Research Article

On Ground States of Discrete $p(k)$-Laplacian Systems in Generalized Orlicz Sequence Spaces

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Using the critical point theory, we establish sufficient conditions on the existence of ground states for discrete $p(k)$-Laplacian systems. Our results considerably generalize some existing ones.

1. Introduction and Main Results

The aim of this paper is to study the existence of ground state for discrete $p(k)$-Laplacian system

$$\Delta \left[a(k) \left| \Delta u(k-1) \right|^{p(k)-2} \Delta u(k-1) \right]$$
$$- q(k) \left| u(k) \right|^{p(k)-2} u(k) + f(k, u(k)) = 0, \tag{1}$$

where $p(k) > 1$, for all $k \in Z$, $a(k)$ and $q(k)$ are real valued on Z. $f : Z \times R \to R$ is continuous in the second variable. Moreover, Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$.

We may think of (1) as being a discrete analogue of the following differential system:

$$\frac{d}{dt} \left(a(t) \left| \dot{u}(t) \right|^{p(t)-2} \dot{u}(t) \right)$$
$$- q(t) \left| u(t) \right|^{p(t)-2} u(t) + f(t, u(t)) = 0. \tag{2}$$

For the case $p(t) \equiv p$, system (2) is a p-Laplacian system, which has been widely studied; to mention a few, see [1, 2]. Even for the special case $p = 2$, system (2) can be regarded as the more general form of Emden-Fowler equation appearing in the study of astrophysics, fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, and chemically reacting system in terms of various special forms

of $f(t, u(t))$ (see, e.g., [3]). The more general differential operator (2), namely, the so-called $p(t)$-Laplacian, has been studied by Fan et al. [4–7]. The $p(t)$-Laplacian operator can be used to describe the physical phenomena with "pointwise different properties." The $p(t)$-Laplacian operator has more complicated properties than that of the p-Laplacian; for example, it is not homogeneous, and this makes some classic theories and methods, such as the theory of Sobolev spaces, not applicable.

With the theory of nonlinear discrete dynamical systems being widely used to study discrete models appearing in many fields such as economics, ecology, computer science, neural networks, and cybernetics [8], the existence of solutions of discrete dynamical systems has become a hot topic; to mention a few, see [9–15]. For the case $p(k) \equiv p$, Iannizzotto and Tersian [16] obtained multiple homoclinic solutions for system (1) by using the critical point theorem, and for the special case $p = 2$, Ma and Guo [13, 14] provided some sufficient conditions on the existence of homoclinic solutions for system (1). For the more general case—$p(k)$-Laplacian system (1)—Chen et al. [17] established some existence criteria to guarantee that the system has at least one or infinitely many homoclinic orbits. Motivated by Liu [2], which discussed the existence of ground state for p-Laplacian system, in this paper, we will consider the existence of ground state for the $p(k)$-Laplacian system (1).

Now we are in a position to state our main results.

Theorem 1. *Assume the following conditions hold:*

    (a) $a(k) > 0$ for all $k \in Z$,

    (q) $q(k) > 0$ for all $k \in Z$ and $q(k) \rightarrow +\infty$ as $|k| \rightarrow +\infty$,

    (p) $1 < p^- = \inf_{k \in Z} p(k) \leq \sup_{k \in Z} p(k) = p^+ < +\infty$ for all $k \in Z$,

    (f) $f(k, u) = f_1(k, u) - f_2(k, u)$ is continuous in u for all $k \in Z$, and $F(k, u) = \int_0^u f(k, s) ds$ for $u \in R$. Moreover,

$$\frac{|f(k, u)|}{q(k)|u|^{p(k)-1}} \longrightarrow 0 \quad \text{as } u \longrightarrow 0 \text{ uniformly in } k \in Z; \tag{3}$$

    (f$_1$) there exists a constant $\beta > p^+$ such that

$$uf_1(k, u) \geq \beta F_1(k, u) > 0 \quad \forall k \in Z, \ u \in R \setminus \{0\}, \tag{4}$$

where $F_1(k, u) = \int_0^u f_1(k, s) ds$ for $u \in R$;

    (f$_2$) there exists a constant $\tau \in (0, \beta)$ such that

$$F_2(k, u) \geq 0, \qquad uf_2(k, u) \leq \tau F_2(k, u)$$
$$\forall k \in Z, \quad u \in R, \tag{5}$$

where $F_2(k, u) = \int_0^u f_2(k, s) ds$ for $u \in R$.
    Then (1) has a ground state solution.

Remark 2. (q) implies that there exists $q_* > 0$ such that $q(k) \geq q_*$ for all $k \in Z$.

