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Research Article

Shrinking Projection Methods for Split Common Fixed-Point Problems in Hilbert Spaces

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Inspired by Moudafi (2011) and Takahashi et al. (2008), we present the shrinking projection method for the split common fixed-point problem in Hilbert spaces, and we obtain the strong convergence theorem. As a special case, the split feasibility problem is also considered.

1. Introduction

Let C and Q be nonempty closed convex sets in real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find

$$x \in C, \quad \text{such that } Ax \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. We use Φ to denote the solution set of the SFP (1). The SFP in finite-dimensional Hilbert space was first introduced by Censor and Elfving [1]. In 2010, Xu [2] considered the SFP in the setting of infinite-dimensional Hilbert space. The SFP has received much attention due to its wide applications in signal processing, image reconstruction, intensity-modulated radiation therapy, and so on (see [3–6]). Several iterative methods can be used to solve the SFP (1). Censor and Elfving [1] constructed the iterative process which involves the computation of the inverse of a matrix. A more popular algorithm that solves the SFP is the CQ algorithm of Byrne [3, 4]; that is, let x_0 be an arbitrary point in H_1:

$$x_{n+1} = P_C \left(I - \gamma A^* \left(I - P_Q\right) A\right) x_n, \tag{2}$$

where $\gamma > 0$ is a parameter and P_C and P_Q are metric projections onto C and Q, respectively.

Let K be a nonempty closed convex subset of a real Hilbert space H and let $T : K \to K$ be a mapping. We denote by $\mathrm{Fix}(T)$ the fixed-point set of T; that is, $\mathrm{Fix}(T) = \{x \in K : Tx = x\}$. A mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ forall $x, y \in K$. A mapping $T : K \to K$ is quasinonexpansive if $\mathrm{Fix}(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ forall $x \in K$ and $y \in \mathrm{Fix}(T)$. It is known that the fixed-point set of a quasinonexpansive mapping is closed and convex (see [7, 8]). There are some quasinonexpansive mappings which are not nonexpansive (see [9–11]). For example, the level set of a continuous convex function is characterized as the fixed-point set of a nonlinear mapping called the subgradient projection, which is not nonexpansive but quasinonexpansive.

Now we focus our attention on the following two-operator split common fixed-point problem (SCFP):

$$\text{find } x^* \in C, \quad \text{such that } Ax^* \in Q, \tag{3}$$

where $A : H_1 \to H_2$ is a bounded linear operator and $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasinonexpansive mappings with $\mathrm{Fix}(U) = C$ and $\mathrm{Fix}(T) = Q$. The solution set of the SCFP (3) is denoted by

$$\Gamma = \left\{x^* \in C : Ax^* \in Q\right\}. \tag{4}$$

As far as we know, the SCFP is introduced by Censor and Segal [12]. By taking $U = P_C$ and $T = P_Q$, the SCFP reduces to the SFP. Hence, the SCFP is a generalization of the SFP. Moudafi [13] considered the following algorithm for the

SCFP: let $x_0 \in H_1$ be arbitrary, $u_k = x_k - \gamma\beta A^*(I-T)Ax_k$ and

$$x_{k+1} = (1-\alpha_k) u_k + \alpha_k U(u_k), \tag{5}$$

where $\beta \in (0,1)$, $\alpha_k \in (0,1)$, and $\gamma \in (0, 1/\lambda\beta)$, with λ being the spectral radius of the operator A^*A. He obtained the weak convergence of the algorithm (5).

In 2008, Takahashi et al. [14] developed the shrinking projection method for the nonexpansive mapping. Let T be a nonexpansive mapping of K into itself such that $\mathrm{Fix}(T) \neq \emptyset$. Let $x_0 \in H$, $C_1 = K$ and $u_1 = P_{C_1}x_0$;

$$\begin{aligned} y_n &= \alpha_n u_n + (1-\alpha_n) T u_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} &= P_{C_{n+1}} x_0, \end{aligned} \tag{6}$$

where $0 \le \alpha_n \le a < 1$. They proved that the sequence $\{u_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}x_0$.

