(b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \right) \tag{30}$$

$$\times \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) u(s,t),$$

for all $(m,n) \in \Omega_{M,N}$. Applying the result of Lemma 1 to inequality (30), we obtain desired estimation (17). □

Lemma 3 (see [39]). *Let $a \geq 0$, $i \geq j \geq 0$, and $i \neq 0$. Then,*

$$a^{j/i} \leq \frac{j}{i} K^{(i-j)/i} a + \frac{i-j}{i} K^{j/i}, \quad \forall K > 0. \tag{31}$$

Theorem 4. *Assume that $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $f(m,n)$, and $g(m,n)$ are defined as in Theorem 2 and that $i \geq j > 0$ and $i \geq r > 0$. If*

$$\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s,t) C(s,t) \exp\left(B(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} F(\tau,\xi) \right) < 1,$$

$$\forall (m,n) \in \Omega_{M,N}, \tag{32}$$

and $u(m,n)$ satisfies difference inequality (8), then

$$u(m,n)$$

$$\leq \left\{ a(m,n) + \exp\left(B(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} F(s,t) \right) \right.$$

$$\times \left[A(m,n) + C(m,n) \right.$$

$$\times \left(\left(\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s,t) A(s,t) \right. \right.$$

$$\times \exp\left(B(s,t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} F(\tau,\xi) \right) \right)$$

$$\times \left(1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s,t) C(s,t) \right.$$

$$\times \exp\left(B(s,t) \right.$$

$$\left. \left. \left. \times \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} F(\tau,\xi) \right) \right)^{-1} \right) \right] \right\}^{1/i},$$

$$\tag{33}$$

for all $(m,n) \in \Omega_{M,N}$, where

$$A(m,n)$$

$$:= b(m,n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s,t) \left(\frac{j}{i} K_1^{(i-j)/i} a(s,t) + \frac{i-j}{i} K_1^{j/i} \right)$$

$$+ c(m,n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s,t) \left(\frac{r}{i} K_2^{(i-r)/i} a(s,t) \right.$$

$$\left. + \frac{i-r}{i} K_2^{r/i} \right), \tag{34}$$

$$B(m,n) := \frac{jb(m,n)}{i}, \qquad C(m,n) := \frac{rc(m,n)}{i}, \tag{35}$$

$$F(m,n) := f(s,t) K_1^{(i-j)/i}, \qquad G(m,n) := g(s,t) K_2^{(i-r)/i}, \tag{36}$$

and K_1, K_2 are arbitrary constants.

Proof. Define a function $v(m, n)$ by

$$
v(m, n) = b(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s, t) u^j(s, t)
$$

$$
+ c(m, n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s, t) u^r(s, t),
$$

(37)

for all $(m, n) \in \Omega_{M,N}$. Then, from (8), we have

$$
u(m, n) \le (a(m, n) + v(m, n))^{1/i}, \quad \forall (m, n) \in \Omega_{M,N}. \quad (38)
$$

Applying Lemma 3 to (38), we obtain

$$
u^j(m, n) \le (a(m, n) + v(m, n))^{j/i}
$$

$$
\le \frac{j}{i} K_1^{(i-j)/i} (a(m, n) + v(m, n)) + \frac{i-j}{i} K_1^{j/i},
$$

$$
u^r(m, n) \le (a(m, n) + v(m, n))^{r/i}
$$

$$
\le \frac{r}{i} K_2^{(i-r)/i} (a(m, n) + v(m, n)) + \frac{i-r}{i} K_2^{r/i},
$$

(39)

for all $(m, n) \in \Omega_{M,N}$. Substituting (39) into (37), we obtain

$$
v(m, n)
$$

$$
\le b(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s, t)
$$

$$
\times \left(\frac{j}{i} K_1^{(i-j)/i} (a(s, t) + v(s, t)) \right.
$$

$$
\left. + \frac{i-j}{i} K_1^{j/i} \right)
$$

$$
+ c(m, n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s, t)
$$

$$
\times \left(\frac{r}{i} K_2^{(i-r)/i} (a(s, t) + v(s, t)) \right.
$$

$$
\left. + \frac{i-r}{i} K_2^{r/i} \right)
$$

$$
= b(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s, t) \left(\frac{j}{i} K_1^{(i-j)/i} a(s, t) \right.
$$