Remark 3. We extend Theorem 3.1 in [14] to the more general case—$p(k)$-Laplacian system. Furthermore, we obtain the existence of the ground state.

    The rest of this paper is organized as follows. In Section 2, we establish the variational structure associated with (1). Some preliminary results are also provided in this section. In Section 3, we give the proof of the main result.

2. Variational Structure and Some Preliminary Results

In this section, we establish a variational structure which enables us to reduce the existence of solutions for (1) to the existence of critical points of the corresponding functional.
    Let S be the set of all two-sided sequences; that is,

$$S = \{u = \{u(k)\} : u(k) \in R, k \in Z\}. \tag{6}$$

Then S is a vector space with $au + bv = \{au(k) + bv(k)\}$ for $u, v \in S$, $a, b \in R$.
    We define $l^{p(k)}$ as the set of all functions $u \in S$ such that

$$l^{p(k)} = \left\{ u \in S : \sum_{k \in Z} |u(k)|^{p(k)} < +\infty \right\} \tag{7}$$

with the norm

$$\|u\|_{p(k)} = \inf \left\{ r > 0 : \sum_{k \in Z} \left| \frac{u(k)}{r} \right|^{p(k)} \leq 1 \right\}. \tag{8}$$

We also define

$$E = \left\{ u \in S : \sum_{k \in Z} \Big[a(k) |\Delta u(k-1)|^{p(k)} \right.$$
$$\left. + q(k) |u(k)|^{p(k)} \Big] < +\infty \right\} \tag{9}$$

with the norm

$$\|u\| = \inf \left\{ r > 0 : \sum_{k \in Z} \left[a(k) \left| \frac{\Delta u(k-1)}{r} \right|^{p(k)} \right. \right.$$
$$\left. \left. + q(k) \left| \frac{u(k)}{r} \right|^{p(k)} \right] \leq 1 \right\}. \tag{10}$$

We call the space E a sequence space; it is a special kind of generalized Orlicz sequence space. For the general theory of generalized Orlicz spaces, see [18, 19].
    Consider the functional I on E defined by

$$I(u) = \sum_{k \in Z} \left[\frac{a(k)}{p(k)} |\Delta u(k-1)|^{p(k)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right.$$
$$\left. - F(k, u(k)) \right]. \tag{11}$$

Using the similar arguments as [17], we have the following lemmas.

Lemma 4. $(l^{p(k)}, \|\cdot\|_{p(k)})$ is a reflexive Banach space. Let $u \in l^{p(k)}$ and

$$\phi(u) = \sum_{k \in Z} |u(k)|^{p(k)}; \tag{12}$$

one has

    (1) *if $\|u\|_{p(k)} > 1$, then $\|u\|_{p(k)}^{p^-} \leq \phi(u) \leq \|u\|_{p(k)}^{p^+}$;*

    (2) *if $\|u\|_{p(k)} < 1$, then $\|u\|_{p(k)}^{p^+} \leq \phi(u) \leq \|u\|_{p(k)}^{p^-}$.*

Lemma 5. $(E, \|\cdot\|)$ is a reflexive Banach space. Let $u \in E$ and

$$\varphi(u) = \sum_{k \in Z} \left[a(k) |\Delta u(k-1)|^{p(k)} + q(k) |u(k)|^{p(k)} \right]; \tag{13}$$

one has

    (1) *if $\|u\| > 1$, then $\|u\|^{p^-} \leq \varphi(u) \leq \|u\|^{p^+}$;*

    (2) *if $\|u\| < 1$, then $\|u\|^{p^+} \leq \varphi(u) \leq \|u\|^{p^-}$.*

Lemma 6. *$I \in C^1(E, R)$ and the Fréchet derivative is given by*

$$
\begin{aligned}
\langle I'(u), v \rangle = \sum_{k \in Z} & \left[a(k) \left| \Delta u(k-1) \right|^{p(k)-2} \right. \\
& \times \Delta u(k-1) \Delta v(k-1) \\
& \left. + q(k) |u(k)|^{p(k)-2} u(k) v(k) \right] \\
& - \sum_{k \in Z} f(k, u(k)) v(k),
\end{aligned}
\tag{14}
$$

for all $u, v \in E$. Moreover, the nonzero critical points of the functional I on E are the nontrivial solutions of (1).

Lemma 7. *(f_1) and (f_2) imply that*

$$
u f(k, u) \geq \beta F(k, u) \quad \forall k \in Z, \ u \in R.
\tag{15}
$$

Moreover, for every $k \in Z$ and $u \in R$, $s^{-\beta} F_1(k, su)$ is nondecreasing on $(0, +\infty)$ and $s^{-\tau} F_2(k, su)$ is nonincreasing on $(0, +\infty)$.

