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Research Article

Zeros of Analytic Continued q-Euler Polynomials and q-Euler Zeta Function

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We study that the q-Euler numbers $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ are analytic continued to $E_q(s)$ and $E_q(s,w)$. We investigate the new concept of dynamics of the zeros of analytic continued polynomials. Finally, we observe an interesting phenomenon of "scattering" of the zeros of $E_q(s,w)$.

1. Introduction

Recently, the computing environment would make more and more rapid progress and there has been increasing interest in solving mathematical problems with the aid of computers. By using software, many mathematicians can explore concepts much easier than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Mathematicians have studied different kinds of the Euler, Bernoulli, Tangent, and Genocchi numbers and polynomials (see [1–14]). Numerical experiments of Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and Tangent polynomials have been the subject of extensive study in recent years and much progress have been made both mathematically and computationally. Using computer, a realistic study for q-Euler polynomials $E_{n,q}(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the q-Euler polynomials $E_{n,q}(x)$ in complex plane. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. In [11], we introduce the q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let q be a complex number with $|q| < 1$. By the meaning of (1) and (2), let us define the q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows (see [11]):

$$F_q(t) = \frac{2}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{1}$$

$$F_q(x,t) = \left(\frac{2}{qe^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2}$$

Observe that if $q \to 1$, then $E_{n,q}(x) = E_n(x)$ and $E_{n,q} = E_n$.

By using computer, the q-Euler numbers $E_{n,q}$ can be determined explicitly. A few of them are

$$\begin{gathered} E_{0,q} = \frac{2}{1+q}, \qquad E_{1,q} = -\frac{2q}{(1+q)^2}, \\ E_{2,q} = -\frac{2q}{(1+q)^2} + \frac{4q^2}{(1+q)^3}, \\ E_{3,q} = -\frac{2q}{(1+q)^2} + \frac{12q^2}{(1+q)^3} - \frac{12q^3}{(1+q)^4}. \end{gathered} \tag{3}$$

Theorem 1. *For $n \in \mathbb{N}_0$, one has*

$$E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} x^{n-l}. \tag{4}$$

By Theorem 1, after some elementary calculations, we have

$$\begin{aligned}\int_a^b E_{n,q}(x)\,dx &= \sum_{l=0}^{n}\binom{n}{l}E_{l,q}\int_a^b x^{n-l}dx\\ &= \frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}E_{l,q}x^{n-l+1}\Big|_a^b \\ &= \frac{E_{n+1,q}(b)-E_{n+1,q}(a)}{n+1}.\end{aligned} \tag{5}$$

Since $E_{n,q}(0)=E_{n,q}$, by (5), we obtain

$$E_{n,q}(x) = E_{n,q} + n\int_0^x E_{n-1,q}(t)\,dt, \quad \text{for } n\in\mathbb{N}. \tag{6}$$

Then, it is easy to deduce that $E_{n,q}(x)$ are polynomials of degree n. Here is the list of the first q-Euler's polynomials:

$$\begin{aligned}E_{0,q}(x) &= \frac{2}{1+q},\\ E_{1,q}(x) &= -\frac{2q}{(1+q)^2}+\frac{2x}{1+q},\\ E_{2,q}(x) &= -\frac{2q}{(1+q)^2}+\frac{4q^2}{(1+q)^3}-\frac{4qx}{(1+q)^2}+\frac{2x^2}{1+q},\\ E_{3,q}(x) &= -\frac{2q}{(1+q)^2}+\frac{12q^2}{(1+q)^3}-\frac{12q^3}{(1+q)^4}+\frac{12q^2x}{(1+q)^3}\\ &\quad -\frac{6qx}{(1+q)^2}-\frac{6qx^2}{(1+q)^2}+\frac{2x^3}{1+q}.\end{aligned} \tag{7}$$

2. Analytic Continuation of Euler Numbers $E_{n,q}$

In this section, we introduced the q-Euler zeta function and Hurwitz q-Euler zeta function. By q-Euler zeta function, we consider the function $E_q(s)$ as the analytic continuation of q-Euler numbers.

