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Research Article

Bounded Doubly Close-to-Convex Functions

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We consider a new class $\mathscr{C}\mathscr{C}(\alpha,\beta)$ of bounded doubly close-to-convex functions. Coefficient bounds, distortion theorems, and radius of convexity for the class $\mathscr{C}\mathscr{C}(\alpha,\beta)$ are investigated. A corresponding class of doubly close-to-starlike functions $\mathcal{S}^*\mathcal{S}(\alpha,\beta)$ is also considered.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}^*$, $\mathscr{K}$, and $\mathscr{C}$ denote the well-known classes of starlike, convex, and close-to-convex functions, respectively.

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{C}(\beta)$ of close-to-convex functions of order $\beta \geq 0$ (see [1]) if there exist $g \in \mathscr{K}$ and $\theta \in \mathbb{R}$ such that

$$\left|\arg\left(e^{i\theta}\frac{f'(z)}{g'(z)}\right)\right| \leq \beta\frac{\pi}{2} \quad (z \in \mathbb{U}). \qquad (2)$$

It is clear that $\mathscr{C}(0) = \mathscr{K}$ and $\mathscr{C}(1) = \mathscr{C}$.

Denote by $\mathscr{B}$ the class of analytic functions ω in $\mathbb{U}$ with $\omega(0) = 0$ and such that $|\omega(z)| < 1$ for all $z \in \mathbb{U}$.

Suppose that f and g are two analytic functions in $\mathbb{U}$. The function f is said to be subordinate to the function g, denoted by $f \prec g$, if there exists a function $\omega \in \mathscr{B}$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$.

Let $\mathscr{P}$ be the well-known class of analytic functions p normalized by $p(0) = 1$ and having positive real part in $\mathbb{U}$.

For a fixed $\alpha > 1/2$ let $\mathscr{P}_\alpha$ denote the subclass of $\mathscr{P}$ defined by

$$\mathscr{P}_\alpha = \left\{p \in \mathscr{P} : \left|p(z) - \alpha\right| < \alpha, z \in \mathbb{U}\right\}. \qquad (3)$$

The class $\mathscr{P}_\alpha$ has been investigated by Goel [2] and also by Libera and Livingston [3].

It is easy to observe that when $\alpha \to \infty$, the class $\mathscr{P}_\alpha$ reduces to the class $\mathscr{P}$.

For $\alpha > 1/2$, the function

$$p_\alpha(z) = \frac{1+z}{1-(1-(1/\alpha))z} \quad (z \in \mathbb{U}) \qquad (4)$$

maps the unit disk $\mathbb{U}$ onto the domain $D_\alpha = \{z \in \mathbb{U} : |z-\alpha| < \alpha\}$. It follows that a function p is in the class $\mathscr{P}_\alpha$ if and only if $p \prec p_\alpha$.

If

$$p_\alpha(z) = 1 + \sum_{n=1}^{\infty} P_n z^n, \qquad (5)$$

then it is easy to check that

$$P_1 = 2 - \frac{1}{\alpha}, \qquad P_2 = \left(2 - \frac{1}{\alpha}\right)\left(1 - \frac{1}{\alpha}\right). \qquad (6)$$

Some properties of the functions belonging to the class $\mathscr{P}_\alpha$ are listed in the next lemma.

Lemma 1 (see [2, 3]). *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be in the class $\mathscr{P}_\alpha$ $(\alpha > 1/2)$. Then*

$$\left|p_n\right| \leq 2 - \frac{1}{\alpha} \quad (n \in \{1, 2, \ldots\}); \qquad (7)$$

$$\frac{1-r}{1+(1-(1/\alpha))r} \leq \left|p(z)\right| \leq \frac{1+r}{1-(1-(1/\alpha))r}; \qquad (8)$$

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{(2-(1/\alpha))r}{1-(1/\alpha)r-(1-(1/\alpha))r^2}, \qquad (9)$$

for $z \in \mathbb{U}$ and $|z| = r < 1$. All the inequalities are sharp.

Let $\mathcal{CC}(\alpha)$ $(\alpha > 1/2)$ denote the class of all functions $f \in \mathcal{A}$ for which there exists a function $g \in \mathcal{K}$ such that $f'/g' \prec p_\alpha$, where p_α is given by (4), or equivalently

$$\left|\frac{f'(z)}{g'(z)} - \alpha\right| < \alpha \quad (z \in \mathbb{U}). \tag{10}$$

It is easy to see that when $\alpha \to \infty$ the class $\mathcal{CC}(\alpha)$ reduces to the class $\mathcal{C}$ of close-to-convex functions.

A slightly different class than the class $\mathcal{CC}(\alpha)$ was investigated in [2].