Let

$$
c_{\min} = \inf \left\{ I(u) : I'(u) = 0, u \in E \setminus \{0\} \right\}.
\tag{16}
$$

Then $u_0 \neq 0$ with $I(u_0) = c_{\min}$ is said to be a ground state solution of (1).

As usual, we make use of the following basic notations. Let H be a Hilbert space and $C^1(H, R)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H.

Definition 8. Let $I \in C^1(H, R)$. A sequence $\{x_j\} \subset H$ is called a Palais-Smale sequence (P.S. sequence) for I if $\{I(x_j)\}$ is bounded and $I'(x_j) \to 0$ as $j \to +\infty$. We say that I satisfies the Palais-Smale condition (P.S. condition) if any P.S. sequence for I possesses a convergent subsequence.

Let B_r be the open ball in H with radius r and center 0, and let ∂B_r denote its boundary.

Lemma 9 (mountain pass lemma). *Let H be a real Hilbert space and $I \in C^1(H, R)$ satisfies the P.S. condition. Assume that $I(0) \leq 0$ and the following two conditions hold.*

(J_1) *There exist constants $a > 0$ and $\rho > 0$ such that $I_{\partial B_\rho} \geq a$.*

(J_2) *There exists an $e \in H \setminus B_\rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$
c = \inf_{h \in \Gamma} \max_{s \in [0,1]} I(h(s)),
\tag{17}
$$

where

$$
\Gamma = \{ h \in C([0, 1], H) : h(0) = 0, h(1) = e \}.
\tag{18}
$$

3. Proof of Main Result

In order to prove Theorem 1, we first prove the following lemmas.

Lemma 10. *The embedding $E \hookrightarrow l^{p(k)}$ is compact.*

Proof. Let $\{u_j\}$ be a bounded sequence in E; that is, there exists $M > 0$ such that $\|u_j\| < M$ for all $j \in Z^+$. By reflexivity, passing to a subsequence we have $u_j \rightharpoonup u$ in E for some $u \in E$. We may assume $u = 0$, in particular $u_j(k) \to 0$ as $j \to +\infty$ for all $k \in Z$. For all $\epsilon > 0$, we can find $h \in Z^+$ such that

$$
q(k) > \frac{1 + M}{\epsilon} \quad \forall |k| > h.
\tag{19}
$$

By continuity of the finite sum, there exists $v_0 \in Z^+$ such that

$$
\sum_{|k| \leq h} \left| u_j(k) \right|^{p(k)} \leq \frac{\epsilon}{1 + M} \quad \forall j > v_0.
\tag{20}
$$

So for all $j \geq v_0$ we have

$$
\begin{aligned}
\sum_{k \in Z} \left| u_j(k) \right|^{p(k)} &\leq \frac{\epsilon}{1 + M} + \frac{\epsilon}{1 + M} \sum_{|k| > h} q(k) \left| u_j(k) \right|^{p(k)} \\
&\leq \frac{\epsilon}{1 + M} + \frac{\epsilon}{1 + M} \sum_{k \in Z} q(k) \left| u_j(k) \right|^{p(k)}.
\end{aligned}
\tag{21}
$$

Since

$$
\sum_{k \in Z} q(k) \left| u_j(k) \right|^{p(k)} \leq \varphi(u_j) \leq
\begin{cases}
\|u_j\|^{p^-}, & \text{if } \|u_j\| < 1, \\
1, & \text{if } \|u_j\| = 1, \\
\|u_j\|^{p^+}, & \text{if } \|u_j\| > 1,
\end{cases}
\tag{22}
$$

letting $M_0 = \max\{M^{p^-}, 1, M^{p^+}\}$, we have

$$
\sum_{k \in Z} \left| u_j(k) \right|^{p(k)} \leq \frac{1 + M_0}{1 + M} \epsilon \quad \forall j \geq v_0.
\tag{23}
$$

Thus, $u_j(k) \to 0$ in $l^{p(k)}$, and the proof is completed. $\square$

Lemma 11. *Assume that $u \in l^{p(k)}$ and $v \in l^{r(k)}$. Moreover, $p(k)$ satisfies condition (p) and $1/r(k) + 1/p(k) = 1$ for all $k \in Z$. Then*

$$
\sum_{k \in Z} u(k) v(k) \leq \left(\frac{1}{p^-} + \frac{1}{r^-} \right) \|u\|_{p(k)} \|v\|_{r(k)},
\tag{24}
$$

where $r^- = \inf\{r(k) : k \in Z\}$ and $r^- = p^+/(p^+ - 1)$.

