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Research Article

A Two-Grid Finite Element Method for a Second-Order Nonlinear Hyperbolic Equation

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We present a two-grid finite element scheme for the approximation of a second-order nonlinear hyperbolic equation in two space dimensions. In the two-grid scheme, the full nonlinear problem is solved only on a coarse grid of size H. The nonlinearities are expanded about the coarse grid solution on the fine gird of size h. The resulting linear system is solved on the fine grid. Some a priori error estimates are derived with the H^1-norm $O(h+H^2)$ for the two-grid finite element method. Compared with the standard finite element method, the two-grid method achieves asymptotically same order as long as the mesh sizes satisfy $h = O(H^2)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary Γ, and consider the initial-boundary value problem for the following second-order nonlinear hyperbolic equation

$$\begin{gathered} u_{tt} - \nabla \cdot (A(u)\nabla u) = f(x,t), \quad x \in \Omega,\ 0 < t \le T, \\ u(x,t) = 0, \quad x \in \Gamma,\ 0 < t \le T, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1}$$

where u_{tt} and u_t denote $\partial^2 u/\partial t^2$ and $\partial u/\partial t$, respectively. $x = (x_1, x_2)$. We assume that $A(u)$ is a symmetric positive definite matrix. $A(u)$ and $A_u(u)$ satisfy the Lipschitz continuous condition with respect to u, where $A_u = \partial A/\partial u$ and

$$|A(u_1) - A(u_2)| \le L|u_1 - u_2|, \tag{2}$$

$$|A_u(u_1) - A_u(u_2)| \le L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}, \tag{3}$$

where L is a positive constant.

Two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems and then use it as the initial guess for the solution on the fine grid. This method involves a nonlinear solution on the coarse grid with grid size H and a linear solution on the fine grid with grid size $h < H$. Two-grid method was first introduced by Xu [1, 2] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial differential equations. Later on, two-grid method was further investigated by many authors. Dawson and Wheeler [3, 4], Chen and Liu [5] have constructed the two-grid method by using finite difference method, mixed finite element method, and piecewise linear finite element method for nonlinear parabolic equations, respectively. Wu and Allen [6] have applied two-grid method combined with mixed finite element method to reaction-diffusion equations. Chen et al. [7–10] have constructed two-grid methods for expanded mixed finite-element solution of semilinear and nonlinear reaction-diffusion equations. Bi and Ginting [11] have studied two-grid finite volume element method for linear and nonlinear elliptic problems. Chen et al. [12], Chen and Liu [13, 14] have studied two-grid methods for semilinear parabolic and second-order hyperbolic equations using finite volume element method.

The finite element analysis for the second-order linear hyperbolic equations was discussed by Dupont [15] and Baker [16]. They have obtained optimal $L^\infty(L^2)$ estimates for the error, $O(h^r)$, using subspaces of piecewise polynomial functions of degree $\le r-1$, for $r \ge 1$. Then Yuan and Wang [17, 18] have studied error estimates for the finite element method of the second-order nonlinear hyperbolic equations and proved the optimal error estimates in the L^2 and H^1 norm. Kumar et al. [19] presented and discussed semidiscrete piecewise linear finite volume approximations for a second-order wave equation and obtained optimal error estimates in L^2, H^1, and L^∞ norms. For second-order hyperbolic equations with a nonlinear reaction term, Chen and Liu [14] have presented a two-grid method using finite volume element method and obtained error estimate in the H^1-norm.

However, as far as we know there is no two-grid finite element convergence analysis for the second-order nonlinear hyperbolic equations (1). In this paper, based on two conforming piecewise linear finite element spaces S_H and S_h on one coarse grid with grid size H and one fine grid with grid size h, respectively, we consider the two-grid finite element discretization techniques for the second-order nonlinear hyperbolic problems. With the proposed techniques, solving the nonlinear problems on the fine-grid space is reduced to solving a linear system on the fine-grid space and a nonlinear system on a much smaller space. This means that solving a nonlinear problem is not much more difficult than solving one linear problem, since $\dim S_H \ll \dim S_h$ and the work for solving the nonlinear problem is relatively negligible. A remarkable fact about this simple approach is, as shown in [1], that the coarse mesh can be quite coarse and still maintain a good accuracy approximation.

