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Research Article

On the Oscillation of Even-Order Half-Linear Functional Difference Equations with Damping Term

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We investigate the oscillatory behavior of solutions of the mth order half-linear functional difference equations with damping term of the form $\Delta[p_n Q(\Delta^{m-1} y_n)] + q_n Q(\Delta^{m-1} y_n) + r_n Q(y_{\tau_n}) = 0$, $n \geq n_0$, where m is even and $Q(s) = |s|^{\alpha-2}s$, $\alpha > 1$ is a fixed real number. Our main results are obtained via employing the generalized Riccati transformation. We provide two examples to illustrate the effectiveness of the proposed results.

1. Introduction

Consider the second order half-linear difference equation:

$$\Delta\left[p_n|\Delta y_n|^{\alpha-2}\Delta y_n\right] + r_n|y_{n+1}|^{\alpha-2}y_{n+1} = 0, \quad n \geq n_0,\ \alpha > 1, \tag{1}$$

where Δ is the forward difference operator and $\{p_n\}$, $\{r_n\}$ are sequences of nonnegative real numbers with $\{p_n\} > 0$. The study of (1) has been initiated by Rehák in [1]. It is well known that there is a close similarity between (1) and the linear second order difference equation. Indeed, if $\{y_n\}$ is a solution of (1), then so is $\{cy_n\}$ for any constant c. Thus, (1) has one half of linearity properties [2].

In the presence of damping, (1) has been extended further to the second order half-linear difference equation with damping term of the form

$$\Delta\left[p_n|\Delta y_n|^{\alpha-2}\Delta y_n\right] + q_n|\Delta y_n|^{\alpha-2}\Delta y_n + r_n|y_{n+1}|^{\alpha-2}y_{n+1} = 0, \quad n \geq n_0 \in \mathbb{N}, \tag{2}$$

where $\{q_n\}$ is a sequence of nonnegative real numbers. It is to be noted that neither (1) nor (2) has involved a delaying term. There are numerous numbers of oscillation criteria established in the literature for the solutions of (1) and (2). Most of these results were obtained by using certain efficient tools among them we name the Riccati transformation, variational principle, and some inequality techniques; see, for instance, the monograph [3] in which many contributions have been cited therein and to the recent papers [4–9].

Let $Q : \mathbb{R} \to \mathbb{R}$ be defined by $Q(s) = |s|^{\alpha-2}s$; $\alpha > 1$ is a fixed real number and $\mathbb{N}_{n_0} = \{n_0, n_0+1, \ldots\}$. Consider the mth order half-linear functional difference equation with damping term of the form

$$\Delta\left[p_n Q\left(\Delta^{m-1} y_n\right)\right] + q_n Q\left(\Delta^{m-1} y_n\right) + r_n Q\left(y_{\tau_n}\right) = 0, \quad n \in \mathbb{N}_{n_0}, \tag{3}$$

where m is even number, and

(H1) $\{p_n\} : \mathbb{N}_{n_0} \to \mathbb{R}^+$ with $\Delta p_n \geq 0$ for all $n \geq n_0$;
(H2) $\{q_n\}$ and $\{r_n\} : \mathbb{N}_{n_0} \to \mathbb{R}$ with $q_n \geq 0$ and $r_n > 0$;
(H3) $\{\tau_n\} : \mathbb{N}_{n_0} \to \mathbb{Z}$ with $\tau_n < n$ and $\lim_{n\to\infty}\tau_n = \infty$.

For close results regarding the continuous counterparts of (1), (2), and (3), the reader is suggested to consult [10–14].

A primary purpose of this paper is to establish sufficient conditions that guarantee the oscillation of solutions of (3). Our main results are obtained via employing the generalized

Riccati transformation. In view of (3), one can easily figure out that it is formulated in more general form so that it includes some particular cases which have been studied in the literature; see [15–23] for more details. To the best of authors' observation, however, no published result has been concerned with the investigation of oscillatory behavior of solutions of (3) or its continuous counterpart. Therefore, our paper is new and presents a new approach.

2. Main Results

We start by recalling the following standard definitions.