$$
\left. + \frac{i-j}{i} K_1^{j/i} \right)
$$

$$
+ c(m, n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s, t)
$$

$$
\times \left(\frac{r}{i} K_2^{(i-r)/i} a(s, t) + \frac{i-r}{i} K_2^{r/i} \right)
$$

$$
+ \frac{jb(m, n)}{i} \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f(s, t) K_1^{(i-j)/i} v(s, t)
$$

$$
+ \frac{rc(m, n)}{i} \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} g(s, t) K_2^{(i-r)/i} v(s, t)
$$

$$
= A(m, n) + B(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} F(s, t) v(s, t)
$$

$$
+ C(m, n) \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s, t) v(s, t),
$$

(40)

for all $(m, n) \in \Omega_{M,N}$, where A, B, C and F, G are defined by (34), (35), and (36), respectively. Since $a(m, n)$, $b(m, n)$, and $c(m, n)$ are nonnegative and nondecreasing in both m and n and by (34), (35), and (36), $A(m, n)$, $B(m, n)$, and $C(m, n)$ are also nonnegative and nondecreasing in both m and n. Using Theorem 2, from (40), we obtain

$$
v(m, n)
$$

$$
\le \exp \left(B(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} F(s, t) \right)
$$

$$
\times \left[A(m, n) + C(m, n) \right.
$$

$$
\times \left(\left(\sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s, t) A(s, t) \right. \right.
$$

$$
\left. \times \exp \left(B(s, t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} F(\tau, \xi) \right) \right)
$$

$$
\times \left(1 - \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} G(s, t) C(s, t) \right.
$$

$$
\left. \left. \left. \times \exp \left(B(s, t) \sum_{\tau=m_0}^{s-1} \sum_{\xi=n_0}^{t-1} F(\tau, \xi) \right) \right)^{-1} \right) \right],
$$

(41)

for all $(m, n) \in \Omega_{M,N}$. Substituting (41) into (38), we get our required estimation (33) of unknown function in (8). □

3. Difference Inequality with Weakly Singular

For the reader's convenience, we present some necessary Lemmas.

Lemma 5 (discrete Jensen inequality [47]). *Let* $A_1, A_2, \ldots,$ A_n *be nonnegative real numbers,* $k > 1$ *a real number, and* n *a natural number. Then,*

$$(A_1 + A_2 + \cdots + A_n)^k \le n^{k-1} \left(A_1^k + A_2^k + \cdots + A_n^k \right). \quad (42)$$

Lemma 6 (discrete Hölder inequality [48]). *Let* a_i, b_i $(i = 1, 2, \ldots, n)$ *be nonnegative real numbers and* p, q *positive numbers such that* $(1/q) + (1/p) = 1$. *Then,*

$$\sum_{i=0}^{n-1} a_i b_i \le \left(\sum_{i=0}^{n-1} a_i^p \right)^{1/p} \left(\sum_{i=0}^{n-1} b_i^q \right)^{1/q}. \quad (43)$$

Lemma 7 (see [15, 49]). *Let* $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, *and* $\sup_{s \in \mathbb{N}, 0 \le s \le n-1} \{\tau_s, s \in \mathbb{N}\} = \tau$. *If* $\beta \in (0.5, 1)$, $\gamma > 1.5 - \beta$, *and* $p = 1/\beta$, *then*

$$\sum_{s=0}^{n-1} (t_n - t_s)^{p(\beta-1)} t_s^{p(\gamma-1)} \tau_s \quad (44)$$

$$\le t_n^\theta \mathscr{B} \left[p(\gamma - 1) + 1, p(\beta - 1) + 1 \right],$$

where $\theta = p(\beta + \gamma - 2) + 1 > 0$ *and* $\mathscr{B}(\xi, \eta) := \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$ *is the well-known* $\mathscr{B}$-*function.*

Now, we consider the weakly singular difference inequality (9).