The rest of this paper is organized as follows. In Section 2, we describe the finite element scheme for the nonlinear second-order hyperbolic problem (1). Section 3 contains the error estimates for the finite element method. Section 4 is devoted to the two-grid finite element and its error analysis. Throughout this paper, the letter C or with its subscript denotes a generic positive constant which does not depend on the mesh parameters and may be different at its different occurrences.

2. Standard Finite Element Method

We adopt the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_\Omega \sum_{|\alpha| \le s} |D^\alpha u|^p dx \right)^{1/p}, \qquad (4)$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible; that is, $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ of functions vanishing on the boundary Γ.

For the variational formulation we multiply (1) by a smooth function v, which vanishes on Γ and find, after integration over Ω and using Green's formula, that $u(\cdot, t) \in H_0^1(\Omega)$, $0 < t \le T$ such that

$$\begin{gathered} (u_{tt}, v) + a(u; u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \qquad (5)$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot;\cdot,\cdot)$ is defined by

$$a(w; u, v) = \int_\Omega A(w) \nabla u \cdot \nabla v \, dx. \qquad (6)$$

Henceforth, it will be assumed that the problem (5) has a unique solution u, and in the appropriate places to follow, additional conditions on the regularity of u which guarantee the convergence results, will be imposed.

Let $\mathcal{T}_h$ be a quasiuniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \mathcal{T}_h$. With the triangulation $\mathcal{T}_h$, we associate the function space S_h consisting of continuous, piecewise linear functions on $\mathcal{T}_h$, vanishing on Γ; that is,

$$S_h = \left\{ v \in C\left(\overline{\Omega}\right) : v \text{ linear in } K \text{ for each } K \in \mathcal{T}_h, v = 0 \text{ on } \Gamma \right\}. \qquad (7)$$

Using the above assumptions on $\mathcal{T}_h$, it is easy to see that S_h is a finite-dimensional subspace of the Hilbert space $H_0^1(\Omega)$ [20].

Thus, the continuous-time finite element approximation is defined as to find a solution $u_h(t) \in S_h$, $0 < t \le T$, such that

$$\begin{gathered} (u_{h,tt}, v_h) + a(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h, \\ u_h(0) = u_0, \qquad u_{h,t}(0) = u_1, \end{gathered} \qquad (8)$$

where $u_{h,tt} = \partial^2 u_h / \partial t^2$. Since we have discretized only in the space variables, this is referred to as a spatially semidiscrete problem. The existence and uniqueness of the solution of (8) have been proved by Yuan and Wang [17].

3. Error Analysis for the Finite Element Method

To describe the error estimates for the finite element scheme (8), we will give some useful lemmas. In [17, 21] it was shown that the bilinear form $a(\cdot;\cdot,\cdot)$ is symmetric and positive definite and the following lemma was proved, which indicates that the bilinear form $a(\cdot;\cdot,\cdot)$ is continuous and coercive on S_h.

Lemma 1. *For h sufficiently small, there exist two positive constants $C_1, C_2 > 0$ such that, for all $u_h, v_h, w_h \in S_h$, the coercive property*

$$a(w_h; u_h, u_h) \ge C_1 \|u_h\|_1^2 \qquad (9)$$

and the boundedness property

$$|a(w_h; u_h, v_h)| \le C_2 \|u_h\|_1 \|v_h\|_1 \qquad (10)$$

hold true.