From (1), we note that

$$\frac{d^k}{dt^k}F_q(t)\Big|_{t=0} = 2\sum_{m=0}^{\infty}(-1)^m q^m m^k = E_{k,q}, \quad (k\in\mathbb{N}). \tag{8}$$

By using the above equation, we are now ready to define q-Euler zeta functions.

Definition 2. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$. Consider

$$\zeta_{E,q}(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{n^s}. \tag{9}$$

Observe that $\zeta_{E,q}(s)$ is a meromorphic function on $\mathbb{C}$. Clearly, $\lim_{q\to 1}\zeta_{E,q}(s) = \zeta_E(s)$ (see [4, 7, 12]). Notice that the q-Euler zeta function can be analytically continued to the whole complex plane, and these zeta function has the values of the q-Euler numbers at negative integers.

Theorem 3. *For $k\in\mathbb{N}$, one has*

$$\zeta_{E,q}(-k) = E_{k,q}. \tag{10}$$

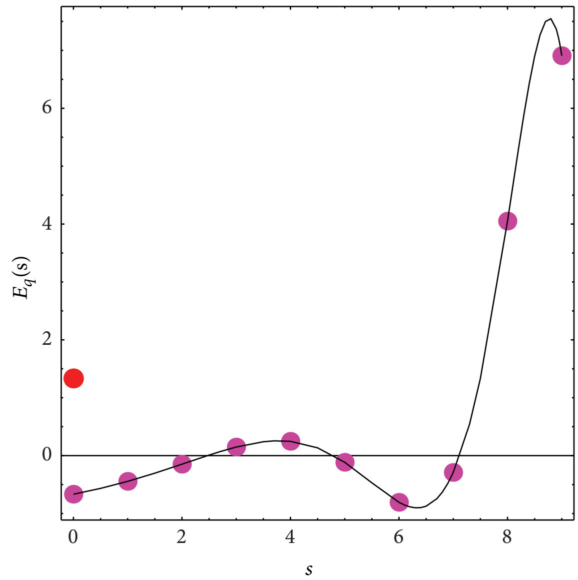

FIGURE 1: The curve $E_q(s)$ runs through the points of all $E_{n,q}$ except $E_{0,q}$.

Observe that $\zeta_{E,q}(s)$ function interpolates $E_{k,q}$ numbers at nonnegative integers.

By using (2), *we note that*

$$\frac{d^k}{dt^k}F_q(x,t)\Big|_{t=0} = 2\sum_{m=0}^{\infty}(-1)^m q^m (x+m)^k = E_{k,q}(x), \quad (k\in\mathbb{N}), \tag{11}$$

$$\left(\frac{d}{dt}\right)^k\left(\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}\right)\Big|_{t=0} = E_{k,q}(x), \quad \textit{for } k\in\mathbb{N}. \tag{12}$$

By (12), *we are now ready to define the Hurwitz-type q-Euler zeta functions.*

Definition 4. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$. Consider

$$\zeta_{E,q}(s,x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(n+x)^s}. \tag{13}$$

Note that $\zeta_{E,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. The relation between $\zeta_{E,q}(s,x)$ and $E_{k,q}(x)$ is given by the following theorem.

Theorem 5. *For $k\in\mathbb{N}$, one has*

$$\zeta_{E,q}(-k,x) = E_{k,q}(x). \tag{14}$$

We now consider the function $E_q(s)$ as the analytic continuation of q-Euler numbers. From the above analytic continuation of q-Euler numbers, we consider

$$\begin{gathered}E_{n,q} \longmapsto E_q(s),\\ \zeta_{E,q}(-n) = E_{n,q} \longmapsto \zeta_{E,q}(-s) = E_q(s).\end{gathered} \tag{15}$$