Definition 1. A nontrivial sequence y_n is called a solution of (3) if it is defined for all $n \geq \sigma$ where $n \in \mathbb{Z}$, $\sigma = \min_{i \geq n_0}\{\tau_i\}$, and $p_n Q(\Delta^{m-1} y_n)$ is differenceable on $\mathbb{N}_{n_0}$ and satisfies (3) for all $n \in \mathbb{N}_{n_0}$.

Definition 2. A nontrivial solution y_n of (3) is said to be oscillatory if the terms of the sequence y_n are not eventually positive or not eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all its solutions oscillate.

To obtain our main results, we need the following essential lemmas. The first of these is the discrete analogue of the well-known Kiguradze's lemma.

Lemma 3 (see [24]). *Let y_n be defined for $n \geq n_0 \in \mathbb{N}$ and $y(n) > 0$ with $\Delta^m y_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer l, $0 \leq l \leq m$ with $(m+l)$ odd for $\Delta^m y_n \leq 0$ and $(m+l)$ even for $\Delta^m y(n) \geq 0$ such that*

(i) *$l \leq m-1$ implies $(-1)^{l+i}\Delta^i y_n > 0$ for all $n \geq n_0$, $l \leq i \leq m-1$,*

(ii) *$l \geq 1$ implies $\Delta^i y_n > 0$ for all large $n \geq n_0$, $1 \leq i \leq l-1$.*

Lemma 4 (see [25]). *Let y_n be defined for $n \geq n_0$ and $y_n > 0$ with $\Delta^m y_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that*

$$y_n \geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1} y_{2^{m-l-1}n}, \quad n \geq n_1, \tag{4}$$

where l is defined as in Lemma 3. Further, if y_n is increasing, then

$$y_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1} y_n, \quad n \geq 2^{m-1}n_1. \tag{5}$$

Lemma 5. *Let y_n satisfy conditions of Lemmas 3 and 4 and $\Delta^{m-1} y_n \Delta^m y_n \leq 0$ for $n \geq n_1 \geq n_0$. Further, if y_n is increasing, then*

$$\Delta y_{n-k} \geq M n^{m-2}\Delta^{m-1} y_n, \quad n \geq n_1, \tag{6}$$

where $M = (1/((m-1)!2^{(m-1)^2})) > 0$.

The proof of Lemma 5 is straightforward and it can be achieved by using the last inequality of Lemma 4.

Lemma 6. *Let y_n be an eventually positive solution of (3). If*

$$(\Lambda 1) \quad \lim_{n\to\infty}\sum_{s=n_1}^{n-1}\left(\frac{1}{p_s}\left(1-\prod_{k=n_1}^{s-1}\left(1-\frac{q_k}{p_k}\right)\right)\right)^{1/(\alpha-1)} = \infty, \tag{7}$$

then $\Delta^{m-1} y_n > 0$, $\Delta^m y_n \leq 0$, and $\Delta y_n > 0$ for all $n \geq n_1 \geq n_0$.

Proof. The fact that y_n is eventually positive solution of (3) implies $y_n > 0$ and $y_{\tau_n} > 0$ for all $n \geq n_1 \geq n_0$. In view of (3), we get

$$\Delta\left[p_n Q\left(\Delta^{m-1} y_n\right)\right] + q_n Q\left(\Delta^{m-1} y_n\right) < 0, \tag{8}$$

which leads to

$$\Delta\left[p_n Q\left(\Delta^{m-1} y_n\right) - \prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1} Q\left(\Delta^{m-1} y_{n_1}\right)\right] < 0. \tag{9}$$

Hence,

$$p_n Q\left(\Delta^{m-1} y_n\right) - \prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1} Q\left(\Delta^{m-1} y_{n_1}\right) \tag{10}$$

is decreasing and $\Delta^{m-1} y_n$ is eventually positive or eventually negative.