Lemma 2. *Let $\widetilde{u} \in S_h$ be the standard Ritz projection such that*

$$a\left(u(x,t);(\widetilde{u}-u)(x,t),v_h\right)=0, \quad \forall v_h \in S_h. \tag{11}$$

Thus $\widetilde{u}$ is the finite element approximation of the solution of the elliptic problem whose exact solution is u. From [21–23], we have

$$\|u-\widetilde{u}\|+h\|u-\widetilde{u}\|_1 \leq Ch^2\|u\|_2, \tag{12}$$

$$\left\|(u-\widetilde{u})_t\right\|+h\left\|(u-\widetilde{u})_t\right\|_1 \leq Ch^2\left\|u_t\right\|_2, \tag{13}$$

for some positive constant C independent of h and u.

And there exists a positive constant C_0 independent of h, such that [21]

$$\|\nabla\widetilde{u}\|_\infty+\left\|\nabla\widetilde{u}_t\right\|_\infty \leq C_0, \quad \text{for } t \leq T. \tag{14}$$

We now turn to describe the estimates for the finite element method. We give the error estimates in the H^1-norm and L^2-norm between the exact solution and the semidiscrete finite element solution.

Theorem 3. *Let u and u_h be the solutions of problem* (1) *and the semidiscrete finite element scheme* (8), *respectively. Under the assumptions given in Section 1, if $u_h(0)=\widetilde{u}_0$ and $u_{h,t}(0)=\widetilde{u}_1$, for $0<t\leq T$, one has*

$$\begin{gathered}\left\|u(t)-u_h(t)\right\|+h\left\|u(t)-u_h(t)\right\|_1 \leq \mathscr{C}h^2, \\ \left\|\left(u(t)-u_h(t)\right)_t\right\| \leq \mathscr{C}h^2,\end{gathered} \tag{15}$$

where $\mathscr{C}=C(\|u\|_{L^2(H^2)},\|u\|_{L^\infty(H^2)},\|u_t\|_{L^2(H^2)},\|u_{tt}\|_{L^2(H^2)})$ independent of h.

Proof. For convenience, let $u-u_h=(u-\widetilde{u})+(\widetilde{u}-u_h)=:\eta+\xi$. Then from (1), (8), and (11), we get the following error equation:

$$\begin{aligned}&\left(\xi_{tt},v_h\right)+a\left(u_h;\xi,v_h\right)\\&\quad=-\left(\eta_{tt},v_h\right)+a\left(u_h;u,v_h\right)-a\left(u;u,v_h\right), \quad \forall v_h \in S_h.\end{aligned} \tag{16}$$

Choosing $v_h=\xi_t$ in (16) and by (11), we get

$$\begin{aligned}&\left(\xi_{tt},\xi_t\right)+a\left(u_h;\xi,\xi_t\right)\\&\quad=-\left(\eta_{tt},\xi_t\right)+a\left(u_h;\widetilde{u},\xi_t\right)-a\left(u;\widetilde{u},\xi_t\right).\end{aligned} \tag{17}$$

For the terms of (17), we have

$$\left(\xi_{tt},\xi_t\right)=\frac{1}{2}\frac{d}{dt}\left(\xi_t,\xi_t\right)=\frac{1}{2}\frac{d}{dt}\left\|\xi_t\right\|^2. \tag{18}$$

$$\begin{aligned}a\left(u_h;\xi,\xi_t\right)&=\int_\Omega A\left(u_h\right)\nabla\xi\cdot\nabla\xi_t dx\\&=\frac{1}{2}\frac{d}{dt}a\left(u_h;\xi,\xi\right)-\frac{1}{2}\int_\Omega\frac{\partial A\left(u_h\right)}{\partial u}\frac{\partial u_h}{\partial t}\nabla\xi\cdot\nabla\xi\,dx.\end{aligned} \tag{19}$$

$$\begin{aligned}&a\left(u_h;\widetilde{u},\xi_t\right)-a\left(u;\widetilde{u},\xi_t\right)\\&\quad=\int_\Omega\left[A\left(u_h\right)-A(u)\right]\nabla\widetilde{u}\cdot\nabla\xi_t dx\\&\quad=\frac{d}{dt}\left(\int_\Omega\left[A\left(u_h\right)-A(u)\right]\nabla\widetilde{u}\cdot\nabla\xi\,dx\right)\\&\qquad-\int_\Omega\left[A\left(u_h\right)-A(u)\right]\nabla\frac{\partial\widetilde{u}}{\partial t}\cdot\nabla\xi\,dx\\&\qquad-\int_\Omega\left(\frac{\partial A\left(u_h\right)}{\partial u}\frac{\partial u_h}{\partial t}-\frac{\partial A(u)}{\partial u}\frac{\partial u}{\partial t}\right)\nabla\widetilde{u}\cdot\nabla\xi\,dx.\end{aligned} \tag{20}$$