Recently, in [4] the authors considered a new class of analytic functions defined in a similar way to the class $\mathcal{C}(\beta)$. For fixed $\beta \geq 0$ and $\gamma \geq 0$ a function $f \in \mathcal{A}$ is called doubly close-to-convex if there exist $g \in \mathcal{C}(\beta)$ and $\phi \in \mathbb{R}$ such that

$$\left|\arg\left(e^{i\phi}\frac{f'(z)}{g'(z)}\right)\right| \leq \gamma\frac{\pi}{2} \quad (z \in \mathbb{U}). \tag{11}$$

Motivated by the ideas from [4] we define a new class of bounded doubly close-to-convex functions.

Definition 2. Let $\alpha > 1/2$ and $\beta > 1/2$ be fixed. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{CC}(\alpha, \beta)$ if there exists a function $g \in \mathcal{CC}(\alpha)$ such that $f'/g' \prec p_\beta$, where p_β is given by (4) with β instead of α, or equivalently

$$\left|\frac{f'(z)}{g'(z)} - \beta\right| < \beta \quad (z \in \mathbb{U}). \tag{12}$$

From the above definition and the definition of the class $\mathcal{CC}(\alpha)$, it follows that $f \in \mathcal{CC}(\alpha, \beta)$ if there exist a function $h \in \mathcal{K}$ and a function $g \in \mathcal{A}$ such that $g'/h' \prec p_\alpha$ and $f'/g' \prec p_\beta$, or equivalently

$$\left|\frac{g'(z)}{h'(z)} - \alpha\right| < \alpha, \quad \left|\frac{f'(z)}{g'(z)} - \beta\right| < \beta \quad (z \in \mathbb{U}). \tag{13}$$

In the next lemma we prove that the new class $\mathcal{CC}(\alpha, \beta)$ is nonempty.

Lemma 3. *Let $\alpha > 1/2$ and $\beta > 1/2$. Then, there exists a function $f \in \mathcal{CC}(\alpha, \beta)$.*

Proof. Define the following three functions:

$$\begin{aligned} f(z) = \int_0^z &(1+\epsilon_1 u)(1+\epsilon_2 u) \\ &\times \left((1+\epsilon_3 u)^2\left[1-(1-(1/\alpha))\epsilon_1 u\right]\right. \\ &\left.\times\left[1-(1-(1/\beta))\epsilon_2 u\right]\right)^{-1} du, \end{aligned} \tag{14}$$

$$\begin{aligned} g(z) &= \int_0^z \frac{1+\epsilon_1 u}{(1+\epsilon_3 u)^2\left[1-(1-(1/\alpha))\epsilon_1 u\right]} du, \\ h(z) &= \frac{z}{1+\epsilon_3 z}, \end{aligned} \tag{15}$$

with $z \in \mathbb{U}$ and $|\epsilon_k| = 1$, $k \in \{1, 2, 3\}$. Since $h \in \mathcal{K}$ (see [5]) and

$$\frac{g'(z)}{h'(z)} = \frac{1+\epsilon_1 z}{1-(1-(1/\alpha))\epsilon_1 z}, \tag{16}$$

it follows that $g \in \mathcal{CC}(\alpha)$. The equality

$$\frac{f'(z)}{g'(z)} = \frac{1+\epsilon_2 z}{1-(1-(1/\beta))\epsilon_2 z} \tag{17}$$

together with $g \in \mathcal{CC}(\alpha)$ shows that the function f defined by (14) belongs to $\mathcal{CC}(\alpha, \beta)$. □

In this paper we obtain distortion theorems, radius of convexity, and coefficient bounds for the class $\mathcal{CC}(\alpha, \beta)$. In the last section of the paper a corresponding class of bounded doubly close-to-starlike functions $\mathcal{S}^*\mathcal{S}(\alpha, \beta)$ is also considered.

2. Distortion Theorems

In this section distortion theorems for the class $\mathcal{CC}(\alpha, \beta)$ are obtained.

Theorem 4. *Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{CC}(\alpha, \beta)$, then*

$$\begin{aligned} &\frac{(1-r)^2}{(1+r)^2\left[1+(1-(1/\alpha))r\right]\left[1+(1-(1/\beta))r\right]} \\ &\quad \leq \left|f'(z)\right| \\ &\quad \leq \frac{(1+r)^2}{(1-r)^2\left[1-(1-(1/\alpha))r\right]\left[1-(1-(1/\beta))r\right]} \end{aligned} \tag{18}$$

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Proof. Let $f \in \mathcal{CC}(\alpha, \beta)$. Then there exists $g \in \mathcal{CC}(\alpha)$ such that $f'(z) = g'(z)q(z)$, where q belongs to the class $\mathcal{P}_\beta$ defined by (3) with β instead of α. Since $q \in \mathcal{P}_\beta$, making use of inequalities (8) from Lemma 1, we obtain

$$\frac{1-r}{1+(1-(1/\beta))r} \leq \left|\frac{f'(z)}{g'(z)}\right| \leq \frac{1+r}{1-(1-(1/\beta))r} \tag{19}$$

for $z \in \mathbb{U}$ and $|z| = r < 1$.