We claim that

$$\Delta^{m-1} y_n > 0, \quad n \geq n_1. \tag{11}$$

Assume, on the contrary, that $\Delta^{m-1} y_n < 0$, $n \geq n_1$. Then, from (10), we obtain

$$\begin{aligned} & p_n\left|\Delta^{m-1} y_n\right|^{\alpha-2}\Delta^{m-1} y_n \\ &\quad - \prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1} \\ &\leq p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1} \\ &\quad - \prod_{k=n_1}^{n_1-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1}, \end{aligned} \tag{12}$$

where $\prod_{k=n_1}^{n_1-1}(1-(q_k/p_k))p_{n_1}|\Delta^{m-1} y_{n_1}|^{\alpha-2}\Delta^{m-1} y_{n_1} = 0$. Therefore, from (12), we have

$$\begin{aligned} & p_n\left|\Delta^{m-1} y_n\right|^{\alpha-2}\Delta^{m-1} y_n \\ &\quad - \prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1} \\ &\leq p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1} \\ &\quad - \prod_{k=n_1}^{n_1-1}\left(1-\frac{q_k}{p_k}\right)p_{n_1}\left|\Delta^{m-1} y_{n_1}\right|^{\alpha-2}\Delta^{m-1} y_{n_1} \\ &\equiv -M_1^{\alpha-1}, \end{aligned} \tag{13}$$

where $M_1 = p_{n_1}^{1/(\alpha-1)}|\Delta^{m-1}y_{n_1}| > 0$. It follows that

$$\left(-\Delta^{m-1}y_n\right)^{\alpha-1} \geq \frac{M_1^{\alpha-1}}{p_n}\left(1-\prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)\right) \tag{14}$$

or

$$\Delta^{m-1}y_n \leq -M_1\left(\frac{1}{p_n}\left(1-\prod_{k=n_1}^{n-1}\left(1-\frac{q_k}{p_k}\right)\right)\right)^{1/(\alpha-1)}. \tag{15}$$

Consequently, we obtain

$$\Delta^{m-2}y_n \leq \Delta^{m-2}y_{n_1} - M_1\sum_{s=n_1}^{n-1}\left(\frac{1}{p_s}\left(1-\prod_{k=n_1}^{s-1}\left(1-\frac{q_k}{p_k}\right)\right)\right)^{1/(\alpha-1)}. \tag{16}$$

Letting $n \to \infty$ in the above inequality, one gets $\lim_{n\to\infty}\Delta^{m-2}y_n = -\infty$. Hence, y_n is an eventually negative function which contradicts that $y_n > 0$. Therefore, inequality (11) holds.

From (3), we get

$$\Delta\left[p_n Q\left(\Delta^{m-1}y_n\right)\right] = p_{n+1}\Delta\left(\Delta^{m-1}y_n\right)^{\alpha-1} + \left(\Delta^{m-1}y_n\right)^{\alpha-1}\Delta p_n \leq 0 \tag{17}$$

from which it follows that

$$\Delta\left(\Delta^{m-1}y_n\right)^{\alpha-1} \leq 0, \quad n \geq n_1 \geq n_0. \tag{18}$$

The above inequality implies that $(\Delta^{m-1}y_n)^{\alpha-1}$ is nonincreasing. Therefore, we can write

$$\begin{aligned}
\Delta\left(\Delta^{m-1}y_n\right)^{\alpha-1} &= \left(\Delta^{m-1}y_{n+1}\right)^{\alpha-1} - \left(\Delta^{m-1}y_n\right)^{\alpha-1} \\
&= \left[\Delta^{m-1}y_{n+1} - \Delta^{m-1}y_n\right] \\
&\quad \times \left[\left(\Delta^{m-1}y_{n+1}\right)^{\alpha-2} + \left(\Delta^{m-1}y_{n+1}\right)^{\alpha-3}\left(\Delta^{m-1}y_n\right)\right. \\
&\quad + \left(\Delta^{m-1}y_{n+1}\right)^{\alpha-4}\left(\Delta^{m-1}y_n\right)^2 \\
&\quad \left.+ \cdots + \left(\Delta^{m-1}y_n\right)^{\alpha-2}\right] \leq 0.
\end{aligned} \tag{19}$$