Integrating (17) from 0 to t, combining with (18)–(20), and noting that $\xi(0)=0$ and $\xi_t(0)=0$, we have

$$\begin{aligned}&\frac{1}{2}\left\|\xi_t\right\|^2+\frac{1}{2}a\left(u_h;\xi,\xi\right)\\&=-\int_0^t\left(\eta_{tt},\xi_t\right)dt\\&\quad+\frac{1}{2}\int_0^t\left(\int_\Omega\frac{\partial A\left(u_h\right)}{\partial u}\frac{\partial u_h}{\partial t}\nabla\xi\cdot\nabla\xi\,dx\right)dt\\&\quad+\int_\Omega\left[A\left(u_h\right)-A(u)\right]\nabla\widetilde{u}\cdot\nabla\xi\,dx\\&\quad-\int_0^t\left(\int_\Omega\left[A\left(u_h\right)-A(u)\right]\nabla\frac{\partial\widetilde{u}}{\partial t}\cdot\nabla\xi\,dx\right)dt\\&\quad-\int_0^t\left(\int_\Omega\left(\frac{\partial A\left(u_h\right)}{\partial u}\frac{\partial u_h}{\partial t}-\frac{\partial A(u)}{\partial u}\frac{\partial u}{\partial t}\right)\nabla\widetilde{u}\cdot\nabla\xi\,dx\right)dt\\&\equiv\sum_{i=1}^5 T_i.\end{aligned} \tag{21}$$

Now let us estimate the right-hand side terms of (21); for T_1, there is

$$\left|T_1\right|\leq C\int_0^t\left\|\eta_{tt}\right\|\left\|\xi_t\right\|dt\leq C\int_0^t\left(\left\|\eta_{tt}\right\|^2+\left\|\xi_t\right\|^2\right)dt. \tag{22}$$

For T_2, by (2), we obtain

$$\begin{aligned}\left|T_2\right|&\leq C\int_0^t\left|\frac{\partial A\left(u_h\right)}{\partial u}\frac{\partial u_h}{\partial t}\right|_\infty\|\nabla\xi\|^2dt\\&\leq CL\int_0^t\left|\frac{\partial u_h}{\partial t}\right|_\infty\|\nabla\xi\|^2dt\leq C\int_0^t\|\xi\|_1^2dt,\end{aligned} \tag{23}$$

where we used the fact that $|\partial u_h/\partial t|_\infty$ is bounded by a positive constant [17].

For T_3, by (14), Schwarz inequality, and (3), we get

$$\begin{aligned}|T_3| &\leq C\|\nabla\tilde{u}\|_\infty \|A(u_h)-A(u)\| \|\nabla\xi\| \\ &\leq C\|\nabla\tilde{u}\|_\infty L\|\xi+\eta\| \|\xi\|_1 \\ &\leq C\left(\|\eta\|^2+\|\xi\|^2\right)+\epsilon\|\xi\|_1^2 \\ &\leq C\left(\|\eta\|^2+\int_0^t\|\xi_t\|^2 dt\right)+\epsilon\|\xi\|_1^2,\end{aligned} \tag{24}$$

with ϵ being a small positive constant. For T_4, similarly we have

$$\begin{aligned}|T_4| &\leq C\int_0^t \|\nabla\tilde{u}_t\|_\infty \|A(u_h)-A(u)\| \|\nabla\xi\| dt \\ &\leq C\int_0^t \|\nabla\tilde{u}_t\|_\infty L\|\xi+\eta\| \|\xi\|_1 dt \\ &\leq C\int_0^t \left(\|\xi\|^2+\|\eta\|^2+\|\xi\|_1^2\right)dt \\ &\leq C\int_0^t \left(\|\eta\|^2+\|\xi\|_1^2\right)dt.\end{aligned} \tag{25}$$