Theorem 8. *Let* $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s \in \mathbb{N}, 0 \le s \le n-1} \{\tau_s, s \in \mathbb{N}\} = \tau$, $\beta \in (0.5, 1)$, *and* $\gamma > 1.5 - \beta$. *Assume that* $i \ge j > 0$, $i \ge r > 0$, $u(n), a(n), b(n), c(n), f(n)$, *and* $g(n)$ *are nonnegative functions on* $\mathbb{N}_0$ *and* $a(n), b(n)$, *and* $c(n)$ *are nondecreasing. If*

$$\sum_{s=0}^{N-1} \widetilde{G}(s) \widetilde{C}(s) \exp \left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) < 1, \quad n \in \mathbb{N}_0, \ n < N, \quad (45)$$

and $u(n)$ *satisfies* (9), *then*

$$u(n)$$
$$\le \left\{ a(n) + \exp \left(\widetilde{B}(n) \sum_{s=0}^{n-1} \widetilde{F}(s) \right) \right.$$
$$\times \left[\widetilde{A}(n) + \widetilde{C}(n) \right.$$
$$\times \left(\left(\sum_{s=0}^{N-1} \widetilde{G}(s) \widetilde{A}(s) \right. \right.$$
$$\times \exp \left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \right)$$
$$\times \left(1 - \sum_{s=0}^{N-1} \widetilde{G}(s) \widetilde{C}(s) \right.$$
$$\left. \left. \left. \left. \times \exp \left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \right)^{-1} \right) \right] \right\}^{1/i},$$
$$n \in \mathbb{N}_0, \ n < N, \quad (46)$$

where

$$\widetilde{A}(n) := \widetilde{b}(n) \sum_{s=0}^{n-1} f^q(s) \left(\frac{j}{i} K_1^{(i-j)/i} \widetilde{a}(s) + \frac{i-j}{i} K_1^{j/i} \right)$$
$$+ \widetilde{c}(n) \sum_{s=n_0}^{N-1} g^q(s) \left(\frac{r}{i} K_2^{(i-r)/i} \widetilde{a}(s) + \frac{i-r}{i} K_2^{r/i} \right),$$

$$\widetilde{B}(n) := \frac{j\widetilde{b}(n)}{i}, \qquad \widetilde{C}(n) := \frac{r\widetilde{c}(n)}{i},$$

$$\widetilde{F}(n) := f^q(n) K_1^{(i-j)/i}, \qquad \widetilde{G}(n) := g^q(n) K_2^{(i-r)/i},$$

$$\widetilde{a}(n) := 3^{q-1} a^q(n),$$

$$\widetilde{b}(n) := 3^{q-1} b^q(n) \tau$$
$$\times \left(t_n^\theta \mathscr{B} \left[p(\gamma-1) + 1, p(\beta-1) + 1 \right] \right)^{q/p},$$

$$\widetilde{c}(n) := 3^{q-1} c^q(n) \tau$$
$$\times \left(t_N^\theta \mathscr{B} \left[p(\gamma-1) + 1, p(\beta-1) + 1 \right] \right)^{q/p},$$

$$\theta := p(\beta + \gamma - 2) + 1, \quad (47)$$

and $p = 1/\beta$, $q = 1/(1-\beta)$, *and* K_1, K_2 *are arbitrary constants.*

Proof. Applying Lemma 6 with $p = 1/\beta$, $q = 1/(1-\beta)$ to (8), we obtain that

$$u^i(n) \le a(n) + b(n) \tau^{(p-1)/p}$$
$$\times \left(\sum_{s=n_0}^{n-1} (t_n - t_s)^{p(\beta-1)} t_s^{p(\gamma-1)} \tau_s \right)^{1/p}$$
$$\times \left(\sum_{s=n_0}^{n-1} f^q(s) u^{qj}(s) \right)^{1/q}$$
$$+ c(n) \tau^{(p-1)/p} \quad (48)$$
$$\times \left(\sum_{s=n_0}^{N-1} (t_N - t_s)^{p(\beta-1)} t_s^{p(\gamma-1)} \tau_s \right)^{1/p}$$
$$\times \left(\sum_{s=n_0}^{n-1} g^q(s) u^{qr}(s) \right)^{1/q},$$

for all $n \in \mathbb{N}_0, n < N$, where $\tau_s < \tau$ is used. Applying Lemma 5 to (48), we have