Proof. Let

$$
r_1 = \|u\|_{p(k)}, \qquad r_2 = \|v\|_{r(k)};
\tag{25}
$$

then

$$\sum_{k \in Z} \frac{u(k) v(k)}{r_1 r_2} \leq \sum_{k \in Z} \left[\frac{1}{p(k)} \left(\frac{u(k)}{r_1} \right)^{p(k)} + \frac{1}{r(k)} \left(\frac{v(k)}{r_2} \right)^{r(k)} \right]$$

$$\leq \frac{1}{p^-} + \frac{1}{r^-}.$$

(26)

The proof is completed. □

Lemma 12. *Assume that all the conditions of Theorem 1 hold. Then the functional I satisfies the P.S. condition.*

Proof. Assume that $\{u_j\}_{j \in N} \subset E$ is a sequence such that $\{I(u_j)\}$ is a bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a positive constant M' such that $|I(u_j)| \leq M'$ for all $j \in Z^+$.

First, we show that $\|u_j\|$ is bounded. Now we may assume that $\|u_j\| > 1$; otherwise, $\|u_j\|$ is bounded obviously. When j is large enough, we have

$$M' + \frac{1}{\beta} \|u_j\| \geq I(u_j) - \frac{1}{\beta} \langle I'(u_j), u_j \rangle$$

$$= \sum_{k \in Z} \left(\frac{1}{p(k)} - \frac{1}{\beta} \right) \left[a(k) \left| \Delta u_j(k-1) \right|^{p(k)} \right.$$

$$\left. + q(k) \left| u_j(k) \right|^{p(k)} \right]$$

$$+ \sum_{k \in Z} \left[\frac{1}{\beta} f(k, u_j) u_j(k) - F(k, u_j(k)) \right]$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\beta} \right) \|u_j\|^{p^-}.$$

(27)

It follows from $\beta > p^+$ and $p^- > 1$ that there exists a constant $M_* > 0$ such that

$$\|u_j\| \leq M_*, \quad \forall j \in Z^+.$$

(28)

By Lemma 10, we can choose a subsequence, still denoted by $\{u_j\}$, such that

$$u_j \rightharpoonup u_* \quad \text{in } E,$$

(29)

$$u_j \longrightarrow u_* \quad \text{in } l^{p(k)},$$

(30)

for some $u_* \in E$.

Next, we prove that

$$\lim_{j \to +\infty} \sum_{k \in Z} \left(f(k, u_j(k)) - f(k, u_*(k)) \right)$$

$$\times \left(u_j(k) - u_*(k) \right) = 0.$$

(31)

By (f), for any $0 < \epsilon < \min\{1/2, 1/2p^+\}$, there exists a positive constant $\rho < 1$ with $q_*^{(1/p^-)} \rho^{(p^+/p^-)} < 1$ such that

$$f(k, u) \leq \epsilon q(k) |u|^{p(k)-1} \quad \forall k \in Z, \ |u| \leq \rho,$$

(32)

Since $u_* \in E$, there exists a positive integer T such that

$$\left| u_*(k) \right| \leq \frac{\rho}{2} \quad \forall k > T;$$

(33)

we have

$$\left| f(k, u_*(k)) \right| \leq \epsilon q(k) \left| u_*(k) \right|^{p(x)-1} \quad \forall k > T.$$

(34)

By (30), there exists $v_1 \in Z^+$ such that

$$\left| u_j(k) - u_*(k) \right| \leq \frac{\rho}{2} \quad \forall j > v_1, \ k \in Z.$$

(35)

This, combined with (33) and (32), gives us

$$\left| u_j(k) \right| \leq \left| u_j(k) - u_*(k) \right| + \left| u_*(k) \right| \leq \rho \quad \forall j > v_1, \ k > T,$$

$$\left| f(k, u_j(k)) \right| \leq \epsilon q(k) \left| u_j(k) \right|^{p(k)-1} \quad \forall j > v_1, \ k > T.$$

(36)