Motivated by the above results, especially by Moudafi [13] and Takahashi et al. [14], in this paper, we present the shrinking projection methods for the split common fixed-point problems. As a special case, the split feasibility problem is also discussed.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. For any $x \in H$, there exists a unique point $P_K x \in K$ such that

$$\|x - P_K x\| \le \|x - y\| \quad \forall y \in K, \tag{7}$$

where K is a nonempty closed convex subset of a real Hilbert space H. The mapping P_K is called the metric projection of H onto K. Note that P_K is a nonexpansive mapping. For $x \in H$ and $z \in K$, we have

$$z = P_K x \Longleftrightarrow \langle x - z, y - z\rangle \le 0 \quad \text{for every } y \in K. \tag{8}$$

We say that a mapping $T : K \to K$ is demiclosed at zero if for any sequence $\{x_n\} \subset K$ which converges weakly to x, the strong convergence of the sequence $\{Tx_n\}$ to zero implies that $Tx = 0$. It is well known that $I - T$ is demiclosed whenever T is nonexpansive. In fact, this property is satisfied for more general mappings (see [15, 16]).

We will use the following notations:

(1) $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x;

(2) $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;

(3) $\omega_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Here are two useful lemmas.

Lemma 1. *Let $x, y \in H$ and let $\lambda \in \mathbb{R}$. One has*

$$\begin{aligned} &\|\lambda x + (1-\lambda) y\|^2 \\ &\quad = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \end{aligned} \tag{9}$$

Lemma 2 (see [17]). *Let K be a closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ satisfies the following conditions:*

(1) $\omega_\omega(x_n) \subset K$,

(2) $\|x_n - u\| \le \|u - q\|$ for all $n \in \mathbb{N}$,

then one has $x_n \to q$.

3. Shrinking Projection Methods

Now we are in a position to give the shrinking projection method for split common fixed-point problem (3).

Theorem 3. *Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two quasinonexpansive mappings with $\mathrm{Fix}(U) = C$ and $\mathrm{Fix}(T) = Q$. Suppose that $I - U$ and $I - T$ are demiclosed at zero and solution set Γ of the SCFP (3) is nonempty. For $u \in H_1$ chosen arbitrarily, $C_1 = H_1$, $h_1 = P_{C_1}u$, define a sequence $\{h_n\}$ by the following algorithm:*

$$\begin{aligned} w_n &= h_n - \gamma A^*(I-T)Ah_n, \\ y_n &= \alpha_n w_n + (1-\alpha_n) U w_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|h_n - z\|\}, \\ h_{n+1} &= P_{C_{n+1}} u. \end{aligned} \tag{10}$$

If the following are satisfied:

(1) $\{\alpha_n\} \subset (0,1)$ and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$,

(2) $0 < \gamma < (1/\lambda_{AA^*})$, where λ_{AA^*} denotes the spectral radius of the operator AA^*,

then the sequence $\{h_n\}$ converges strongly to $P_\Gamma u$.

Proof. We first show that $\Gamma \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that Γ is contained in $C_1 = H_1$. Suppose that $\Gamma \subset C_k$ for some $k \in \mathbb{N}$. We have, for any $p \in \Gamma \subset C_k$,

$$\begin{aligned} &\|y_k - p\|^2 \\ &\quad = \|\alpha_k w_k + (1-\alpha_k) U w_k - p\|^2 \\ &\quad \le \alpha_k\|w_k - p\|^2 + (1-\alpha_k)\|w_k - p\|^2 \\ &\quad = \|h_k - \gamma A^*(I-T)Ah_k - p\|^2 \\ &\quad = \|h_k - p\|^2 - 2\gamma\langle A^*(I-T)Ah_k, h_k - p\rangle \\ &\qquad + \gamma^2\|A^*(I-T)Ah_k\|^2 \\ &\quad = \|h_k - p\|^2 + 2\gamma\langle TAh_k - Ah_k, Ah_k - Ap\rangle \end{aligned}$$