$$u^i(n) \le a(n) + b(n) \tau^{(p-1)/p}$$
$$\times \left(t_n^\theta \mathscr{B} \left[p(\gamma-1) + 1, p(\beta-1) + 1 \right] \right)^{1/p}$$
$$\times \left(\sum_{s=n_0}^{n-1} f^q(s) u^{qj}(s) \right)^{1/q}$$
$$+ c(n) \tau^{(p-1)/p} \quad (49)$$
$$\times \left(t_N^\theta \mathscr{B} \left[p(\gamma-1) + 1, p(\beta-1) + 1 \right] \right)^{1/p}$$
$$\times \left(\sum_{s=n_0}^{n-1} g^q(s) u^{qr}(s) \right)^{1/q},$$

for all $n \in \mathbb{N}_0$, $n < N$. By discrete Jensen inequality (42) with $n = 2$, $k = q$, from (49), we obtain that

$$
\begin{aligned}
u^{qi}(n) &\leq 3^{q-1} a^q(n) + 3^{q-1} b^q(n) \tau^{q(p-1)/p} \\
&\quad \times \left(t_n^\theta \mathscr{B} \left[p(\gamma - 1) + 1, p(\beta - 1) + 1 \right] \right)^{q/p} \\
&\quad \times \sum_{s=n_0}^{n-1} f^q(s) u^{qj}(s) + 3^{q-1} c^q(n) \tau^{q(p-1)/p} \\
&\quad \times \left(t_N^\theta \mathscr{B} \left[p(\gamma - 1) + 1, p(\beta - 1) + 1 \right] \right)^{q/p} \\
&\quad \times \sum_{s=n_0}^{n-1} g^q(s) u^{qr}(s) \\
&= 3^{q-1} a^q(n) + 3^{q-1} b^q(n) \tau \\
&\quad \times \left(t_n^\theta \mathscr{B} \left[p(\gamma - 1) + 1, p(\beta - 1) + 1 \right] \right)^{q/p} \\
&\quad \times \sum_{s=n_0}^{n-1} f^q(s) u^{qj}(s) + 3^{q-1} c^q(n) \tau \\
&\quad \times \left(t_N^\theta \mathscr{B} \left[p(\gamma - 1) + 1, p(\beta - 1) + 1 \right] \right)^{q/p} \\
&\quad \times \sum_{s=n_0}^{n-1} g^q(s) u^{qr}(s) \\
&= \tilde{a}(n) + \tilde{b}(n) \sum_{s=n_0}^{n-1} f^q(s) u^{qj}(s) \\
&\quad + \tilde{c}(n) \sum_{s=n_0}^{n-1} g^q(s) u^{qr}(s), \\
&\qquad\qquad n \in \mathbb{N}_0, \ n < N.
\end{aligned}
\tag{50}
$$

Applying Theorem 4 to (50), we have

$$
\begin{aligned}
&u(n) \\
&\leq \Bigg\{ a(n) + \exp\left(\tilde{B}(n) \sum_{s=0}^{n-1} \tilde{F}(s) \right) \\
&\quad \times \Bigg[\tilde{A}(n) + \tilde{C}(n) \\
&\quad \times \Bigg(\left(\sum_{s=0}^{N-1} \tilde{G}(s) \tilde{A}(s) \right. \\
&\qquad \times \exp\left(\tilde{B}(s) \sum_{\tau=0}^{s-1} \tilde{F}(\tau) \right) \Bigg) \\
&\qquad \times \left(1 - \sum_{s=0}^{N-1} \tilde{G}(s) \tilde{C}(s) \right. \\
&\qquad \left. \left. \times \exp\left(\tilde{B}(s) \sum_{\tau=0}^{s-1} \tilde{F}(\tau) \right) \right)^{-1} \right) \Bigg] \Bigg\}^{1/i}, \\
&\qquad\qquad\qquad n \in \mathbb{N}_0, \ n < N.
\end{aligned}
\tag{51}
$$

This is our required estimation (46) of unknown function in (9). $\qquad\square$

4. Applications

In this section, we apply our results to discuss the boundedness of solutions of an iterative difference equation with a weakly singular kernel.