Then for $j > v_1$,

$$\sum_{|k| > T} \left| f(k, u_j(k)) - f(k, u_*(k)) \right| \left| u_j(k) - u_*(k) \right|$$

$$\leq \epsilon \sum_{|k| > T} \left(q(k) \left| u_j(k) \right|^{p(k)-1} + q(k) \left| u_*(k) \right|^{p(k)-1} \right)$$

$$\times \left(\left| u_j(k) \right| + \left| u_*(k) \right| \right)$$

$$\leq \epsilon \sum_{k \in Z} q(k)^{1/r(k)} \left| u_j(k) \right|^{p(k)-1} q(k)^{1/p(k)} \left| u_j(k) \right|$$

$$+ \epsilon \sum_{k \in Z} q(k)^{1/r(k)} \left| u_j(k) \right|^{p(k)-1} q(k)^{1/p(k)} \left| u_*(k) \right|$$

$$+ \epsilon \sum_{k \in Z} q(k)^{1/r(k)} \left| u_*(k) \right|^{p(k)-1} q(k)^{1/p(k)} \left| u_j(k) \right|$$

$$+ \epsilon \sum_{k \in Z} q(k)^{1/r(k)} \left| u_*(k) \right|^{p(k)-1} q(k)^{1/p(k)} \left| u_*(k) \right|,$$

(37)

where $1/r(k) + 1/p(k) = 1$ for all $k \in Z$.

Let $v_1 = \{v_1(k)\}$ and $v_1(k) = q(k)^{1/r(k)} |u_j(k)|^{p(k)-1}$, $v_2 = \{v_2(k)\}$ and $v_2(k) = q(k)^{1/p(k)} |u_j(k)|$, $h_1 = \{h_1(k)\}$ and $h_1(k) = q(k)^{1/r(k)} |u_*(k)|^{p(k)-1}$, and $h_2 = \{h_2(k)\}$ and $h_2(k) = q(k)^{1/p(k)} |u_*(k)|$.

It is easy to check that $v_1, h_1 \in l^{r(k)}$ and $v_2, h_2 \in l^{p(k)}$. Then using Lemma 11, for $j > v_1$, we have

$$\sum_{|k| > T} \left| f(k, u_j(k)) - f(k, u_*(k)) \right| \left| u_j(k) - u_*(k) \right|$$

$$\leq \left(\frac{\epsilon}{p^-} + \frac{\epsilon}{r^-} \right)$$

$$\times \left(\|v_1\|_{r(k)} \|v_2\|_{p(k)} + \|v_1\|_{r(k)} \|h_2\|_{p(k)} \right.$$

$$\left. + \|h_1\|_{r(k)} \|v_2\|_{p(k)} + \|h_1\|_{r(k)} \|h_2\|_{p(k)} \right).$$

(38)

Now we show that $\|v_1\|_{r(k)}$ is bounded. We may assume that $\|v_1\|_{r(k)} > 1$; otherwise, $\|v_1\|_{r(k)}$ is bounded obviously. By Lemmas 4 and 5, we have

$$\|v_1\|_{r(k)} \leq \left[\sum_{k \in Z} q(k) \left| u_j(k) \right|^{p(k)} \right]^{1/r^-} \leq \|u_j\|^{p^+/r^-} \leq M_*^{p^+/r^-}. \tag{39}$$

Let $M_1 = \{1, M_*^{p^+/r^-}\}$; then $\|v_1\|_{r(k)} \leq M_1$; that is, $\|v_1\|_{r(k)}$ is bounded. Using the similar arguments as above, we obtain that $\|v_2\|_{p(k)}$, $\|h_1\|_{r(k)}$, and $\|h_2\|_{p(k)}$ are bounded; that is, there exist three positive constants M_2, M_3, and M_4 such that

$$\|v_2\|_{p(k)} \leq M_2, \quad \|h_1\|_{r(k)} \leq M_3, \quad \|h_2\|_{p(k)} \leq M_4. \tag{40}$$

This, combined with (38), gives us

$$\sum_{|k|>T} \left| f\left(k, u_j(k)\right) - f\left(k, u_*(k)\right) \right| \left| u_j(k) - u_*(k) \right|$$
$$\leq \epsilon \left(\frac{1}{p^-} + \frac{1}{r^-} \right)$$
$$\times \left(M_1 M_2 + M_1 M_4 + M_3 M_2 + M_3 M_4 \right) \quad \forall j > \nu_1. \tag{41}$$

By continuity of the finite sum and (30), there exists $\nu_2 \in Z^+$ such that

$$\sum_{|k| \leq T} \left| f\left(k, u_j(k)\right) - f\left(k, u_*(k)\right) \right|$$
$$\times \left| u_j(k) - u_*(k) \right| \leq \epsilon \quad \forall j \geq \nu_2. \tag{42}$$

Let $\nu = \max\{\nu_1, \nu_2\}$. Combining (41) and (42) together, we have

$$\sum_{k \in Z} \left| f\left(k, u_j(k)\right) - f\left(k, u_*(k)\right) \right| \left| u_j(k) - u_*(k) \right|$$
$$\leq \epsilon \left(\frac{1}{p^-} + \frac{1}{r^-} \right)$$
$$\times \left(M_1 M_2 + M_1 M_4 + M_3 M_2 + M_3 M_4 \right) + \epsilon \quad \forall j > \nu. \tag{43}$$

Thus, (31) holds.