For T_5, by Lemma 2, we obtain

$$\begin{aligned}|T_5| &\leq C\int_0^t \|\nabla\tilde{u}\|_\infty \left\|\frac{\partial u_h}{\partial t}-\frac{\partial u}{\partial t}\right\| \|\nabla\xi\| dt \\ &\leq C\int_0^t \left(\|\xi_t\|+\|\eta_t\|\right)\|\xi\|_1 dt \\ &\leq C\int_0^t \left(\|\xi_t\|^2+\|\eta_t\|^2+\|\xi\|_1^2\right)dt.\end{aligned} \tag{26}$$

By Lemma 1, from (21)–(26), we get

$$\begin{aligned}&\|\xi_t\|^2+C_0\|\xi\|_1^2 \\ &\quad\leq C_1\left[\int_0^t\left(\|\eta_{tt}\|^2+\|\eta_t\|^2+\|\eta\|^2\right)dt+\|\eta\|^2\right] \\ &\qquad + C_2\int_0^t\left(\|\xi_t\|^2+\|\xi\|_1^2\right)dt+\epsilon\|\xi\|_1^2.\end{aligned} \tag{27}$$

Choosing proper ϵ and kicking the last term into the left-hand side of (27), and applying Gronwall's lemma, for $t \leq T$, we have

$$\begin{aligned}&\|\xi_t\|^2+\|\xi\|_1^2 \\ &\quad\leq C_1\left[\int_0^T\left(\|\eta_{tt}\|^2+\|\eta_t\|^2+\|\eta\|^2\right)dt+\|\eta\|^2\right] \\ &\quad\leq Ch^4\left[\int_0^T\left(\|u_{tt}\|_2^2+\|u_t\|_2^2+\|u\|_2^2\right)dt+\|u\|_2^2\right].\end{aligned} \tag{28}$$

Together with (12) and (13), this yields (15). □

4. Two-Grid Finite Element Method

In this section, we will present a two-grid finite element algorithm for problem (1) based on two different finite element spaces. The idea of the two-grid method is to reduce the nonlinear problem on a fine grid into a linear system on the fine grid by solving a nonlinear problem on a coarse grid. The basic mechanisms are two quasiuniform triangulations of Ω, $\mathcal{T}_H$ and $\mathcal{T}_h$, with two different mesh sizes H and h $(H > h)$, and the corresponding piecewise linear finite element spaces S_H and S_h which will be called the coarse-grid and the fine-grid spaces, respectively.

To solve problem (1), we introduce two-grid algorithms into finite element method. This method involves a nonlinear solution on the coarse grid space and a linear solution on the fine grid space. We present the two-grid finite element method with two steps.

Algorithm 4. Consider the following.

Step 1. On the coarse grid $\mathcal{T}_H$, find $u_H \in S_H$, such that

$$\begin{gathered}(u_{H,tt}, v_H)+a(u_H; u_H, v_H)=(f, v_H), \quad \forall v_H \in S_H, \\ u_H(0)=\tilde{u}_0, \qquad u_{H,t}(0)=\tilde{u}_1.\end{gathered} \tag{29}$$

Step 2. On the fine grid $\mathcal{T}_h$, find $u_h \in S_h$, such that

$$\begin{gathered}(u_{h,tt}, v_h)+a(u_H; u_h, v_h)=(f, v_h), \quad \forall v_h \in S_h, \\ u_h(0)=\tilde{u}_0, \qquad u_{h,t}(0)=\tilde{u}_1.\end{gathered} \tag{30}$$

Now we consider the error estimates in the H^1-norm for the two-grid finite element method Algorithm 4.