Example 9. Suppose that $u(n)$ satisfies the difference equation

$$
\begin{aligned}
x^3(n) &= a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{-0.3} t_s^{-0.1} \tau_s f(s) x^2(s) \\
&\quad + c(n) \sum_{s=0}^{N-1} (t_N - t_s)^{-0.3} t_s^{-0.1} \tau_s g(s) x(s),
\end{aligned}
\tag{52}
$$

where $t_0 = 0$, $\tau_s = t_{s+1} - t_s > 0$, $\sup_{s \in \mathbb{N}, 0 \leq s \leq n-1}\{\tau_s, s \in \mathbb{N}\} = \tau$, $u(n)$, $a(n)$, $b(n)$, $c(n)$, $f(n)$, and $g(n)$ are nonnegative functions on $\mathbb{N}_0$, and $a(n)$, $b(n)$, and $c(n)$ are nondecreasing. From (52), we have

$$
\begin{aligned}
|x(n)|^3 &\leq a(n) + b(n) \sum_{s=0}^{n-1} (t_n - t_s)^{-0.3} t_s^{-0.1} \tau_s f(s) |x(s)|^2 \\
&\quad + c(n) \sum_{s=0}^{N-1} (t_N - t_s)^{-0.3} t_s^{-0.1} \tau_s g(s) |x(s)|.
\end{aligned}
\tag{53}
$$

Let $p = 10/7$, $q = 10/3$, and K_1, K_2 are arbitrary constants, and

$$
\theta := \frac{3}{7}, \qquad \tilde{a}(n) := 3^{7/3} a^{10/3}(n),
$$

$$
\tilde{b}(n) := 3^{7/3} b^{10/3}(n) \tau \left(t_n^\theta \mathscr{B} \left[\frac{6}{7}, \frac{4}{7} \right] \right)^{7/3},
$$

$$
\tilde{c}(n) := 3^{7/3} c^{10/3}(n) \tau \left(t_N^\theta \mathscr{B} \left[\frac{6}{7}, \frac{4}{7} \right] \right)^{7/3},
$$

$$
\begin{aligned}
\tilde{A}(n) &:= \tilde{b}(n) \sum_{s=0}^{n-1} f^{10/3}(s) \left(\frac{2}{3} K_1^{1/3} \tilde{a}(s) + \frac{1}{3} K_1^{2/3} \right) \\
&\quad + \tilde{c}(n) \sum_{s=n_0}^{N-1} g^{10/3}(s) \left(\frac{1}{3} K_2^{2/3} \tilde{a}(s) + \frac{2}{3} K_2^{1/3} \right),
\end{aligned}
\tag{54}
$$

$$
\tilde{B}(n) := \frac{2\tilde{b}(n)}{3}, \qquad \tilde{C}(n) := \frac{\tilde{c}(n)}{3},
$$

$$
\tilde{F}(n) := f^{10/3}(n) K_1^{1/3}, \qquad \tilde{G}(n) := g^{10/3}(n) K_2^{2/3}.
$$

If

$$
\sum_{s=0}^{N-1} \tilde{G}(s) \tilde{C}(s) \exp\left(\tilde{B}(s) \sum_{\tau=0}^{s-1} \tilde{F}(\tau) \right) < 1,
\tag{55}
$$

$$
n \in \mathbb{N}_0, \ n < N.
$$

Applying Theorem 8 to (53), we obtain the estimation of the solutions of difference equation (52)

$$
\begin{aligned}
|x(n)| \\
\leq \Bigg\{ &a(n) + \exp\left(\widetilde{B}(n) \sum_{s=0}^{n-1} \widetilde{F}(s) \right) \\
&\times \Bigg[\widetilde{A}(n) + \widetilde{C}(n) \\
&\quad \times \Bigg(\Bigg(\sum_{s=0}^{N-1} \widetilde{G}(s)\,\widetilde{A}(s) \\
&\qquad \times \exp\left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \Bigg) \\
&\quad \times \Bigg(1 - \sum_{s=0}^{N-1} \widetilde{G}(s)\,\widetilde{C}(s) \\
&\qquad \times \exp\left(\widetilde{B}(s) \sum_{\tau=0}^{s-1} \widetilde{F}(\tau) \right) \Bigg)^{-1} \Bigg) \Bigg] \Bigg\}^{1/i}, \\
&\qquad\qquad\qquad\qquad n \in \mathbb{N}_0, \ n < N.
\end{aligned}
\tag{56}
$$

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

References

[1] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Annals of Mathematics*, vol. 20, no. 4, pp. 292–296, 1919.