Finally, we show that $\{u_j\}$ possesses a convergent subsequence. Since $I'(u_j) \to 0$ and $u_j \rightharpoonup u_*$, it follows at once that

$$\lim_{j \to +\infty} \left\langle I'\left(u_j\right) - I'\left(u_*\right), u_j - u_* \right\rangle = 0. \tag{44}$$

This, combined with (31), gives us

$$\lim_{j \to +\infty} \left\{ \sum_{k \in Z} a(k) \left[\left| \Delta u_j(k-1) \right|^{p(k)-2} \Delta u_j(k-1) \right. \right.$$
$$\left. - \left| \Delta u_*(k-1) \right|^{p(k)-2} \Delta u_*(k-1) \right]$$
$$\times \left(\Delta u_j(k-1) - \Delta u_*(k-1) \right)$$
$$+ \sum_{k \in Z} q(k) \left[\left| u_j(k) \right|^{p(k)-2} u_j(k) \right.$$
$$\left. - \left| u_*(k) \right|^{p(k)-2} u_*(k) \right]$$
$$\left. \times \left(u_j(k) - u_*(k) \right) \right\} = 0. \tag{45}$$

The following two inequalities are taken from [20] and will play an important role in the proof of our main result:

$$\left[\left(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) (\xi - \eta) \right]$$
$$\times \left(|\xi| + |\eta| \right)^{2-p} \geq |\xi - \eta|^2, \quad 1 < p < 2, \tag{46}$$
$$2^p \left[\left(|\xi|^{p-2} \xi - |\eta|^{p-2} \eta \right) (\xi - \eta) \right] \geq |\xi - \eta|^p, \quad p \geq 2,$$

for every ξ and η in R. We define

$$R_j(k) = a(k) \left[\left| \Delta u_j(k-1) \right|^{p(k)-2} \Delta u_j(k-1) \right.$$
$$\left. - \left| \Delta u_*(k-1) \right|^{p(k)-2} \Delta u_*(k-1) \right]$$
$$\times \left(\Delta u_j(k-1) - \Delta u_*(k-1) \right) \quad \forall k \in Z.$$
$$Q_j(k) = q(k) \left[\left| u_j(k) \right|^{p(k)-2} u_j(k) - \left| u_*(k) \right|^{p(k)-2} u_*(k) \right]$$
$$\times \left(u_j(k) - u_*(k) \right) \quad \forall k \in Z. \tag{47}$$

This, combined with (45), produces at once

$$\lim_{j \to +\infty} \sum_{k \in Z} R_j(k) = 0, \qquad \lim_{j \to +\infty} \sum_{k \in Z} Q_j(k) = 0. \tag{48}$$

Now we show that $\varphi(u_j - u_*) \to 0$ as $j \to +\infty$. That is,

$$\lim_{j \to +\infty} \sum_{k \in Z} a(k) \left| \Delta u_j(k-1) - \Delta u_*(k-1) \right|^{p(k)} = 0, \tag{49}$$

$$\lim_{j \to +\infty} \sum_{k \in Z} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)} = 0. \tag{50}$$

Let us first prove (50). Since $\{u_j\}$ and u_* are bounded in E, there exists a constant $M^* > 1$ such that $\varphi(u_j) \leq M^*$ and $\varphi(u_*) \leq M^*$ for all $j \in Z^+$. We denote

$$W_1 = \{k \in Z : 1 < p(k) < 2\},$$
$$W_2 = \{k \in Z : p(k) \geq 2\}. \tag{51}$$

By (46), we have

$$\sum_{k \in W_2} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)}$$

$$\leq \sum_{k \in W_2} 2^{p^+} q(k) \left(\left| u_j(k) \right|^{p(k)-2} u_j(k) \right.$$

$$\left. - \left| u_*(k) \right|^{p(k)-2} u_*(k) \right) \tag{52}$$

$$\times \left(u_j(k) - u_*(k) \right) \leq 2^{p^+} \sum_{k \in Z} Q_j(k),$$

$$\sum_{k \in W_1} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)}$$

$$= \sum_{k \in W_1} q(k) \left[\left| u_j(k) - u_*(k) \right|^2 \right]^{p(k)/2}$$

$$\leq \sum_{k \in W_1} \left(Q_j(k) \right)^{p(k)/2}$$

$$\times \left\{ q(k) \left[\left| u_j(k) \right| + \left| u_*(k) \right| \right] \right\}^{p(k)} \right\}^{(2-p(k))/2}$$