Theorem 5. *Let u and u_h be the solutions of problem* (1) *and the two-grid scheme Algorithm 4, respectively. Under the assumptions given in Section 1, if $u_h(0)=\tilde{u}_0$ and $u_{h,t}(0)=\tilde{u}_1$, for $0 < t \leq T$, we have*

$$\|u(t)-u_h(t)\|_1 \leq \mathcal{C}\left(h+H^2\right), \tag{31}$$

where $\mathcal{C} = C(\|u\|_{L^2(H^2)}, \|u\|_{L^\infty(H^2)}, \|u\|_{L^\infty(W^{1,\infty})}, \|u_t\|_{L^2(H^2)}, \|u_t\|_{L^\infty(W^{1,\infty})}, \|u_{tt}\|_{L^2(H^2)})$ independent of h.

Proof. Once again, we set $u-u_h=(u-\tilde{u})+(\tilde{u}-u_h)=:\eta+\xi$ and choose $v_h=\xi_t$. Then for Algorithm 4, we get the error equation

$$\begin{aligned}&(\xi_{tt}, \xi_t)+a(u_H; \xi, \xi_t) \\ &\quad= -(\eta_{tt}, \xi_t)+a(u_H; \tilde{u}, \xi_t)-a(u; \tilde{u}, \xi_t).\end{aligned} \tag{32}$$

Similarly as the proof of Theorem 3, we get

$$\begin{aligned}
&\frac{1}{2}\|\xi_t\|^2+\frac{1}{2}a(u_H;\xi,\xi)\\
&=-\int_0^t(\eta_{tt},\xi_t)\,dt+\frac{1}{2}\int_0^t\left(\int_\Omega\frac{\partial A(u_H)}{\partial u}\frac{\partial u_H}{\partial t}\nabla\xi\cdot\nabla\xi\,dx\right)dt\\
&\quad+\int_\Omega[A(u_H)-A(u)]\nabla\tilde{u}\cdot\nabla\xi\,dx\\
&\quad-\int_0^t\left(\int_\Omega[A(u_H)-A(u)]\nabla\frac{\partial\tilde{u}}{\partial t}\cdot\nabla\xi\,dx\right)dt\\
&\quad-\int_0^t\left(\int_\Omega\left(\frac{\partial A(u_H)}{\partial u}\frac{\partial u_H}{\partial t}-\frac{\partial A(u)}{\partial u}\frac{\partial u}{\partial t}\right)\nabla\tilde{u}\cdot\nabla\xi\,dx\right)dt\\
&\equiv\sum_{i=1}^5T_i'.
\end{aligned}\tag{33}$$

For T_1' and T_2', we can estimate them similarly as in Theorem 3. So our main task is to deal with T_3'–T_5'. By (3), we have

$$\begin{aligned}
\left|T_3'\right|&\le C\|\nabla\tilde{u}\|_\infty\|A(u_H)-A(u)\|\|\nabla\xi\|\\
&\le C\|\nabla\tilde{u}\|_\infty L\|u_H-u\|\|\xi\|_1\\
&\le C\|u_H-u\|^2+\epsilon\|\xi\|_1^2,
\end{aligned}\tag{34}$$

with ϵ being a small positive constant

$$\begin{aligned}
\left|T_4'\right|&\le C\int_0^t\|\nabla\tilde{u}_t\|_\infty\|A(u_H)-A(u)\|\|\nabla\xi\|\,dt\\
&\le C\int_0^t\|\nabla\tilde{u}_t\|_\infty L\|u_H-u\|\|\xi\|_1dt\\
&\le C\int_0^t\left(\|u_H-u\|^2+\|\xi\|_1^2\right)dt,
\end{aligned}\tag{35}$$

$$\begin{aligned}
\left|T_5'\right|&\le C\int_0^t\|\nabla\tilde{u}\|_\infty\left\|\frac{\partial u_H}{\partial t}-\frac{\partial u}{\partial t}\right\|\|\nabla\xi\|\,dt\\
&\le C\int_0^t\|(u_H-u)_t\|\|\xi\|_1dt\\
&\le C\int_0^t\left(\|(u_H-u)_t\|^2+\|\xi\|_1^2\right)dt.
\end{aligned}\tag{36}$$