Since $(\Delta^{m-1}y_n)^{\alpha-1}$ is nonincreasing and positive, then from the above inequality, we have

$$\begin{aligned}
\Delta\left(\Delta^{m-1}y_n\right)^{\alpha-1} &\leq \left[\Delta^{m-1}y_{n+1} - \Delta^{m-1}y_n\right](\alpha-1)\left(\Delta^{m-1}y_n\right)^{\alpha-2} \\
&= (\alpha-1)\Delta\left(\Delta^{m-1}y_n\right)\left(\Delta^{m-1}y_n\right)^{\alpha-2} \\
&\leq (\alpha-1)\Delta^m y_n\left(\Delta^{m-1}y_n\right)^{\alpha-2} \leq 0
\end{aligned} \tag{20}$$

by which we have

$$\Delta^m y_n \leq 0. \tag{21}$$

In virtue of (21) and Lemma 3, we deduce that since m is even then l is odd. Hence $\Delta y_n > 0$ for $n \geq n_1 \geq n_0$. The proof is complete. □

Theorem 7. *Let condition* (Λ1) *hold. Further, assume that there exists a constant $\lambda > \alpha - 1$ such that*

$$(\Lambda 2)\quad \limsup_{n\to\infty}\frac{1}{(n-n_1)^\lambda}\sum_{k=n_1}^{n-1}\left[(n-k)^\lambda r_k - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}X_{nk}^\alpha Y_{nk}^{1-\alpha}\right] = \infty, \tag{22}$$

where

$$\begin{aligned}
X_{nk} &= \left((n+1-k)^\lambda - (n-k)^\lambda\left(1+\frac{q_k}{p_{k+1}}\right)\right), \\
Y_{nk} &= (\alpha-1)M\tau_k^{m-2}(n-k)^\lambda\frac{p_k}{p_{k+1}} > 0,
\end{aligned} \tag{23}$$

and M is as in Lemma 5. Then, (3) *is oscillatory.*

Proof. For the sake of contradiction, assume that (1) has a nonoscillatory solution y_n. Without loss of generality, we assume that y_n is eventually positive (the proof is similar when y_n is eventually negative). That is, $y_n > 0, y_{\tau_n} > 0$ and $y_{\tau_{n-k}} > 0$ for all $n \geq n_1 \geq n_0$. By Lemma 6, we have $\Delta^{m-1}y_n > 0$, $\Delta^m y_n \leq 0$, and $\Delta y_n > 0$ for $n \geq n_1$. Consider the function

$$w_n = \frac{p_n Q\left(\Delta^{m-1}y_n\right)}{Q\left(y_{\tau_{n-k}}\right)} = p_n\left(\frac{\Delta^{m-1}y_n}{y_{\tau_{n-k}}}\right)^{\alpha-1} > 0, \quad n \geq n_1. \tag{24}$$

Taking into account that $\Delta y_n > 0$ and y_n is increasing and $\tau_{n-k} < \tau_n$, we deduce that $\Delta^m y_n \leq 0$ and $\Delta^{m-1}y_n$ is nonincreasing. Lemmas 3 and 4, (1), and (24) yield

$$\begin{aligned}
\Delta w_n &= \frac{-q_n\left(\Delta^{m-1}y_n\right)^{\alpha-1} - r_n\left(y_{\tau_n}\right)^{\alpha-1}}{\left(y_{\tau_{n+1-k}}\right)^{\alpha-1}} - \frac{p_n\left(\Delta^{m-1}y_n\right)^{\alpha-1}\Delta\left(y_{\tau_{n-k}}\right)^{\alpha-1}}{\left(y_{\tau_{n-k}}\right)^{\alpha-1}\left(y_{\tau_{n+1-k}}\right)^{\alpha-1}} \\
&\leq -r_n - \frac{q_n}{p_{n+1}}w_{n+1} - \frac{(\alpha-1)p_n\left(\Delta^{m-1}y_n\right)^{\alpha-1}\left(y_{\tau_{n-k}}\right)^{\alpha-2}\Delta y_{\tau_{n-k}}}{\left(y_{\tau_{n-k}}\right)^{\alpha-1}\left(y_{\tau_{n+1-k}}\right)^{\alpha-1}}
\end{aligned}$$