$$\leq \sum_{k \in W_1} \left(Q_j(k) \right)^{p(k)/2}$$

$$\times \left[2^{p^+} q(k) \left(\left| u_j(k) \right|^{p(k)} + \left| u_*(k) \right|^{p(k)} \right) \right]^{(2-p(k))/2}$$

$$\leq 2^{p^+(2-p^-)/2}$$

$$\times \sum_{k \in Z} \left(Q_j(k) \right)^{p(k)/2}$$

$$\times \left[q(k) \left| u_j(k) \right|^{p(k)} + q(k) \left| u_*(k) \right|^{p(k)} \right]^{(2-p(k))/2}. \tag{53}$$

Let

$$p_1(k) = \frac{2}{p(k)}, \qquad r_1(k) = \frac{2}{2-p(k)},$$

$$v_{j*}(k) = \left(Q_j(k) \right)^{p(k)/2},$$

$$w_{j*}(k) = \left[q(k) \left| u_j(k) \right|^{p(k)} + q(k) \left| u_*(k) \right|^{p(k)} \right]^{(2-p(k))/2}. \tag{54}$$

Then it is easy to check that $v_{j*} = \{v_{j*}(k)\} \in l_{p_1(k)}$ and $w_{j*} = \{w_{j*}(k)\} \in l_{r_1(k)}$. By Lemma 11 and (53), we have

$$\sum_{k \in W_1} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)}$$

$$\leq 2^{p^+(2-p^-)/2} \left(1 + \frac{p^+}{2} - \frac{p^-}{2} \right) \left\| v_{j*} \right\|_{p_1(k)} \left\| w_{j*} \right\|_{r_1(k)}. \tag{55}$$

Since

$$\lim_{j \to +\infty} \sum_{k \in Z} Q_j(k) = 0,$$

$$\sum_{k \in Z} \left[q(k) \left| u_j(k) \right|^{p(k)} + q(k) \left| u_*(k) \right|^{p(k)} \right] \tag{56}$$

$$\leq \varphi\left(u_j\right) + \varphi\left(u_*\right) \leq 2M^*.$$

It is easy to see that $\left\| v_{j*} \right\|_{p_1(k)} \to 0$ as $j \to +\infty$ and $\left\| w_{j*} \right\|_{r_1(k)}$ is bounded for all $j \in Z^+$. This, combined with (52) and (55), gives us

$$\lim_{j \to +\infty} \sum_{k \in Z} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)}$$

$$= \lim_{j \to +\infty} \sum_{k \in W_1} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)} \tag{57}$$

$$+ \lim_{j \to +\infty} \sum_{k \in W_2} q(k) \left| u_j(k) - u_*(k) \right|^{p(k)} = 0.$$

Using the similar arguments, we have (49). So $\varphi(u_j - u_*) \to 0$ as $j \to +\infty$. By Lemma 5, it follows that $\|u_j - u_*\| \to 0$ as $j \to +\infty$, and the proof is completed. $\qquad\square$

Proof of Theorem 1. The proof consists of two steps.

Step 1. We use Lemma 9 to show that (1) has a nontrivial solution in E.

First we prove that I satisfies (J_1) of Lemma 9. It follows from (32) that

$$|F(k, u(k))| \leq \frac{1}{2p^+} q(k) |u(k)|^{p(k)}, \tag{58}$$

for $|u(k)| \leq \rho$ and $k \in Z$. Then, let $\delta = q_*^{(1/p^-)} \rho^{(p^+/p^-)} < 1$, for all $u \in \partial B_\delta \cap E$, we have $|u(k)| \leq \rho$ and

$$I(u) = \sum_{k \in Z} \left[\frac{a(k)}{p(k)} |\Delta u(k-1)|^{p(k)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right]$$

$$- \sum_{k \in Z} F(k, u(k))$$

$$\geq \sum_{k \in Z} \frac{1}{p^+} \left[a(k) |\Delta u(k-1)|^{p(k)} + q(k) |u(k)|^{p(k)} \right]$$

$$- \sum_{k \in Z} \frac{1}{2p^+} q(k) |u(k)|^{p(k)}$$

$$\geq \sum_{k \in Z} \frac{1}{2p^+} \left[a(k) |\Delta u(k-1)|^{p(k)} + q(k) |u(k)|^{p(k)} \right]$$

$$\geq \frac{1}{2p^+} \|u\|^{p^+} = \frac{1}{2p^+} \delta^{p^+} > 0, \tag{59}$$

and hence I satisfies (J_1) of Lemma 9.