$$\begin{aligned}
&+\gamma^2\langle(I-T)Ah_k, AA^*(I-T)Ah_k\rangle\\
&\le \|h_k-p\|^2+\gamma^2\lambda_{AA^*}\|(I-T)Ah_k\|^2\\
&\quad+\gamma\left[\|TAh_k-Ap\|^2-\|TAh_k-Ah_k\|^2-\|Ah_k-Ap\|^2\right]\\
&\le \|h_k-p\|^2+\gamma^2\lambda_{AA^*}\|(I-T)Ah_k\|^2\\
&\quad-\gamma\|TAh_k-Ah_k\|^2\\
&=\|h_k-p\|^2+\gamma(\gamma\lambda_{AA^*}-1)\|(I-T)Ah_k\|^2\\
&\le\|h_k-p\|^2.
\end{aligned} \tag{11}$$

It follows that $p \in C_{k+1}$. Thus, we get $\Gamma \subset C_n$ for all $n \in \mathbb{N}$.

Next we show that C_n is closed and convex for all $n \in \mathbb{N}$. The set $C_1 = H_1$ is obviously closed and convex. Suppose that C_k is closed and convex. We see that C_{k+1} is closed and convex since $\|y_n - z\| \le \|h_n - z\|$ is equivalent to

$$\|y_n\|^2-\|h_n\|^2-2\langle y_n-h_n, z\rangle\le 0. \tag{12}$$

It follows that C_n is closed and convex for all $n \in \mathbb{N}$. Therefore, we obtain that the sequence $\{h_n\}$ is well defined.

From $h_n = P_{C_n}u$, we have

$$\langle u-h_n, h_n-y\rangle\ge 0\quad \forall y\in C_n. \tag{13}$$

Recalling that $\Gamma \subset C_n$, one has

$$\langle u-h_n, h_n-p\rangle\ge 0\quad \forall p\in \Gamma. \tag{14}$$

Hence,

$$\begin{aligned}
0&\le\langle u-h_n,h_n-p\rangle\\
&=\langle u-h_n,h_n-u+u-p\rangle\\
&\le-\|u-h_n\|^2+\|u-h_n\|\|u-p\|.
\end{aligned} \tag{15}$$

This implies that

$$\|u-h_n\|\le\|u-p\|, \tag{16}$$

which yields that $\{h_n\}$ is bounded.

From $h_n = P_{C_n}u$ and $h_{n+1} = P_{C_{n+1}}u \in C_{n+1} \subset C_n$, we get

$$\begin{aligned}
0&\le\langle u-h_n,h_n-h_{n+1}\rangle\\
&\le-\|u-h_n\|^2+\|u-h_n\|\|u-h_{n+1}\|,
\end{aligned} \tag{17}$$

which gives that

$$\|u-h_n\|\le\|u-h_{n+1}\|. \tag{18}$$

Hence,

$$\text{the limit } \lim_{n\to\infty}\|u-h_n\| \text{ exists.} \tag{19}$$

It follows from (17) that

$$\begin{aligned}
&\|h_n-h_{n+1}\|^2\\
&\quad=\|h_n-u\|^2+2\langle h_n-u,u-h_{n+1}\rangle\\
&\qquad+\|u-h_{n+1}\|^2\\
&\quad=\|h_n-u\|^2+2\langle h_n-u,u-h_n+h_n-h_{n+1}\rangle\\
&\qquad+\|u-h_{n+1}\|^2\\
&\quad=-\|h_n-u\|^2+2\langle h_n-u,h_n-h_{n+1}\rangle\\
&\qquad+\|u-h_{n+1}\|^2\\
&\quad\le-\|h_n-u\|^2+\|u-h_{n+1}\|^2.
\end{aligned} \tag{20}$$