$$\begin{aligned}
&\le -r_n - \frac{q_n}{p_{n+1}} w_{n+1} \\
&\quad - \frac{(\alpha-1)\, p_n\left(\Delta^{m-1} y_n\right)^{\alpha-1}\left(y_{\tau_{n-k}}\right)^{\alpha-2} M\tau_n^{m-2}\Delta^{m-1} y_{\tau_n}}{\left(y_{\tau_{n-k}}\right)^{\alpha-1}\left(y_{\tau_{n+1-k}}\right)^{\alpha-1}} \\
&\le -r_n - \frac{q_n}{p_{n+1}} w_{n+1} - \frac{(\alpha-1)\, p_n\left(\Delta^{m-1} y_n\right)^{\alpha} M\tau_n^{m-2}}{\left(y_{\tau_{n+1-k}}\right)^{\alpha}} \\
&\le -r_n - \frac{q_n}{p_{n+1}} w_{n+1} - \frac{(\alpha-1)\, M\tau_n^{m-2} p_n\left(\Delta^{m-1} y_{n+1}\right)^{\alpha}}{\left(y_{\tau_{n+1-k}}\right)^{\alpha}} \\
&= -r_n - \frac{q_n}{p_{n+1}} w_{n+1} \\
&\quad - \frac{p_n\,(\alpha-1)\, M\tau_n^{m-2} p_{n+1}\left(\Delta^{m-1} y_{n+1}\right)^{\alpha}}{p_{n+1}\left(y_{\tau_{n+1-k}}\right)^{\alpha}} \\
&= -r_n - \frac{q_n}{p_{n+1}} w_{n+1} - (\alpha-1)\, M\tau_n^{m-2}\frac{p_n}{p_{n+1}} w_{n+1}^{\alpha/(\alpha-1)}.
\end{aligned} \tag{25}$$

Multiplying by $(n-k)^\lambda$ and summing up from n_1 to $n-1$, we obtain

$$\begin{aligned}
\sum_{k=n_1}^{n-1}(n-k)^\lambda r_k &\le -\sum_{k=n_1}^{n-1}(n-k)^\lambda \Delta w_k \\
&\quad - \sum_{k=n_1}^{n-1}(n-k)^\lambda \frac{q_k}{p_{k+1}} w_{k+1} \\
&\quad - \sum_{k=n_1}^{n-1}(\alpha-1)\, M\tau_k^{m-2}(n-k)^\lambda \frac{p_k}{p_{k+1}} w_{k+1}^{\alpha/(\alpha-1)} \\
&\le (n-n_1)^\lambda w_{n_1} - w_n \\
&\quad + \sum_{k=n_1}^{n-1}\Bigg((n+1-k)^\lambda \\
&\qquad -(n-k)^\lambda\left(1+\frac{q_k}{p_{k+1}}\right)\Bigg) w_{k+1} \\
&\quad - \sum_{k=n_1}^{n-1}(\alpha-1)\, M\tau_k^{m-2}(n-k)^\lambda \frac{p_k}{p_{k+1}} w_{k+1}^{\alpha/(\alpha-1)} \\
&\le (n-n_1)^\lambda w_{n_1} \\
&\quad + \sum_{k=n_1}^{n-1}\Bigg((n+1-k)^\lambda \\
&\qquad -(n-k)^\lambda\left(1+\frac{q_k}{p_{k+1}}\right)\Bigg) w_{k+1} \\
&\quad - \sum_{k=n_1}^{n-1}(\alpha-1)\, M\tau_k^{m-2}(n-k)^\lambda \frac{p_k}{p_{k+1}} w_{k+1}^{\alpha/(\alpha-1)}
\end{aligned} \tag{26}$$

or

$$\begin{aligned}
&\frac{1}{(n-n_1)^\lambda}\left[\sum_{k=n_1}^{n-1}(n-k)^\lambda r_k - \sum_{k=n_1}^{n-1}\left(X_{nk} w_{k+1} - Y_{nk} w_{k+1}^{\alpha/(\alpha-1)}\right)\right] \\
&\quad \le w_{n_1},
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
X_{nk} &= \left((n+1-k)^\lambda - (n-k)^\lambda\left(1+\frac{q_k}{p_{k+1}}\right)\right), \\
Y_{nk} &= (\alpha-1)\, M\tau_k^{m-2}(n-k)^\lambda \frac{p_k}{p_{k+1}} > 0.
\end{aligned} \tag{28}$$