Next, we prove that I satisfies (J_2) of Lemma 9. Let $e = \{e(k)\} \in E$ and

$$
e(k) = \begin{cases} 0, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0. \end{cases} \tag{60}
$$

Then

$$
F(k, e(k)) = \begin{cases} 0, & \text{if } k \neq 0, \\ F(0, 1), & \text{if } k = 0. \end{cases} \tag{61}
$$

By Lemma 7, for $s > 1$, we have

$$
\begin{aligned}
&\sum_{k \in Z} F_1(k, se(k)) = F_1(0, s) \geq s^\beta F_1(0, 1), \quad F_1(0, 1) > 0, \\
&\sum_{k \in Z} F_2(k, se(k)) = F_2(0, s) \leq s^\tau F_2(0, 1), \quad F_2(0, 1) \geq 0.
\end{aligned} \tag{62}
$$

Then

$$
\begin{aligned}
I(se) &= \sum_{k \in Z} \left[\frac{a(k)}{p(k)} |s\Delta e(k-1)|^{p(k)} + \frac{q(k)}{p(k)} |se(k)|^{p(k)} \right] \\
&\quad - \sum_{k \in Z} \left[F_1(k, se(k)) - F_2(k, se(k)) \right] \\
&\leq \frac{a(0)}{p(0)} s^{p(0)} + \frac{a(1)}{p(1)} s^{p(1)} + \frac{q(0)}{p(0)} s^{p(0)} \\
&\quad - s^\beta F_1(0, 1) + s^\tau F_2(0, 1).
\end{aligned} \tag{63}
$$

Since $F_1(0, 1) > 0$ and $p(0), p(1), \tau$ are smaller than β, we can choose s^* large enough such that $I(s^*e) < 0$. So we have verified all assumptions of Lemma 9; we know that I possesses a critical value $\alpha \geq (1/2p^+)\delta^{p^+} > 0$, where

$$
\begin{aligned}
&\alpha = \inf_{h \in \Gamma} \max_{s \in [0,1]} I(h(s)), \\
&\Gamma = \{ h \in C([0, 1], E) : h(0) = 0, h(1) = s^*e \}.
\end{aligned} \tag{64}
$$

A critical point u^* of I corresponding to α is nonzero as $\alpha > 0$.

Step 2. We prove that (1) has a ground state in E.
   Let

$$
K = \{ u \in E : I'(u) = 0, u \neq 0 \} \tag{65}
$$

be the critical set of I. Obviously, K is a nonempty set. Denote

$$
\tilde{c} = \inf \{ I(u) : u \in K \}. \tag{66}
$$

Since $u \in K$, we have

$$
I(u) = I(u) - \frac{1}{\beta} \langle I'(u), u \rangle \geq \left(\frac{1}{p^+} - \frac{1}{\beta} \right) \varphi(u) \geq 0. \tag{67}
$$

Then $0 \leq \tilde{c} \leq I(u)$.

Suppose that $\{u_j\} \subset K$ such that $I(u_j) \to \tilde{c}$. Obviously, $\{u_j\}$ is a P.S. sequence. By Lemma 12, we can choose a subsequence, still denoted by $\{u_j\}$, such that

$$
\lim_{j \to \infty} \|u_j - u_0\| = 0, \quad u_0 \in E. \tag{68}
$$

Then $I(u_j) \to I(u_0)$ and $I(u_0) = \tilde{c}$. Now we prove that u_0 is nonzero. If $u_0 = 0$, then there exists a positive integer W such that for all $j > W$, we have

$$
\|u_j\| \leq q_*^{(1/p^-)} \rho^{(p^+/p^-)} < 1. \tag{69}
$$

By (32), it follows that

$$
\begin{aligned}
\langle I'(u_j), u_j \rangle &= \sum_{k \in Z} \left[a(k) |\Delta u_j(k-1)|^{p(k)} + q(k) |u_j(k)|^{p(k)} \right. \\
&\quad \left. - f(k, u_j(k)) u_j(k) \right] \\
&\geq \sum_{k \in Z} \left[a(k) |\Delta u_j(k-1)|^{p(k)} \right. \\
&\quad \left. + q(k) |u_j(k)|^{p(k)} \right] \\
&\quad - \frac{1}{2} \sum_{k \in Z} q(k) |u_j(k)|^{p(k)} \\
&\geq \frac{1}{2} \varphi(u_j) \geq \frac{1}{2} \|u_j\|^{p^+} > 0,
\end{aligned} \tag{70}
$$

which is in contradiction to $I'(u_j) = 0$. Thus, u_0 is the ground state solution of (1). The proof is completed. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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