The function g belongs to the class $\mathcal{CC}(\alpha)$ and thus, there exists a function $h \in \mathcal{K}$ such that $g'(z) = h'(z)p(z)$, where $p \in \mathcal{P}_\alpha$. Using once more the inequalities (8) from Lemma 1, we have

$$\frac{1-r}{1+(1-(1/\alpha))r} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{1+r}{1-(1-(1/\alpha))r} \tag{20}$$

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Moreover, $h \in \mathcal{K}$ implies that (see [5, 6])

$$\frac{1}{(1+r)^2} \leq \left|h'(z)\right| \leq \frac{1}{(1-r)^2} \tag{21}$$

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Combining the inequalities (19), (20), and (21), we obtain the desired inequality (18). □

Theorem 5. *Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathscr{CC}(\alpha, \beta)$, then*

$$\begin{aligned} |f(z)| \le 4\alpha\beta \left[\frac{r}{1-r} + (\alpha+\beta-1)\log(1-r)\right] \\ - \frac{\alpha\beta}{\alpha-\beta}\left\{(2\alpha-1)^2 \log\left[1-\left(1-\frac{1}{\alpha}\right)r\right]\right. \\ \left. -(2\beta-1)^2\log\left[1-\left(1-\frac{1}{\beta}\right)r\right]\right\}, \end{aligned} \tag{22}$$

$$\begin{aligned} |f(z)| \ge 4\alpha\beta \left[\frac{r}{1+r} - (\alpha+\beta-1)\log(1+r)\right] \\ + \frac{\alpha\beta}{\alpha-\beta}\left\{(2\alpha-1)^2 \log\left[1+\left(1-\frac{1}{\alpha}\right)r\right]\right. \\ \left. -(2\beta-1)^2\log\left[1+\left(1-\frac{1}{\beta}\right)r\right]\right\} \end{aligned} \tag{23}$$

if $\alpha \ne \beta$ and

$$\begin{aligned} |f(z)| \le 4\alpha^2 \left[\frac{r}{1-r} + (2\alpha-1)\log(1-r)\right] \\ + (2\alpha-1)\left\{\frac{r}{1-(1-(1/\alpha))r}\right. \\ \left. - 4\alpha^2\log\left[1-\left(1-\frac{1}{\alpha}\right)r\right]\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} |f(z)| \ge 4\alpha^2 \left[\frac{r}{1+r} - (2\alpha-1)\log(1+r)\right] \\ + (2\alpha-1)\left\{\frac{r}{1+(1-(1/\alpha))r}\right. \\ \left. + 4\alpha^2\log\left[1+\left(1-\frac{1}{\alpha}\right)r\right]\right\} \end{aligned} \tag{25}$$

if $\alpha = \beta$.

Proof. Let $f \in \mathscr{CC}(\alpha, \beta)$. Integrating along the straight line segment from origin to $z = re^{i\theta}$ $(0 < r < 1)$ the right-hand side of inequality (18) we obtain

$$\begin{aligned} &|f(z)| \\ &\le \int_0^r \left|f'\left(\rho e^{i\theta}\right)\right| d\rho \\ &\le \int_0^r \frac{(1+\rho)^2}{(1-\rho)^2\left[1-(1-(1/\alpha))\rho\right]\left[1-(1-(1/\beta))\rho\right]} d\rho, \end{aligned} \tag{26}$$

which leads to inequalities (22) and (24).

To prove the lower bound of $|f(z)|$ we proceed in the following way. Let $\delta > 0$ be the radius of the open disk contained entirely in $f(\mathbb{U})$. Consider z_0 with $|z_0| = r < 1$ such that $|f(z_0)| = \min_{|z|=r}|f(z)|$. The minimum increases with r and is less than δ. Hence, the linear segment Γ which connects the origin with the point $f(z_0)$ will be covered entirely by the values of $f(z)$. Denote by γ the arc in $\mathbb{U}$ which is mapped by $w = f(z)$ in Γ. Making use of the left-hand side of inequality (18) we get

$$\begin{aligned} &|f(z)| \\ &\ge |f(z_0)| = \int_\Gamma dw \ge \int_\gamma \left|f'(z)\right| |dz| \\ &\ge \int_0^r \frac{(1-\rho)^2}{(1+\rho)^2\left[1+(1-(1/\alpha))\rho\right]\left[1+(1-(1/\beta))\rho\right]} d\rho. \end{aligned} \tag{27}$$

After simple calculations we obtain the inequalities (23) and (25). Thus, the proof of our theorem is completed. □

3. Radius of Convexity

In this section we obtain the radius of the disk which is mapped onto a convex domain by the functions belonging to $\mathscr{CC}(\alpha, \beta)$.