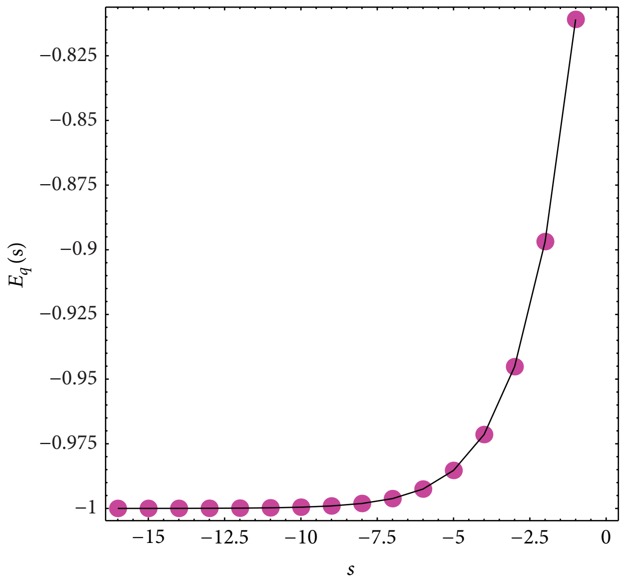

Figure 2: The curve $E_q(s)$ runs through the points $E_{-n,q}$ for $q = 1/2$.

All the q-Euler numbers $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of q-Euler numbers evaluated at n (see Figure 1). Consider

$$E_{n,q} = E_q(n), \quad \text{for } n \geq 1$$
$$\text{except } E_q(0) = \frac{-2q}{1+q}, \quad \text{but } E_{0,q} = \frac{2}{1+q}. \tag{16}$$

In Figure 1, we choose $q = 1/2$. In fact, we can express $E_q'(s)$ in terms of $\zeta_{E,q}'(s)$, the derivative of $\zeta_E(s)$. Consider

$$E_q(s) = \zeta_{E,q}(-s), \qquad E_q'(s) = -\zeta_{E,q}'(-s),$$
$$E_q'(2n+1) = -\zeta_{E,q}'(-2n-1), \quad \text{for } n \in \mathbb{N}_0. \tag{17}$$

From the relation (17), we can define the other analytic continued half of q-Euler numbers:

$$E_q(s) = \zeta_{E,q}(-s), \qquad E_q(-s) = \zeta_{E,q}(s)$$
$$\Longrightarrow E_q(-n) = \zeta_{E,q}(n), \quad n \in \mathbb{N}. \tag{18}$$

By (18), we obtain

$$\lim_{n\to\infty} E_{-n,q} = \zeta_{E,q}(n) = -2q. \tag{19}$$

The curve $E_q(s)$ runs through the points $E_{-n,q} = E_q(-n)$ and grows $\sim -2q$ asymptotically as $-n \to \infty$ (see Figure 2).

3. Analytic Continuation of Euler Polynomials $E_{n,q}(x)$

In this section, we observe the analytic continued q-Euler polynomials. Looking back at (9) and (18), for consistency

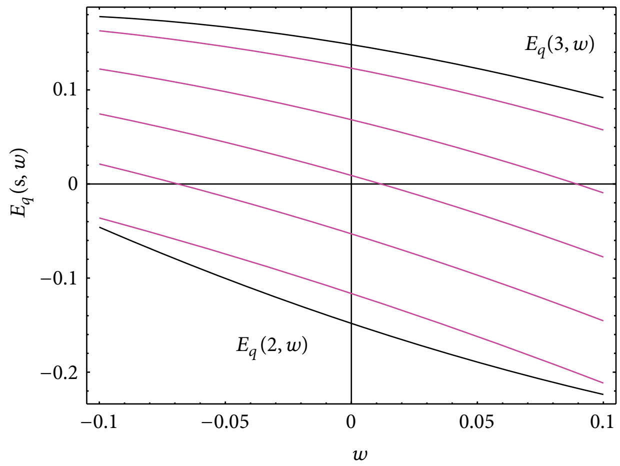

Figure 3: The curve of $E_q(s, w)$, $1 \leq s \leq 2$, and $-0.1 \leq w \leq 0.1$.

with the definition of $E_{n,q}(x) = E_q(n, x)$, q-Euler polynomials should be analogously redefined as

$$E_q(0, x) = -qE_{0,q}(x),$$
$$E_q(n, x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} x^{n-l}. \tag{20}$$

