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Research Article

Existence of Periodic Solutions and Stability of Zero Solution of a Mathematical Model of Schistosomiasis

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A mathematical model on schistosomiasis governed by periodic differential equations with a time delay was studied. By discussing boundedness of the solutions of this model and construction of a monotonic sequence, the existence of positive periodic solution was shown. The conditions under which the model admits a periodic solution and the conditions under which the zero solution is globally stable are given, respectively. Some numerical analyses show the conditional coexistence of locally stable zero solution and periodic solutions and that it is an effective treatment by simply reducing the population of snails and enlarging the death ratio of snails for the control of schistosomiasis.

1. Introduction

Schistosomiasis, a disease caused by a wormlike parasite, is one of the most prevalent parasitic diseases in the tropical and subtropical regions of the developing world. In China, despite remarkable achievements in schistosomiasis control over the past five decades, the disease still remains a major public health concern. It mainly prevails in 381 counties (cities, districts) of 12 provinces, autonomous regions, and municipalities along and to the south of the Yangtze River valley and caused a number of above 100 million victims (Chen, 2008 [1]).

The mathematical models in schistosomiasis appeared in the 1960s (Macdonald, 1965 [2], Hairston, 1965 [3]). Since then, a number of mathematical models of the transmission dynamics of schistosomes have been developed (Williams et al., 2002 [4], Liang et al., 2005 [5], and references therein).

Schistosomiasis is a serious infectious and parasitic disease transmitted through the medium of water. Male and female helminthes must mate in a host (e.g., humans, ducks, etc.). Thereafter, some of fertilized eggs leave the host in its feces. Upon contact with fresh water, it hatches and attempts to penetrate a snail. Once a snail is infected, a large number of larvae are produced and swim freely in search of a host for reproduction. It might penetrate the skin of a host or be ingested with water or food grown in the water (Lucas, 1983 [6], Hoppensteadt, and Peskin, 1992 [7]). According to the process above Lucas, 1983 [6], provided a modified version of MacDonald's model:

$$\begin{aligned} \frac{di}{dt} &= -i\delta + C(m) B(S-i), \\ \frac{dm}{dt} &= -rm + i\frac{A}{S}, \end{aligned} \tag{1}$$

where (also see Wu, 2005 [8]) m is the mean number of worms upon each host and i is the number of infected snails. The constants r and δ are death probabilities of worms and snails, and S is the number (fixed) of snails. The A indicates the ratio of infection caused in final host population by a snail per unit time, while the B means the ratio of infection in snail population caused by a worm per unit time. The $C(m)$ represents the probability of a snail infected once per time unit. If we suppose that $C(m) = m^2/(m+1)$ (Hoppensteadt and Peskin 1992 [7]) and consider that there is a delay $T \geq 0$

due to the development period of helminthes, then there is the following system:

$$\frac{dx(t)}{dt} = -rx(t) + \frac{A}{S}y(t),$$
$$\frac{dy(t)}{dt} = -\delta y(t) + B(S - y(t))\frac{x^2(t-T)}{1+x(t-T)}. \tag{2}$$

In fact, as reported by Lu et al., 2005 [9], the number of snails S and the death ratio of snails δ are related to time; for example,

$$\begin{aligned} S(t) &= 6.060 + 0.750\cos\frac{\pi t}{6} - 2.897\cos\frac{2\pi t}{6} \\ &\quad + 4.036\sin\frac{\pi t}{6} - 0.441\sin\frac{2\pi t}{6}, \\ \delta(t) &= 0.048 - 0.030\cos\frac{\pi t}{12} - 0.006\cos\frac{2\pi t}{12} \\ &\quad - 0.047\sin\frac{\pi t}{12} + 0.021\sin\frac{2\pi t}{12}, \end{aligned} \tag{3}$$

which is generated by fitting the data reported in Lu et al., 2005 [9]. Therefore, the system (2) might be modified by

$$\frac{dx(t)}{dt} = -rx(t) + \frac{A}{S(t)}y(t),$$
$$\frac{dy(t)}{dt} = -\delta(t)y(t) + B(S(t) - y(t))\frac{x^2(t-T)}{1+x(t-T)}. \tag{4}$$

It is easy to check that system (4) does not belong to the situations studied in the literature (Tang and Kuang, 1997 [10], Tang and Zhou, 2006 [11], and Fan and Zou, 2004 [12]), though the systems therein are more general.