Let

$$F\left(w_{k+1}\right) = X_{nk} w_{k+1} - Y_{nk} w_{k+1}^{\alpha/(\alpha-1)}. \tag{29}$$

Then, F has maximum value at $w_{k+1} = ((\alpha-1)/\alpha)^{\alpha-1} X_{nk}^{\alpha-1} Y_{nk}^{1-\alpha}$. That is,

$$F_{\max} = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} X_{nk}^\alpha Y_{nk}^{1-\alpha}. \tag{30}$$

Therefore, (27) can be rewritten as

$$\frac{1}{(n-n_1)^\lambda}\sum_{k=n_1}^{n-1}\left[(n-k)^\lambda r_k - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} X_{nk}^\alpha Y_{nk}^{1-\alpha}\right] \le w_{n_1}. \tag{31}$$

Hence, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{(n-n_1)^\lambda}\sum_{k=n_1}^{n-1}\left[(n-k)^\lambda r_k - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} X_{nk}^\alpha Y_{nk}^{1-\alpha}\right] \\
&\quad \le w_{n_1}
\end{aligned} \tag{32}$$

which contradicts condition (Λ2). The proof is complete. □

Theorem 8. *Let condition* (Λ1) *hold. Further, assume that there exists a function* $\delta_n : \mathbb{N} \to \mathbb{R}^+$ *such that*

$$\begin{aligned}
(\Lambda 3)\quad &\limsup_{n\to\infty}\sum_{k=n_1}^{n-1}\left[\delta_{k+1} r_k - \frac{1}{\alpha^\alpha}\left(\frac{\Delta\delta_k}{\delta_k}\right)^\alpha \left(M\tau_k^{m-2}\right)^{1-\alpha}\right] \\
&\quad = \infty, \quad n_1 \ge n_0,
\end{aligned} \tag{33}$$

where M *is as in Lemma 5. Then,* (3) *is oscillatory.*

Proof. For the sake of contradiction, assume that (3) has a nonoscillatory solution y_n. Without loss of generality, we assume that y_n is eventually positive (the proof is similar when y_n is eventually negative). That is, $y_n > 0$, $y_{\tau_n} > 0$ and $y_{\tau_{n-k}} > 0$ for all $n \ge n_1 \ge n_0$. By Lemma 6, we have

$\Delta^{m-1} y_n > 0$, $\Delta^m y_n \le 0$, and $\Delta y_n > 0$ for $n \ge n_1$. Consider the function

$$w_n = \delta_n p_n \left(\frac{\Delta^{m-1} y_n}{y_{\tau_{n-k}}} \right)^{\alpha-1} > 0, \quad n \ge n_1. \tag{34}$$

By utilizing the same approach as in the proof of Theorem 7, we arrive at

$$\Delta w_n \le -\delta_{n+1} r_n + \frac{1}{\alpha^\alpha} \left(\frac{\Delta \delta_n}{\delta_n} \right)^\alpha \left(M \tau_n^{m-2} \right)^{1-\alpha}. \tag{35}$$

Summing up (35) from n_1 to $n-1$, we have

$$\sum_{k=n_1}^{n-1} \left[\delta_{k+1} r_k - \frac{1}{\alpha^\alpha} \left(\frac{\Delta \delta_k}{\delta_k} \right)^\alpha \left(M \tau_k^{m-2} \right)^{1-\alpha} \right] \le w_{n_1}. \tag{36}$$

Letting $n \to \infty$ in the above inequality and taking the upper limit, we get a contradiction to (Λ3). The proof is complete. □

Remark 9. In view of the statements of Theorems 7 and 8, one can easily deduce that condition (Λ3) is a generalization of (Λ2).

Example 10. Consider the fourth order half-linear functional difference equation with damping

$$\Delta \left[n \left(\Delta^3 y_n \right)^2 \right] + n \left(\Delta^3 y_n \right)^2 + \frac{1}{n} y_{n-1}^2 = 0, \quad n \ge 2, \tag{37}$$

where $p_n = n$, $q_n = n$, $r_n = 1/n$, $\tau_n = n-1$, $m = 4$, and $\alpha = 3$. It is easy to see that conditions (H1)–(H3) are satisfied. It remains to check the validity of conditions Λ1 and Λ2.