Let $\Gamma(s)$ be the gamma function. The analytic continuation can be then obtained as

$$n \longmapsto s \in \mathbb{R}, \qquad x \longmapsto w \in \mathbb{C},$$
$$E_{0,q} \longmapsto E_q(0) = -\frac{1}{q}\zeta_{E,q}(0),$$
$$E_{k,q} \longmapsto E_q(k + s - [s]) = \zeta_{E,q}(-(k + (s - [s]))),$$
$$\binom{n}{k} \longmapsto \frac{\Gamma(1+s)}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)}$$
$$\Longrightarrow E_{n,q}(w) \longmapsto E_q(s, w)$$
$$= \sum_{k=-1}^{[s]} \frac{\Gamma(1+s) E_q(k + s - [s]) w^{[s]-k}}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)}$$
$$= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s) E_q((k-1) + s - [s]) w^{[s]+1-k}}{\Gamma(k + (s - [s]))\Gamma(2 + [s] - k)}, \tag{21}$$

where $[s]$ gives the integer part of s, and so $s - [s]$ gives the fractional part.

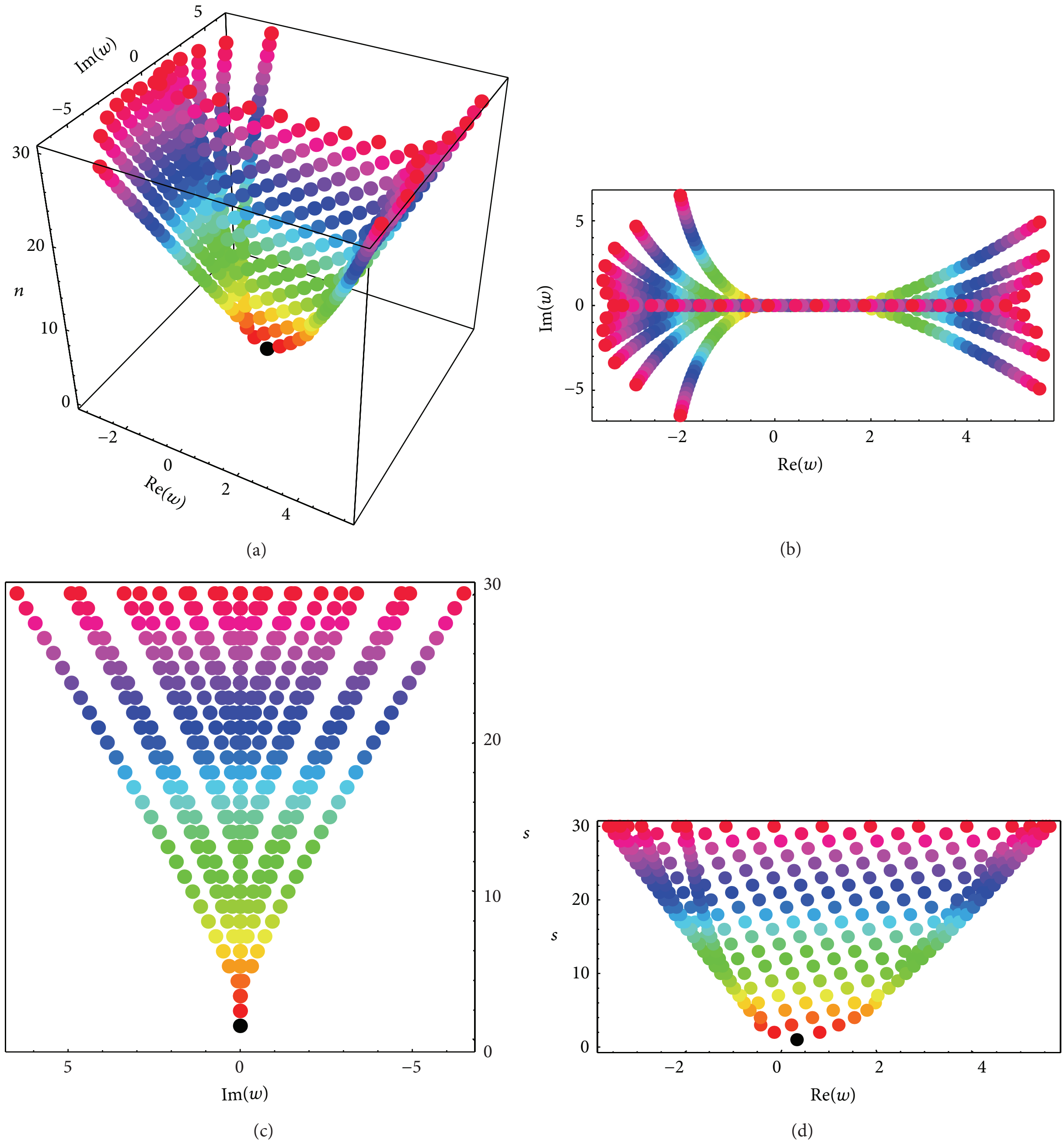

FIGURE 4: Stacks of zeros of $E_q(n, x)$ for $1 \le n \le 30$.

By (21), we have analytic continuation of q-Euler polynomials for $q = 1/2$. Consider

$$E_q(0, w) \approx 1.33333,$$
$$E_q(1, w) \approx -0.44444 + 1.33333w,$$
$$E_q(2, w) \approx -0.14814 - 0.88888w + 1.33333w^2,$$
$$E_q(2.1, w) \approx -0.11635 - 0.87711w - 0.74865w^2 - 0.03078w^3,$$
$$E_q(2.3, w) \approx -0.05297 - 0.83227w - 0.91011w^2 - 0.11583w^3,$$
$$E_q(2.5, w) \approx 0.00907 - 0.75822w - 1.06129w^2 - 0.23359w^3,$$
$$E_q(2.7, w) \approx 0.06832 - 0.65450w - 1.19317w^2 - 0.38417w^3,$$
$$E_q(2.9, w) \approx 0.123069 - 0.521512w - 1.29600w^2 - 0.56561w^3,$$