The aim of this paper is to study the existence of periodic solution of the system (4). The periodic solutions of the models of schistosomiasis relate to the periodic phenomenon in epidemiology and control of schistosomiasis. As an application, we will discuss what conditions enable the zero solution of the model stable. We will also perform numerical simulations, which indicate that the conditions of existence of periodic solution can be further improved. To begin with, we study the boundedness of solutions of model (4). And then we will prove the existence of periodic solutions of this model and discuss the stability of zero solution.

2. Preliminaries

The following assumptions apply to the whole paper concerning model (4).

(H) The constants A, B, and r are positive, and $\delta(t)$, $S(t)$ are periodic continuous positive functions with period of ω:

$$\delta_0 = \min_{t\in[0,\omega]}\delta(t) > 0, \qquad S_0 = \min_{t\in[0,\omega]}S(t) > 0. \tag{5}$$

The following results will be used in next sections.

Lemma 1. *If (H) holds, then the solution $(x(t), y(t))$ of periodic system (4) with initial value*

$$x(\theta) = x_0(\theta) > 0, \quad \theta \in [-T, 0],$$
$$y(0) = y_0 > 0 \tag{6}$$

is positive; that is, $x(t) \geq 0$, $y(t) \geq 0$, and $x(t) + y(t) > 0$ for $t \in I_0$, the interval of existence of $(x(t), y(t))$.

Proof. Let $(x(t), y(t))$ be the solution of (4) and (6). By the first equation of (4), we have

$$x(t) = x_0(0)e^{-rt} + \int_0^t \frac{A}{S(\theta)}y(\theta)e^{-r(t-\theta)}d\theta. \tag{7}$$

So, $x(t) > 0$ as long as $y(t) \geq 0$.

By the continuity of solutions, there is a $t_1 > 0$ such that $y(t) > 0$ for $t \in [0, t_1)$. If we suppose that there exists a $t^* > 0$ such that

$$y(t) > 0, \quad t \in [0, t^*), \quad y(t^*) = 0,$$
$$y(t) < 0, \quad t \in (t^*, t^* + \tau_1), \tag{8}$$

where τ_1 is a positive number. By (7), $x(t) > 0$ holds for $t \in [-T, t^*]$. By the continuity of solutions again, there is a $\tau_2 > 0$ such that $x(t) > 0$ for $t \in [t^*, t^* + \tau_2)$. Therefore in interval $[t^*, t^* + \tau)$ $(\tau = \min(\tau_1, \tau_2))$, there hold $y(t) < 0$ and

$$\begin{aligned} \frac{dy(t)}{dt} &= -\delta(t)y(t) \\ &\quad + B(S(t) - y(t))\frac{x^2(t-T)}{1+x(t-T)} > -\delta(t)y(t), \\ &\qquad t \in [t^*, t^* + \tau), \end{aligned} \tag{9}$$

which implies that $y(t) \geq u(t)$ for $t \in (t^*, t^* + \tau)$ where $u(t)$ is the solution of differential equation $u'(t) = -\delta(t)u(t)$ and $u(t^*) = 0$. This is a contradiction to (8). Thus the t^* does not exist. The proof is completed. □

Lemma 2. *Suppose that (H) holds. Then the interval of existence of the solution of periodic systems (4) and (6) is $[0, +\infty)$.*

Proof. By Lemma 1, the solution $(x(t), y(t))$ of (4) and (6) is positive. Suppose that $[0, t^*)$ is the maximal interval of existence of $(x(t), y(t))$; that is, $\lim_{t\to t^*} x(t) = +\infty$ or $\lim_{t\to t^*} y(t) = +\infty$.