Theorem 6. *Let $\alpha > 1/2$ and $\beta > 1/2$. Suppose that $f \in \mathscr{CC}(\alpha, \beta)$. Then, the function f maps the disk $\{z \in \mathbb{C} : |z| < r_0 < 1\}$ onto a convex domain, where r_0 is the smallest positive root of the equation*

$$\begin{aligned} (\alpha+\beta-\alpha\beta-1)r^4 + 4(\alpha\beta-\alpha-\beta+1)r^3 \\ + (10\alpha\beta-5\alpha-5\beta+1)r^2 + 4\alpha\beta r - \alpha\beta = 0. \end{aligned} \tag{28}$$

Proof. Let $f \in \mathscr{CC}(\alpha, \beta)$. Then, there exists $g \in \mathscr{CC}(\alpha)$ such that

$$f'(z) = q(z)g'(z) \quad (z \in \mathbb{U}), \tag{29}$$

where $q \in \mathscr{P}_\beta$. Since $g \in \mathscr{CC}(\alpha)$, there exists a function $h \in \mathscr{K}$ such that

$$g'(z) = p(z)h'(z) \quad (z \in \mathbb{U}), \tag{30}$$

where $p \in \mathscr{P}_\alpha$. Moreover, since $h \in \mathscr{K}$ it follows that there exists a function $k \in \mathscr{S}^*$ (see [5, 6]) such that

$$k(z) = zh'(z) \quad (z \in \mathbb{U}). \tag{31}$$

Combining the equalities (29), (30), and (31), we get

$$zf'(z) = q(z)p(z)k(z) \quad (z \in \mathbb{U}). \tag{32}$$

By taking logarithmic derivative in (32), we obtain

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zq'(z)}{q(z)} + \frac{zp'(z)}{p(z)} + \frac{zk'(z)}{k(z)} \tag{33}$$

which leads to

$$\begin{aligned} &\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \\ &\ge \min\Re\left(\frac{zk'(z)}{k(z)}\right) - \max\left|\frac{zq'(z)}{q(z)}\right| - \max\left|\frac{zp'(z)}{p(z)}\right|. \end{aligned} \tag{34}$$

For $k \in \mathcal{S}^*$, we have (see [7])

$$\Re\left(\frac{zk'(z)}{k(z)}\right) \geq \frac{1-r}{1+r} \tag{35}$$

with $z \in \mathbb{U}$ and $|z| = r < 1$.

Since $p \in \mathcal{P}_\alpha$ and $q \in \mathcal{P}_\beta$, making use of inequality (9) from Lemma 1, we obtain

$$\begin{aligned} \left|\frac{zp'(z)}{p(z)}\right| &\leq \frac{(2-(1/\alpha))\,r}{1-(1/\alpha)\,r-(1-(1/\alpha))\,r^2}, \\ \left|\frac{zq'(z)}{q(z)}\right| &\leq \frac{(2-(1/\beta))\,r}{1-(1/\beta)\,r-(1-(1/\beta))\,r^2} \end{aligned} \tag{36}$$

with $z \in \mathbb{U}$ and $|z| = r < 1$.

Substituting (35) and (36) in (34) we have

$$\begin{aligned} \Re\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \frac{1-r}{1+r} &- \frac{(2-(1/\alpha))\,r}{1-(1/\alpha)\,r-(1-(1/\alpha))\,r^2} \\ &- \frac{(2-(1/\beta))\,r}{1-(1/\beta)\,r-(1-(1/\beta))\,r^2}. \end{aligned} \tag{37}$$

It follows that the function f is convex whenever the expression in the right-hand side of (37) is positive. The numerator of this expression can be written as $P(r) = (1 - r)Q(r)$, where

$$\begin{aligned} Q(r) = {} & (\alpha+\beta-\alpha\beta-1)\,r^4 + 4\,(\alpha\beta-\alpha-\beta+1)\,r^3 \\ & + (10\alpha\beta-5\alpha-5\beta+1)\,r^2 + 4\alpha\beta r - \alpha\beta. \end{aligned} \tag{38}$$

We observe that $Q(0) = -\alpha\beta < 0$ and $Q(1) = 4(2\alpha - 1)(2\beta - 1) > 0$. It follows that the smallest root r_0 of $Q(r) = 0$ and also of $P(r) = 0$ lies between 0 and 1 and, thus, the theorem is proved. □

4. Coefficient Estimates

In order to find coefficient estimates for the class $\mathcal{CC}(\alpha,\beta)$, we will find first the coefficient estimates for the class $\mathcal{CC}(\alpha)$.