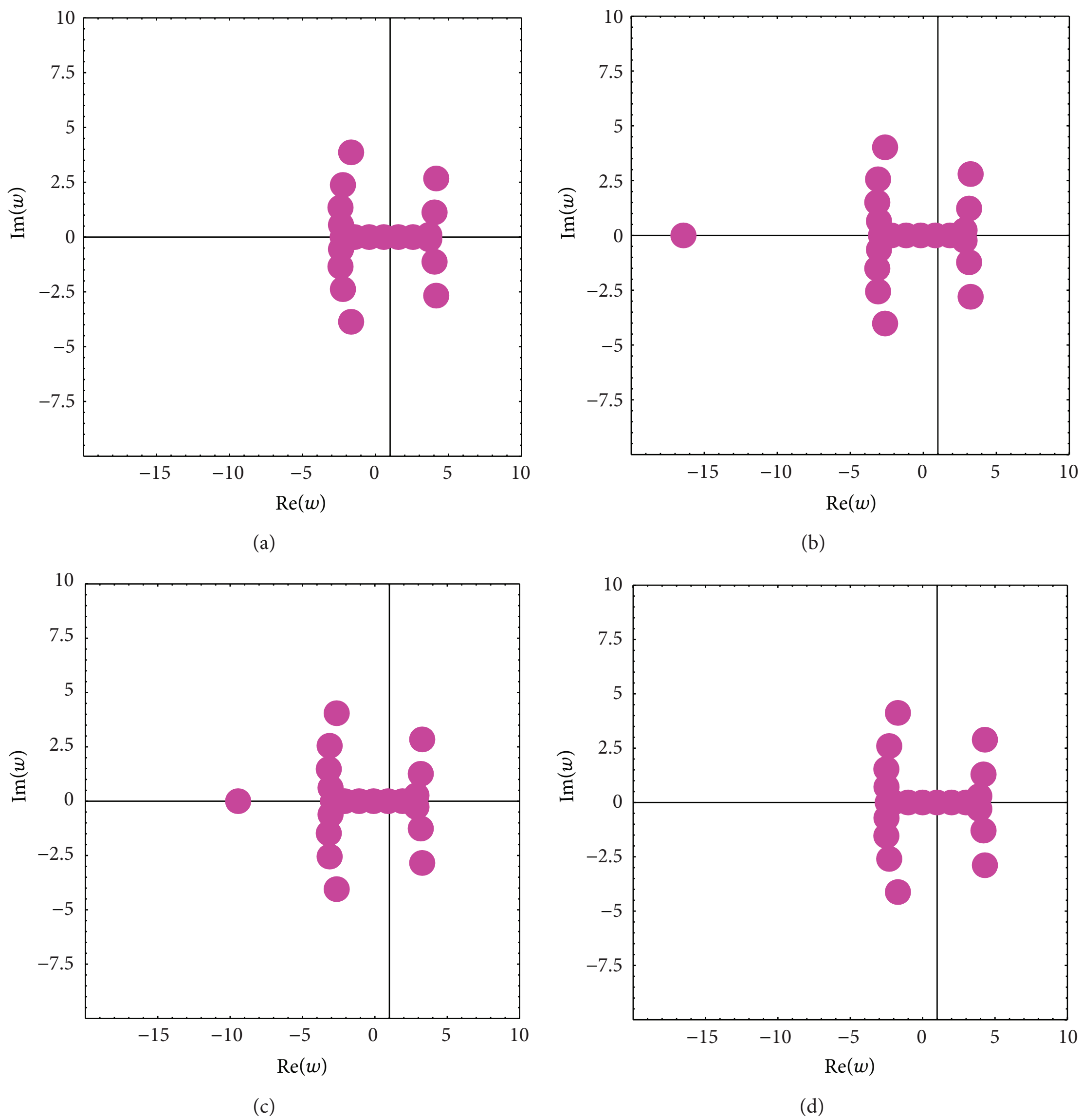

FIGURE 5: Zeros of $E_q(s,w)$ for $s = 20, 20.6, 20.8, 21$.

$$E_q(3,w) \approx 0.14814 - 0.444444w - 1.33333w^2 + 1.33333w^3. \quad (22)$$

By using (22), we plot the deformation of the curve $E_q(2,w)$ into the curve of $E_q(3,w)$ via the real analytic continuation $E_q(s,w)$, $2 \le s \le 3$, $w \in \mathbb{R}$ (see Figure 3).

Next, we investigate the beautiful zeros of the $E_q(n,w)$ by using a computer. We plot the zeros of $E_q(n,w)$ for $n \in \mathbb{N}$, $q = 1/2$, and $w \in \mathbb{C}$ (Figure 4).

In Figure 4, we observe that $E_q(n,w)$, $w \in \mathbb{C}$, has $\mathrm{Im}(w) = 0$ reflection symmetry analytic complex functions (Figure 4). The obvious corollary is that the zeros of $E_q(n,w)$ will also inherit these symmetries. Consider

$$\text{if } E_q(n,w_0) = 0, \quad \text{then } E_q(n,w_0^*) = 0, \quad (23)$$

where $*$ denotes complex conjugation.

Finally, we investigate the beautiful zeros of the $E_q(s,w)$ by using a computer. We plot the zeros of $E_q(s,w)$ for $s = 20, 20.6, 20.8, 21$, $q = 1/2$, and $w \in \mathbb{C}$ (Figure 5). In Figure 5(a), we choose $s = 20$. In Figure 5(b), we choose $s = 20.6$. In Figure 5(c), we choose $s = 20.8$. In Figure 5(d), we choose $s = 21$.