Substituting the estimates of T_i' in (33) and by Lemma 1, we obtain

$$\begin{aligned}
&\|\xi_t\|^2+C_0\|\xi\|_1^2\\
&\quad\le C_1\int_0^t\left(\|\eta_{tt}\|^2+\|u_H-u\|^2+\|(u_H-u)_t\|^2\right)dt\\
&\qquad+C_2\|u_H-u\|^2\\
&\qquad+C_3\int_0^t\left(\|\xi_t\|^2+\|\xi\|_1^2\right)dt+\epsilon\|\xi\|_1^2.
\end{aligned}\tag{37}$$

Choosing proper ϵ and kicking the last term into the left-hand side of (33), and applying Gronwall's lemma, for $t\le T$, we have

$$\begin{aligned}
&\|\xi_t\|^2+\|\xi\|_1^2\\
&\quad\le C_1\int_0^t\left(\|\eta_{tt}\|^2+\|u_H-u\|^2+\|(u_H-u)_t\|^2\right)dt\\
&\qquad+C_2\|u_H-u\|^2.
\end{aligned}\tag{38}$$

By Theorem 3, we obtain

$$\|\xi_t\|^2+\|\xi\|_1^2\le\mathscr{C}\left(h^4+H^4\right),\tag{39}$$

where $\mathscr{C}=C(\|u\|_{L^2(H^2)},\|u\|_{L^\infty(H^2)},\|u\|_{L^\infty(W^{1,\infty})},\|u_t\|_{L^2(H^2)},\|u_t\|_{L^\infty(W^{1,\infty})},\|u_{tt}\|_{L^2(H^2)})$ independent of h. Thus,

$$\|\xi_t\|+\|\xi\|_1\le\mathscr{C}\left(h^2+H^2\right).\tag{40}$$

By (12) and the triangular inequality, the proof is complete. □

Remark 6. In order to give the fully discrete scheme, we further discretize time t of the semidiscrete two-grid finite element method in this section. We consider a time step Δt and approximate the solutions at $t^n=n\Delta t$, $\Delta t=T/N$, $n=0,1,\dots,N$. Denote $u_h^n=u_h(t^n)$, $u_{h,tt}^n=(u_h^{n+1}-2u_h^n+u_h^{n-1})/(\Delta t)^2$, $u_{h,t}^n=(u_h^{n+1}-u_h^n)/\Delta t$, we can get the fully discrete two-grid finite element scheme for (1). For simplicity and convenience, we only give the fully discrete scheme for Algorithm 4.

Algorithm 4′. Consider the following.

Step 1. On the coarse grid $\mathscr{T}_H$, find $u_H^n\in S_H$ ($n=1,2,\dots$), such that

$$\begin{gathered}
\left(u_{H,tt}^n,v_H\right)+a\left(u_H^{n+1};u_H^{n+1},v_H\right)=\left(f^{n+1},v_H\right),\quad\forall v_H\in S_H,\\
u_H^0=\tilde{u}_0,\qquad u_{H,t}^0=\tilde{u}_1.
\end{gathered}\tag{41}$$

Step 2. On the fine grid $\mathscr{T}_h$, find $u_h^n\in S_h$ ($n=1,2,\dots$), such that

$$\begin{gathered}
\left(u_{h,tt}^n,v_h\right)+a\left(u_H^{n+1};u_h^{n+1},v_h\right)=\left(f^{n+1},v_h\right),\quad\forall v_h\in S_h,\\
u_h^0=\tilde{u}_0,\qquad u_{h,t}^0=\tilde{u}_1.
\end{gathered}\tag{42}$$

We can get the same kind of estimate as Theorem 5 with the result $\|u^n-u_h^n\|_1\le\mathscr{C}((\Delta t)^2+h+H^2)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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