[2] R. Bellman, "The stability of solutions of linear differential equations," *Duke Mathematical Journal*, vol. 10, pp. 643–647, 1943.

[3] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[4] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1992.

[5] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, vol. 197 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1998.

[6] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.

[7] W. Cheung, "Some new nonlinear inequalities and applications to boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 9, pp. 2112–2128, 2006.

[8] W. S. Wang, "A generalized retarded Gronwall-like inequality in two variables and applications to BVP," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 144–154, 2007.

[9] A. Abdeldaim and M. Yakout, "On some new integral inequalities of Gronwall-Bellman-Pachpatte type," *Applied Mathematics and Computation*, vol. 217, no. 20, pp. 7887–7899, 2011.

[10] Y. S. Lu, W. S. Wang, X. L. Zhou, and Y. Huang, "Generalized nonlinear Volterra-Fredholm type integral inequality with two variables," *Journal of Applied Mathematics*, vol. 2014, Article ID 359280, 14 pages, 2014.

[11] K. Cheng and C. Guo, "New explicit bounds on Gamidov type integral inequalities for functions in two variables and their applications," *Abstract and Applied Analysis*, vol. 2014, Article ID 539701, 9 pages, 2014.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, New York, NY, USA, 1981.

[13] M. Medveď, "A new approach to an analysis of Henry type integral inequalities and their Bihari type versions," *Journal of Mathematical Analysis and Applications*, vol. 214, no. 2, pp. 349–366, 1997.

[14] M. Medveď, "Nonlinear singular integral inequalities for functions in two and n independent variables," *Journal of Inequalities and Applications*, vol. 5, no. 3, pp. 287–308, 2000.

[15] Q. H. Ma and E. H. Yang, "Estimates on solutions of some weakly singular Volterra integral inequalities," *Acta Mathematicae Applicatae Sinica*, vol. 25, no. 3, pp. 505–515, 2002.

[16] Y. Wu and S. Deng, "Generalization on some weakly singular Volterra integral inequalities," *Journal of Sichuan University (Natural Science Edition)*, vol. 41, no. 3, pp. 472–479, 2004.

[17] K. M. Furati and N. Tatar, "Behavior of solutions for a weighted Cauchy-type fractional differential problem," *Journal of Fractional Calculus*, vol. 28, pp. 23–42, 2005.

[18] H. Ye, J. Gao, and Y. Ding, "A generalized Gronwall inequality and its application to a fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1075–1081, 2007.

[19] W. Cheung, Q. H. Ma, and S. Tseng, "Some new nonlinear weakly singular integral inequalities of Wendroff type with applications," *Journal of Inequalities and Applications*, vol. 2008, Article ID 909156, 12 pages, 2008.

[20] Q. Ma and J. Pečarić, "Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 894–905, 2008.

[21] S. Deng and C. Prather, "Generalization of an impulsive nonlinear singular Gronwall-Bihari inequality with delay," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 34, 11 pages, 2008.

[22] Y. Wu, "A new type of weakly singular Volterra integral inequalities," *Acta Mathematicae Applicatae Sinica*, vol. 31, no. 4, pp. 584–591, 2008.

[23] S. Mazouzi and N.-E. Tatar, "New bounds for solutions of a singular integro-differential inequality," *Mathematical Inequalities & Applications*, vol. 13, no. 2, pp. 427–435, 2010.

[24] H. Wang and K. Zheng, "Some nonlinear weakly singular integral inequalities with two variables and applications," *Journal of Inequalities and Applications*, vol. 2010, Article ID 345701, 12 pages, 2010.

[25] H. Ye and J. Gao, "Henry-Gronwall type retarded integral inequalities and their applications to fractional differential equations with delay," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4152–4160, 2011.

[26] Q. H. Ma and E. H. Yang, "Bounds on solutions to some nonlinear Volterra integral inequalities with weakly singular kernels," *Annals of Differential Equations*, vol. 27, no. 3, pp. 283–292, 2011.

[27] K. Zheng, "Bounds on some new weakly singular Wendroff-type integral inequalities and applications," *Journal of Inequalities and Applications*, vol. 2013, article 159, 2013.