Since

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q^{-1}}(1-x)\frac{(-1)^n t^n}{n!} &= \frac{2}{q^{-1}e^{-t}+1}e^{(1-x)(-t)} \\ &= q\left(\frac{2}{qe^t+1}\right)e^{xt} \\ &= \sum_{n=0}^{\infty} qE_n(x)\frac{t^n}{n!},\end{aligned} \quad (24)$$

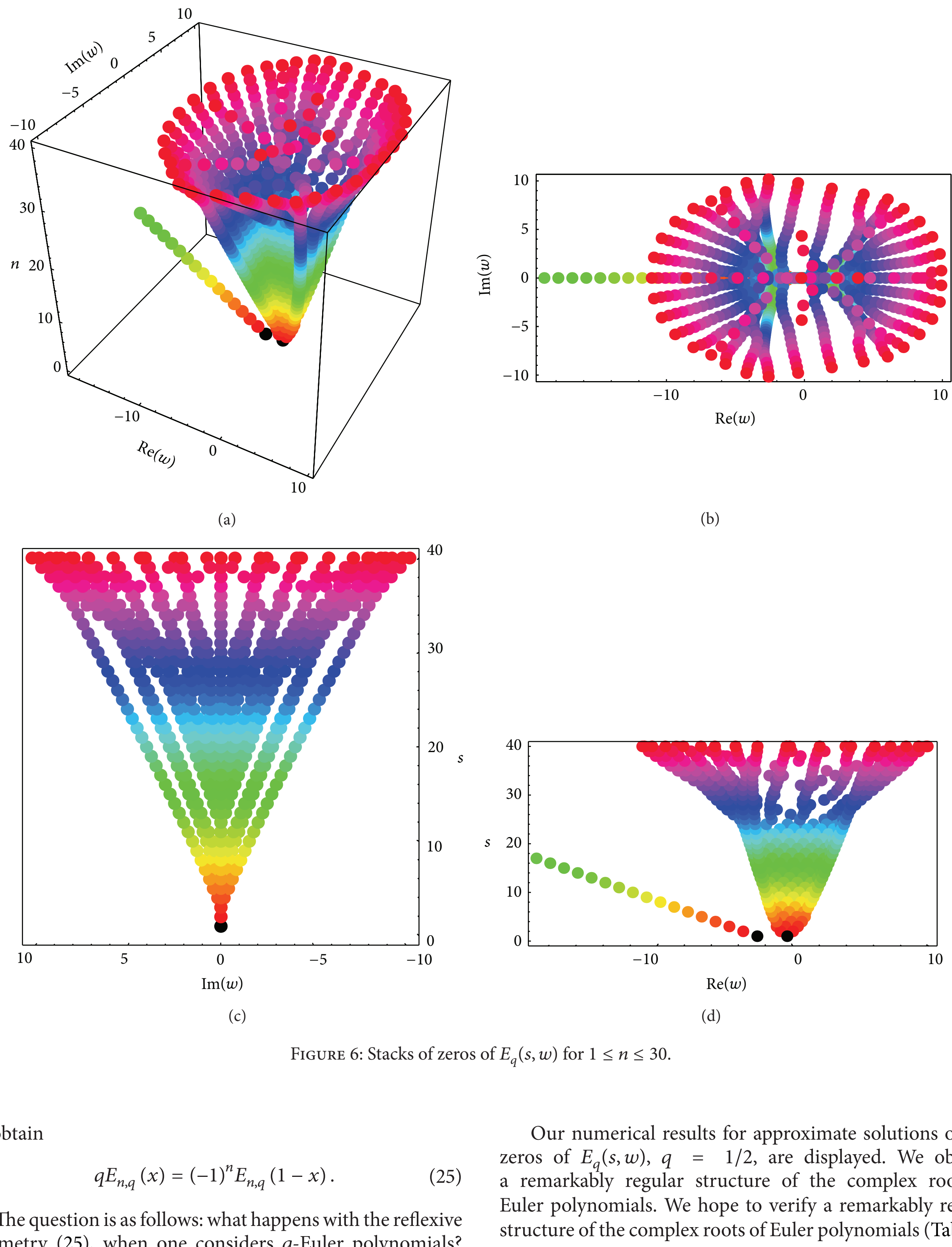

FIGURE 6: Stacks of zeros of $E_q(s,w)$ for $1 \le n \le 30$.

we obtain

$$qE_{n,q}(x) = (-1)^n E_{n,q}(1-x). \tag{25}$$

The question is as follows: what happens with the reflexive symmetry (25), when one considers q-Euler polynomials? Prove that $E_q(n,w)$, $w \in \mathbb{C}$, has not $\mathrm{Re}(w) = 1/2$ reflection symmetry analytic complex functions (Figure 4). However, we observe that $E_q(s,w)$, $w \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (Figure 5).