Theorem 7. *Let $\alpha > 1/2$. If the function f given by* (1) *is in the class $\mathcal{CC}(\alpha)$, then*

$$|a_n| \leq 1 + \frac{(2\alpha-1)(n-1)}{2\alpha} \quad (n \in \{2,3,\ldots\}). \tag{39}$$

Proof. Since $f \in \mathcal{CC}(\alpha)$ we have

$$f'(z) = p(z)\,g'(z) \quad (z \in \mathbb{U}), \tag{40}$$

where

$$\begin{aligned} g(z) &= z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}, \\ p(z) &= 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}_\alpha. \end{aligned} \tag{41}$$

Equating the coefficients of z^n on both sides of (40), we find the following relation between the coefficients:

$$\begin{aligned} na_n = {} & nb_n + p_{n-1} + 2b_2 p_{n-2} \\ & + \cdots + (n-1)\,b_{n-1}p_1 \quad (n \in \{2,3,\ldots\}). \end{aligned} \tag{42}$$

For $g \in \mathcal{K}$ we have $|b_k| \leq 1$, $k \geq 2$ (see [5, 6]). In virtue of inequality (7) of Lemma 1 we have $|p_k| \leq 2 - (1/\alpha)$, $k \in \{1,2,\ldots\}$.

Making use of (42), we find

$$\begin{aligned} n|a_n| &\leq n + \left(2-\frac{1}{\alpha}\right)\sum_{k=1}^{n-1} k \\ &= n + \left(2-\frac{1}{\alpha}\right)\frac{n(n-1)}{2} \quad (n \in \{2,3,\ldots\}) \end{aligned} \tag{43}$$

and, thus, we get the desired inequality (39). □

When $\alpha \to \infty$ we find the well-known coefficient estimates of close-to-convex functions (see [5, 6]).

In the next theorem we obtain the coefficient estimates for the class $\mathcal{CC}(\alpha,\beta)$.

Theorem 8. *Let $\alpha > 1/2$ and $\beta > 1/2$. If the function f given by* (1) *is in the class $\mathcal{CC}(\alpha,\beta)$, then*

$$\begin{aligned} |a_n| \leq {} & 1 + \left(4-\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{n-1}{2} \\ & + \left(2-\frac{1}{\alpha}\right)\left(2-\frac{1}{\beta}\right)\frac{(n-1)(n-2)}{6} \quad (n \in \{2,3,\ldots\}). \end{aligned} \tag{44}$$

Proof. Let $f \in \mathcal{CC}(\alpha,\beta)$. Then, there exist

$$\begin{aligned} g(z) &= z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{CC}(\alpha), \\ q(z) &= 1 + \sum_{n=1}^{\infty} q_n z^n \in \mathcal{P}_\beta \end{aligned} \tag{45}$$

such that

$$f'(z) = q(z)\,g'(z) \quad (z \in \mathbb{U}). \tag{46}$$

Comparing the coefficients of z^n on both sides of the above equality, we obtain the next relation:

$$\begin{aligned} na_n = {} & nb_n + q_{n-1} + 2b_2 q_{n-2} \\ & + \cdots + (n-1)\,b_{n-1}q_1 \quad (n \in \{2,3,\ldots\}). \end{aligned} \tag{47}$$

Since $g \in \mathcal{CC}(\alpha)$ and $q \in \mathcal{P}_\beta$, from (39) and (7), we get

$$\begin{aligned} |b_k| &\leq 1 + \frac{(2\alpha-1)(k-1)}{2\alpha} \quad (k \in \{2,3,\ldots\}), \\ |q_k| &\leq 2 - \frac{1}{\beta} \quad (k \in \{1,2,\ldots\}). \end{aligned} \tag{48}$$

From (47) in connection with (48), we obtain

$$\begin{aligned}
&|a_n| \\
&\leq 1 + \frac{(2\alpha-1)(n-1)}{2\alpha} \\
&\quad + \frac{1}{n}\left(2-\frac{1}{\beta}\right)\left[1+\sum_{k=2}^{n-1} k\left(1+\frac{(2\alpha-1)(k-1)}{2\alpha}\right)\right] \\
&= 1 + \frac{(2\alpha-1)(n-1)}{2\alpha} \\
&\quad + \frac{1}{n}\left[\left(2-\frac{1}{\beta}\right)\left(1+\sum_{k=2}^{n-1} k\right)+\left(2-\frac{1}{\beta}\right)\frac{2\alpha-1}{2\alpha}\sum_{k=2}^{n-1} k(k-1)\right] \\
&= 1 + \frac{(2\alpha-1)(n-1)}{2\alpha} + \frac{2\beta-1}{2\beta}(n-1) \\
&\quad + \frac{(2\alpha-1)(2\beta-1)}{6\alpha\beta}(n-1)(n-2)
\end{aligned} \tag{49}$$

which leads to inequality (44). □

5. Maximum Value of $|a_3-(2/3)a_2^2|$

The problem of finding sharp upper bounds for the functional $|a_3-\mu a_2^2|$ for a family of analytic functions is known as the Fekete-Szegö problem. For the classes $\mathscr{K}$ and $\mathscr{C}$, the following estimates are known (see, e.g., [8–11]):