[28] K. Cheng, C. Guo, and M. Tang, "Some nonlinear Gronwall-Bellman-GAMidov integral inequalities and their weakly singular analogues with applications," *Abstract and Applied Analysis*, vol. 2014, Article ID 562691, 9 pages, 2014.

[29] T. E. Hull and W. A. J. Luxemburg, "Numerical methods and existence theorems for ordinary differential equations," *Numerische Mathematik*, vol. 2, pp. 30–41, 1960.

[30] D. Willett and J. S. W. Wong, "On the discrete analogues of some generalizations of Gronwall's inequality," *Monatshefte für Mathematik*, vol. 69, no. 4, pp. 362–367, 1965.

[31] S. Sugiyama, "On the stability problems of difference equations," *Bulletin of Science and Engineering Research Laboratory, Waseda University*, vol. 45, pp. 140–144, 1969.

[32] B. G. Pachpatte and S. G. Deo, "Stability of discrete time systems with retarded argument," *Utilitas Mathematica*, vol. 4, pp. 15–33, 1973.

[33] B. G. Pachpatte, "Finite difference inequalities and discrete time control systems," *Indian Journal of Pure and Applied Mathematics*, vol. 9, no. 12, pp. 1282–1290, 1978.

[34] P. Y. H. Pang and R. P. Agarwal, "On an integral inequality and its discrete analogue," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 2, pp. 569–577, 1995.

[35] B. G. Pachpatte, "On some new inequalities related to certain inequalities in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 128–144, 1995.

[36] B. G. Pachpatte, "A note on certain integral inequality," *Tamkang Journal of Mathematics*, vol. 33, no. 4, pp. 353–358, 2002.

[37] W. S. Cheung and J. Ren, "Discrete non-linear inequalities and applications to boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 708–724, 2006.

[38] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, vol. 205 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[39] F. C. Jiang and F. W. Meng, "Explicit bounds on some new nonlinear integral inequalities with delay," *Journal of Computational and Applied Mathematics*, vol. 205, no. 1, pp. 479–486, 2007.

[40] Q. Ma and W. Cheung, "Some new nonlinear difference inequalities and their applications," *Journal of Computational and Applied Mathematics*, vol. 202, no. 2, pp. 339–351, 2007.

[41] W. Sheng and W. N. Li, "Bounds on certain nonlinear discrete inequalities," *Journal of Mathematical Inequalities*, vol. 2, no. 2, pp. 279–286, 2008.

[42] W. Wang, "A generalized sum-difference inequality and applications to partial difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 695495, 12 pages, 2008.

[43] W. S. Wang, "Estimation on certain nonlinear discrete inequality and applications to boundary value problem," *Advances in Difference Equations*, vol. 2009, Article ID 708587, 8 pages, 2009.

[44] S. Deng, "Nonlinear discrete inequalities with two variables and their applications," *Applied Mathematics and Computation*, vol. 217, no. 5, pp. 2217–2225, 2010.

[45] Q. H. Ma, "Estimates on some power nonlinear Volterra-Fredholm type discrete inequalities and their applications," *Journal of Computational and Applied Mathematics*, vol. 233, no. 9, pp. 2170–2180, 2010.

[46] H. Zhou, D. Huang, W. Wang, and J. Xu, "Some new difference inequalities and an application to discrete-time control systems," *Journal of Applied Mathematics*, vol. 2012, Article ID 214609, 14 pages, 2012.

[47] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, University of Katowice, Katowice, Poland, 1985.

[48] E. H. Yang, Q. H. Ma, and M. C. Tan, "Discrete analogues of a new class of nonlinear Volterra singular integral inequalities," *Journal of Jinan University*, vol. 28, no. 1, pp. 1–6, 2007.

[49] K. Zheng, H. Wang, and C. Guo, "On nonlinear discrete weakly singular inequalities and applications to Volterra-type difference equations," *Advances in Difference Equations*, vol. 2013, article 239, 2013.

[50] C. Huang, W. S. Wang, and X. Zhou, "A class of iterative nonlinear difference inequality with weakly singularity," *Journal of Applied Mathematics*, vol. 2014, Article ID 236965, 9 pages, 2014.