Thus, we get

$$\lim_{n\to\infty}\|h_n-h_{n+1}\|=0. \tag{21}$$

The fact that $h_{n+1} = P_{C_{n+1}}u \in C_{n+1}$ gives

$$\|y_n-h_{n+1}\|\le\|h_n-h_{n+1}\|\longrightarrow 0. \tag{22}$$

The expressions (21) and (22) yield

$$\lim_{n\to\infty}\|y_n-h_n\|=0. \tag{23}$$

We will prove that $\omega_\omega(h_n) \subset \Gamma$. Without loss of generality, we assume that $h_n \rightharpoonup h^*$. It follows from (11) that

$$\begin{aligned}
&\gamma(1-\gamma\lambda_{AA^*})\|(I-T)Ah_n\|^2\\
&\quad\le\|h_n-p\|^2-\|y_n-p\|^2\\
&\quad\le(\|h_n-p\|+\|y_n-p\|)\|h_n-y_n\|.
\end{aligned} \tag{24}$$

This together with (23) implies that

$$\lim_{n\to\infty}\|(I-T)Ah_n\|=0. \tag{25}$$

We have $Ah^* \in \mathrm{Fix}(T) = Q$ since $I - T$ is demiclosed at zero.

Using (**??**) and (25), we get $w_n \rightharpoonup h^*$. For any $p \in \Gamma$, one has

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad=\|\alpha_nw_n+(1-\alpha_n)Uw_n-p\|^2\\
&\quad=\alpha_n\|w_n-p\|^2+(1-\alpha_n)\|Uw_n-p\|^2\\
&\qquad-\alpha_n(1-\alpha_n)\|Uw_n-w_n\|^2\\
&\quad\le\|w_n-p\|^2-\alpha_n(1-\alpha_n)\|Uw_n-w_n\|^2\\
&\quad\le\|h_n-p\|^2-\alpha_n(1-\alpha_n)\|Uw_n-w_n\|^2,
\end{aligned} \tag{26}$$

which implies that

$$\begin{aligned}\alpha_n(1-\alpha_n)\|Uw_n-w_n\|^2 &\le \|h_n-p\|^2-\|y_n-p\|^2\\ &\le (\|h_n-p\|+\|y_n-p\|)\|h_n-y_n\|.\end{aligned} \tag{27}$$

Therefore, one has

$$\lim_{n\to\infty}\|Uw_n-w_n\|=0. \tag{28}$$

It follows that $h^* \in \mathrm{Fix}(U) = C$ since $I-U$ is demiclosed at zero. Thus, we have obtained $\omega_\omega(h_n) \in \Gamma$. According to Lemma 2, we see that $h_n \to P_\Gamma u$. □

By Theorem 3, we immediately obtain the shrinking projection method for the split feasibility problem.

Theorem 4. *Let H_1 and H_2 be real Hilbert spaces and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A: H_1 \to H_2$ be a bounded linear operator. Suppose that the solution set Φ of the SFP (1) is nonempty. For $u \in H_1$ chosen arbitrarily, $C_1 = H_1$, $h_1 = P_{C_1}u$, define a sequence $\{h_n\}$ by the following algorithm:*

$$\begin{aligned} w_n &= h_n-\gamma A^*(I-P_Q)Ah_n,\\ y_n &= \alpha_n w_n+(1-\alpha_n)P_C w_n,\\ C_{n+1} &= \{z\in C_n : \|y_n-z\|\le\|h_n-z\|\},\\ h_{n+1} &= P_{C_{n+1}}u.\end{aligned} \tag{29}$$

If the following are satisfied:

(1) $\{\alpha_n\} \subset (0,1)$ *and* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$,

(2) $0 < \gamma < 1/\lambda_{AA^*}$, *where* λ_{AA^*} *denotes the spectral radius of the operator* AA^*,

then the sequence $\{h_n\}$ converges strongly to $P_\Phi u$.

Remark 5. Letting $u = 0$ in Theorems 3 and 4, we obtain the shrinking projection methods for minimum-norm solutions of corresponding problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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