Stacks of zeros of $E_q(s,w)$ for $s = n + 1/2$, $1 \le n \le 30$, forming a 3D structure are presented (Figure 6).

Our numerical results for approximate solutions of real zeros of $E_q(s,w)$, $q = 1/2$, are displayed. We observe a remarkably regular structure of the complex roots of Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of Euler polynomials (Table 1).

Next, we calculated an approximate solution satisfying $E_q(s,w)$, $q = 1/2$, $w \in \mathbb{R}$. The results are given in Table 2.

In Figure 7, we plot the real zeros of the q-Euler polynomials $E_q(s,w)$ for $s = n+(1/2)$, $q = 1/2$, and $w \in \mathbb{C}$ (Figure 7).

q-Euler polynomials $E_q(n,w)$ are polynomials of degree n. Thus, $E_q(n,w)$ has n zeros and $E_q(n+1,w)$ has $n+1$

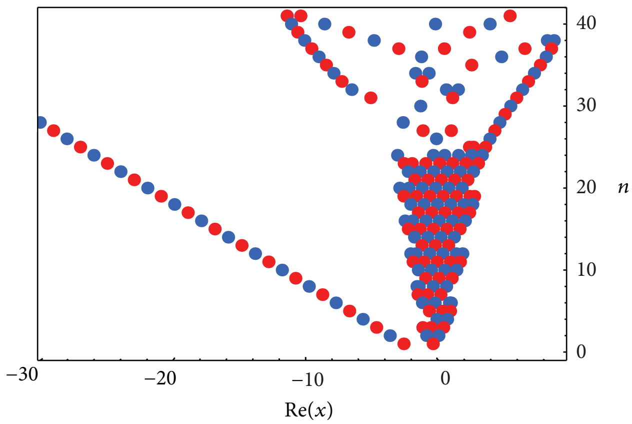

Figure 7: Real zeros of $E_q(s, w)$.

Table 1: Numbers of real and complex zeros of $E_q(s, w)$.

s	Real zeros	Complex zeros
1.5	2	0
2.5	3	0
3.5	4	0
4.5	3	2
5.5	4	2
6.5	5	2
7.5	4	4
8.5	5	4
9.5	4	6
10	4	6
10.6	5	6
10.8	5	6
11	5	6

Table 2: Approximate solutions of $E_q(s, w) = 0$, $w \in \mathbb{R}$.

s	w
6	−0.471899, 0.51601, 1.52785, and 1.95754
6.5	−7.71893, −1.22075, −0.268389, 0.814309, and 0.947618
7	−0.842392, −0.0529263, and 0.947074
7.5	−8.73262, −1.55376, −0.838007, and 0.16246
8	−1.00171, −0.627539, 0.377875, and 1.37783
8.5	−9.74602, −1.62998, −1.46812, −0.406677, and 0.592636
9	−0.191227, 0.808773, and 1.80456
9.5	−10.7592, −0.975608, 0.0242117, and 1.0081
10	−0.758565, 0.239647, 1.23965, and 2.19578
10.6	−8.91801, −1.48957, −0.501829, 0.498175, and 1.39252
10.8	−5.27281, −1.41108, −0.415653, 0.58436, and 1.4511
11	−1.22287, −0.329477, 0.670523, 1.67056, and 2.5067

zeros. When discrete n is analytic continued to continuous parameter s, it naturally leads to the following question.

How does $E_q(s, w)$, the analytic continuation of $E_q(n, w)$, pick up an additional zero as s increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued Euler polynomials—the idea of looking at how the zeros move about in the w complex plane as we vary the parameter s.

To have a physical picture of the motion of the zeros in the complex w plane, imagine that each time, as s increases gradually and continuously by one, an additional real zero flies in from positive infinity along the real positive axis, gradually slowing down as if "it is flying through a viscous medium."

For more studies and results on this subject you may see [5, 10–12].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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