For $n \ge 2$, we have

$$\Gamma 1 := \sum_{s=n_1}^{n-1} \left(\frac{1}{p_s} \left(1 - \prod_{\nu=n_1}^{s-1} \left(1 - \frac{q_\nu}{p_\nu} \right) \right) \right)^{1/(\alpha-1)} = \sum_{s=2}^{n-1} \left(\frac{1}{s} \right)^{1/2}. \tag{38}$$

It is clear that $\Gamma 1 \to \infty$ as $n \to \infty$. Therfore, condition (Λ1) holds. For $n \ge 2$ and $\lambda = 3 > \alpha - 1 = 2$, we have

$$\begin{aligned} \Gamma 2 &:= \frac{1}{(n-n_1)^\lambda} \sum_{k=n_1}^{n-1} \left[(n-k)^\lambda r_k - \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} X_{nk}^\alpha Y_{nk}^{1-\alpha} \right] \\ &= \frac{1}{(n-2)^3} \sum_{k=2}^{n-1} \left[\frac{(n-k)^3}{k} - \frac{4}{27} X_{nk}^3 Y_{nk}^2 \right], \end{aligned} \tag{39}$$

where

$$\begin{aligned} X_{nk}^3 &= \left[\frac{(k+1)(n+1-k)^3 - (2k+1)(n-k)^3}{k+1} \right]^3, \\ Y_{nk}^2 &= \frac{(k+1)^2}{4M^2 k^2 (k-2)^8 (n-k)^6}. \end{aligned} \tag{40}$$

It is clear that $\Gamma 2 \to \infty$ as $n \to \infty$. Then, condition (Λ2) holds. Thus, by the conclusion of Theorem 7, (37) is oscillatory.

Example 11. Consider the sixth order half-linear functional difference equation with damping

$$\Delta \left[n \left(\Delta^5 y_n \right)^2 \right] + n \left(\Delta^5 y_n \right)^2 + n^2 y_{n-1}^2 = 0, \quad n \ge 2, \tag{41}$$

where $p_n = n$, $q_n = n$, $r_n = n^2$, $\tau_n = n-1$, $m = 6$, and $\alpha = 3$. It is easy to see that conditions (H1)–(H3) are satisfied. In Example 10, we have seen that (Λ1) is satisfied. It remains to check the validity of condition (Λ3).

For $n \ge 2$ and $\delta_n = n$, we have

$$\begin{aligned} \Gamma 3 &:= \sum_{k=n_1}^{n-1} \left[\delta_{k+1} r_k - \frac{1}{\alpha^\alpha} \left(\frac{\Delta \delta_k}{\delta_k} \right)^\alpha \left(M \tau_k^{m-2} \right)^{1-\alpha} \right] \\ &= \sum_{k=2}^{n-1} \left[k^2 (k+1) - \frac{1}{27 M^2 k^3 (k-1)^8} \right] \\ &= \sum_{k=2}^{n-1} \left[\frac{27 M^2 k^5 (k+1)(k-1)^8 - 1}{27 M^2 k^3 (k-1)^8} \right]. \end{aligned} \tag{42}$$

It is clear that $\Gamma 3 \to \infty$ as $n \to \infty$. Then, condition (Λ3) holds. Thus, by the conclusion of Theorem 8, (41) is oscillatory.

Remark 12. It is not possible to decide the oscillatory behavior of solutions of (37) and (41) by using any of the results reported in [12, 13]. This implies that the results of our paper extend and generalize some known theorems.

Remark 13. The main results of this paper remain valid for nondelay difference equations of the form

$$\begin{gathered} \Delta \left[p_n Q \left(\Delta^{m-1} y_n \right) \right] + q_n Q \left(\Delta^{m-1} y_n \right) + r_n Q \left(y_n \right) = 0, \\ n \in \mathbb{N}_{n_0}. \end{gathered} \tag{43}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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