$$\max_{f\in\mathscr{K}}\left|a_3-\mu a_2^2\right| = \max\left\{\frac{1}{3}, |\mu-1|\right\}, \tag{50}$$

$$\max_{f\in\mathscr{C}}\left|a_3-\mu a_2^2\right| = \begin{cases} 3-4\mu, & 0\leq\mu\leq\dfrac{1}{3} \\ \dfrac{1}{3}+\dfrac{4}{9\mu}, & \dfrac{1}{3}\leq\mu\leq\dfrac{2}{3} \\ 1, & \dfrac{2}{3}\leq\mu\leq 1. \end{cases} \tag{51}$$

In this section, the case $\mu = 2/3$ of the Fekete-Szegö problem will be considered, first for the class $\mathscr{CC}(\alpha)$ and then, for the class $\mathscr{CC}(\alpha,\beta)$.

In order to prove our results we need the following lemma due to Keogh and Merkes [9].

Lemma 9. *Let $\omega(z) = e_1z+e_2z^2+\cdots$ be in the class $\mathscr{B}$ and let $\lambda\in\mathbb{C}$. Then*

$$\left|e_2-\lambda e_1^2\right| \leq \max\{1,|\lambda|\}. \tag{52}$$

Equality may be attained for $\omega(z) = z^2$ and $\omega(z) = z$.

Theorem 10. *Let $\alpha > 1/2$. If $f\in\mathscr{CC}(\alpha)$ is of the form* (1), *then*

$$\left|a_3-\frac{2}{3}a_2^2\right| \leq 1-\frac{1}{3\alpha}. \tag{53}$$

Proof. Let $f\in\mathscr{CC}(\alpha)$. Then there exists $g\in\mathscr{K}$ such that $f'(z)/g'(z)\prec p_\alpha(z)$, where $p_\alpha(z)$ is given by (4). Let $g(z) = z+b_2z^2+b_3z^3+\cdots$. Define

$$p(z) = \frac{f'(z)}{g'(z)} = 1+p_1z+p_2z^2+\cdots \quad (z\in\mathbb{U}). \tag{54}$$

Since $p\prec p_\alpha$, there exists $\omega(z) = e_1z+e_2z^2+\cdots\in\mathscr{B}$ such that

$$\begin{aligned} p(z) &= p_\alpha(\omega(z)) \\ &= 1+P_1e_1z+\left(P_1e_2+P_2e_1^2\right)z^2+\cdots \quad (z\in\mathbb{U}), \end{aligned} \tag{55}$$

where P_1 and P_2 are given by (6). Combining (54) and (55), after simple calculations, we get

$$a_2 = b_2+\frac{p_1}{2}, \qquad a_3 = b_3+\frac{p_2}{3}+\frac{2}{3}b_2p_1, \tag{56}$$

$$p_1 = P_1e_1, \qquad p_2 = P_1e_2+P_2e_1^2. \tag{57}$$

Substituting (6) and (57) in (56) we obtain

$$\begin{gathered} a_2 = b_2+\left(1-\frac{1}{2\alpha}\right)e_1, \\ a_3 = b_3+\frac{2}{3}\left(2-\frac{1}{\alpha}\right)b_2e_1+\frac{2}{3}\left(2-\frac{1}{\alpha}\right)\left[e_2+\left(1-\frac{1}{\alpha}\right)e_1^2\right] \end{gathered} \tag{58}$$

so that

$$a_3-\frac{2}{3}a_2^2 = \left(b_3-\frac{2}{3}b_2^2\right)+\frac{1}{3}\left(2-\frac{1}{\alpha}\right)\left(e_2-\frac{1}{2\alpha}e_1^2\right). \tag{59}$$

Since $g\in\mathscr{K}$, making use of (50) with $\mu = 2/3$, we have

$$\left|b_3-\frac{2}{3}b_2^2\right| \leq \frac{1}{3}. \tag{60}$$

In virtue of Lemma 9 and taking into account that $\alpha > 1/2$, we get

$$\left|e_2-\frac{1}{2\alpha}e_1^2\right| \leq 1. \tag{61}$$

Combining (59), (60), and (61), we obtain

$$\begin{aligned} \left|a_3-\frac{2}{3}a_2^2\right| &\leq \left|b_3-\frac{2}{3}b_2^2\right|+\frac{1}{3}\left(2-\frac{1}{\alpha}\right)\left|e_2-\frac{1}{2\alpha}e_1^2\right| \\ &\leq \frac{1}{3}+\frac{1}{3}\left(2-\frac{1}{\alpha}\right) = 1-\frac{1}{3\alpha} \end{aligned} \tag{62}$$

and, thus, the proof is completed. □

It is easy to observe that when $\alpha\to\infty$ inequality (53) reduces to $|a_3-(2/3)a_2^2|\leq 1$ which is the same with (51) for $\mu = 2/3$.

Theorem 11. *Let $\alpha > 1/2$ and $\beta > 1/2$. If f of the form* (1) *belongs to the class $\mathscr{CC}(\alpha,\beta)$, then*

$$\left|a_3-\frac{2}{3}a_2^2\right| \leq \frac{1}{3}\left(5-\frac{1}{\alpha}-\frac{1}{\beta}\right). \tag{63}$$

Proof. Since $f \in \mathcal{CC}(\alpha,\beta)$, there exists $g \in \mathcal{CC}(\alpha)$ such that $f'(z)/g'(z) \prec p_\beta(z)$, where $p_\beta(z)$ is given by (4) with β instead of α. Let $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and $p(z)$ defined by

$$p(z) = \frac{f'(z)}{g'(z)} = 1 + p_1z + p_2z^2 + \cdots \quad (z \in \mathbb{U}). \tag{64}$$

From $p \prec p_\beta$ it follows that there exists $\omega(z) = e_1z + e_2z^2 + \cdots \in \mathcal{B}$ such that $p(z) = p_\beta(\omega(z))$.

Using the same method as in the proof of Theorem 10, we obtain

$$a_3 - \frac{2}{3}a_2^2 = \left(b_3 - \frac{2}{3}b_2^2\right) + \frac{1}{3}\left(2 - \frac{1}{\beta}\right)\left(e_2 - \frac{1}{2\beta}e_2^2\right). \tag{65}$$

Since $g \in \mathcal{CC}(\alpha)$, from (53), we have

$$\left|b_3 - \frac{2}{3}b_2^2\right| \le 1 - \frac{1}{3\alpha}. \tag{66}$$

Moreover, for $\beta > 1/2$, we get from Lemma 9 that

$$\left|e_2 - \frac{1}{2\beta}e_2^2\right| \le 1. \tag{67}$$

Combining (65), (66), and (67), the inequality (63) follows. □

6. Bounded Doubly Close-to-Starlike Functions

Let $\alpha > 1/2$. Consider $\mathcal{S}^*\mathcal{S}(\alpha)$ the class of all functions $f \in \mathcal{A}$ for which there exists a function $g \in \mathcal{S}^*$ such that $f/g \prec p_\alpha$, with p_α given by (4), or equivalently

$$\left|\frac{f(z)}{g(z)} - \alpha\right| < \alpha \quad (z \in \mathbb{U}). \tag{68}$$

It is easy to observe that when $\alpha \to \infty$ the class $\mathcal{S}^*\mathcal{S}(\alpha)$ reduces to the class of close-to-starlike functions defined by Reade [12].

For $\alpha > 1/2$ and $\beta > 1/2$ we denote by $\mathcal{S}^*\mathcal{S}(\alpha,\beta)$ the class of functions $f \in \mathcal{A}$ for which there exists a function $g \in \mathcal{S}^*\mathcal{S}(\alpha)$ such that $f/g \prec p_\beta$, with p_β given by (4), or equivalently

$$\left|\frac{f(z)}{g(z)} - \beta\right| < \beta \quad (z \in \mathbb{U}). \tag{69}$$

Connections between the classes $\mathcal{S}^*\mathcal{S}(\alpha)$ and $\mathcal{CC}(\alpha)$ and also between $\mathcal{S}^*\mathcal{S}(\alpha,\beta)$ and $\mathcal{CC}(\alpha,\beta)$ are given in the next theorem.

Theorem 12. *Let $\alpha > 1/2$ and $\beta > 1/2$. Then, the following relationships hold:*

$$f(z) \in \mathcal{CC}(\alpha) \quad iff \ zf'(z) \in \mathcal{S}^*\mathcal{S}(\alpha), \tag{70}$$

$$f(z) \in \mathcal{CC}(\alpha,\beta) \quad iff \ zf'(z) \in \mathcal{S}^*\mathcal{S}(\alpha,\beta), \tag{71}$$

$$f(z) \in \mathcal{S}^*\mathcal{S}(\alpha) \quad iff \ \int_0^z \frac{f(t)}{t}dt \in \mathcal{CC}(\alpha), \tag{72}$$

$$f(z) \in \mathcal{S}^*\mathcal{S}(\alpha,\beta) \quad iff \ \int_0^z \frac{f(t)}{t}dt \in \mathcal{CC}(\alpha,\beta). \tag{73}$$

Proof. It is well known that a function $g(z) \in \mathcal{K}$ if and only if $zg'(z) \in \mathcal{S}^*$.

The definition of the class $\mathcal{CC}(\alpha)$ implies that $f \in \mathcal{CC}(\alpha)$ if and only if there exists $g \in \mathcal{K}$ such that $f'(z)/g'(z) \prec p_\alpha(z)$.

The relation (70) follows from

$$\frac{zf'(z)}{zg'(z)} = \frac{f'(z)}{g'(z)} \prec p_\alpha(z), \quad zg'(z) \in \mathcal{S}^*. \tag{74}$$

In the same way, $f \in \mathcal{CC}(\alpha,\beta)$ if and only if there exists $g \in \mathcal{CC}(\alpha)$ such that $f'(z)/g'(z) \prec p_\beta(z)$. We have

$$\frac{zf'(z)}{zg'(z)} = \frac{f'(z)}{g'(z)} \prec p_\beta(z) \tag{75}$$

and taking into account (70), the relation (71) follows.

The proofs of (72) and (73) are similar and will be omitted. □

The condition (71) of Theorem 12 together with Lemma 3 and (14) shows that the function

$$g(z) = \frac{z(1+\epsilon_1z)(1+\epsilon_2z)}{(1+\epsilon_3z)^2[1-(1-(1/\alpha))\epsilon_1z][1-(1-(1/\beta))\epsilon_2z]} \quad (z \in \mathbb{U}), \tag{76}$$

where $|\epsilon_k| = 1$, $k \in \{1,2,3\}$ belongs to the class $\mathcal{S}^*\mathcal{S}(\alpha,\beta)$ and, thus, this class is nonempty.

Combining Theorem 12 with Theorems 4 and 8 the next properties of the class $\mathcal{S}^*\mathcal{S}(\alpha,\beta)$ can be easily obtained.

Corollary 13. *Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{S}^*\mathcal{S}(\alpha,\beta)$, then*

$$\frac{r(1-r)^2}{(1+r)^2[1+(1-(1/\alpha))r][1+(1-(1/\beta))r]} \le |f(z)| \le \frac{r(1+r)^2}{(1-r)^2[1-(1-(1/\alpha))r][1-(1-(1/\beta))r]} \tag{77}$$

for $z \in \mathbb{U}$ and $|z| = r < 1$.

Corollary 14. *Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{S}^*\mathcal{S}(\alpha, \beta)$ is given by* (1)*, then*

$$\begin{aligned}|a_n| \leq n + \left(4 - \frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{n(n-1)}{2} \\ + \left(2 - \frac{1}{\alpha}\right)\left(2 - \frac{1}{\beta}\right)\frac{n(n-1)(n-2)}{6} \\ (n \in \{2, 3, \ldots\}).\end{aligned} \quad (78)$$

Making use of Theorem 12, we can also obtain an upper bound of $|a_3 - (1/2)a_2^2|$ for functions in the class $\mathcal{S}^*\mathcal{S}(\alpha, \beta)$.

Corollary 15. *Let $\alpha > 1/2$ and $\beta > 1/2$. If $f \in \mathcal{S}^*\mathcal{S}(\alpha, \beta)$ is given by* (1)*, then*

$$\left|a_3 - \frac{1}{2}a_2^2\right| \leq 5 - \frac{1}{\alpha} - \frac{1}{\beta}. \quad (79)$$

Proof. Let $f \in \mathcal{S}^*\mathcal{S}(\alpha, \beta)$. Then, from (73), the function $F(z) = z + b_2z^2 + b_3z^3 + \cdots$ given by

$$F(z) = \int_0^z \frac{f(t)}{t}dt \quad (80)$$

belongs to the class $\mathcal{C}\mathcal{C}(\alpha, \beta)$. Comparing the coefficients of z^2 and z^3 on both sides of the above equality, we obtain

$$a_2 = 2b_2, \qquad a_3 = 3b_3 \quad (81)$$

so that

$$\left|a_3 - \frac{1}{2}a_2^2\right| = \left|3b_3 - 2b_2^2\right| = 3\left|b_3 - \frac{2}{3}b_2^2\right|. \quad (82)$$

Now, the inequality (79) follows as an application of Theorem 11. □

Once again making use of Theorem 12, we have that $f \in \mathcal{S}^*\mathcal{S}(\alpha, \beta)$ if and only if

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zF''(z)}{F'(z)} \quad (z \in \mathbb{U}) \quad (83)$$

for some $F \in \mathcal{C}\mathcal{C}(\alpha, \beta)$. Therefore, a radius of convexity for $\mathcal{C}\mathcal{C}(\alpha, \beta)$ will correspond to a radius of starlikeness for $\mathcal{S}^*\mathcal{S}(\alpha, \beta)$.

The next result follows easily from Theorem 6.

Corollary 16. *Let $\alpha > 1/2$ and $\beta > 1/2$. Suppose that $f \in \mathcal{S}^*\mathcal{S}(\alpha, \beta)$. Then, the function f maps the disk $\{z \in \mathbb{C} : |z| < r_0 < 1\}$ onto a starlike domain, where r_0 is the smallest positive root of* (28) *in Theorem 6.*

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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