If $\lim_{t\to t^*} y(t) = +\infty$ is true, then there is $t_0^* < t^*$ such that

$$y(t_0^*) = S^0 = \max_{t\in[0,\omega]}S(t) > 0,$$
$$y(t) > S^0, \quad t \in (t_0^*, t^*), \tag{10}$$

which imply that

$$\frac{dy(t)}{dt} = -\delta(t)y(t) + B(S(t) - y(t))\frac{x^2(t-T)}{1+x(t-T)} < -\delta(t)y(t), \quad t \in [t_0^*, t^*). \tag{11}$$

It follows comparison principle that

$$\begin{aligned} y(t) \le u(t) &= y(t_0^*)\exp\left(-\int_{t_0^*}^{t}\delta(s)\,ds\right) \\ &= S^0\exp\left(-\int_{t_0^*}^{t}\delta(s)\,ds\right), \quad t \in (t_0^*, t^*) \end{aligned} \tag{12}$$

holds, where $u(t)$ is the solution of differential equation $u'(t) = -\delta(t)u(t)$ and $u(t_0^*) = y(t_0^*) = S^0$. Obviously, (12) contradicts (10) due to $\delta(t) \ge \delta_0 > 0$ in (H), which means that $\lim_{t\to t^*} y(t) = +\infty$ does not hold. Since the function $(A/S(u))y(u)e^{r(u)}$ is continuous, one gets that

$$\lim_{t\to t^*} x(t) = \lim_{t\to t^*}\left[e^{-rt}\int_0^t \frac{A}{S(u)}y(u)e^{ru}du\right] < +\infty. \tag{13}$$

It completes the proof. □

Now we give a result of boundedness of solutions.

Theorem 3. *Suppose that (H) holds. Let $(x(t), y(t))$ be a solution of (4) and (6); then there is a number $M > 0$, which is independent of $(x(t), y(t))$, such that*

$$\limsup_{t\to+\infty} x(t) \le M, \qquad \limsup_{t\to+\infty} y(t) \le M. \tag{14}$$

Proof. We claim that $\limsup_{t\to+\infty} y(t) \le S^0 = \max_{[0,\omega]} S(t)$. In fact, if it is false, then there are two cases.

(i) One has a $T_0 > 0$ such that $y_0(t) > S^0$ for $t \ge T_0$. By the second equation of (4),

$$\frac{dy_0(t)}{dt} = -\delta(t)y_0(t) + B(S(t) - y_0(t))\frac{x^2(t-T)}{1+x(t-T)} < -\delta(t)y_0(t), \quad t \ge T_0, \tag{15}$$

which implies that for, $t \ge T_0$, the inequality $y_0(t) < y_0(T_0)e^{-\int_T^t \delta(s)ds} \to 0$, $(t \to +\infty)$ is true. This is a contradiction.

(ii) The solution $y_0(t)$ is oscillatory with respect to S^0. Then there a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that

$$y_0(t_j) > S^0, \quad \frac{d}{dt}y_0(t_j) = 0, \quad j = 1, 2, \ldots. \tag{16}$$

By the second equation of (4), we get

$$\begin{aligned} 0 &= \frac{d}{dt}y_0(t_j) \\ &= -\delta(t_j)y_0(t_j) + B(S(t_j) - y_0(t_j))\frac{x^2(t_j - T)}{1 + x(t_j - T)} \\ &< -\delta(t_j)y_0(t_j) < 0, \quad j = 1, 2, \ldots, \end{aligned} \tag{17}$$

which is a contradiction. Thus the following inequality is true:

$$\limsup_{t\to+\infty} y(t) \le S^0 = \max_{[0,\omega]} S(t). \tag{18}$$

By (18), one has a $T_1 > 0$ large enough such that $y(t) \le S^0$ for $t \ge T_1$. The first equation of (4) gives that

$$\begin{aligned} \frac{dx(t)}{dt} &= -rx(t) + \frac{A}{S(t)}y(t) \le -rx(t) + \frac{A}{S(t)}S^0 \\ &\le -rx(t) + \frac{AS^0}{S_0}, \quad t \ge T_1, \end{aligned} \tag{19}$$

which implies that

$$\limsup_{t\to+\infty} x(t) \le \frac{A_0S^0}{rS_0}. \tag{20}$$

It is obvious that (14) follows (18) and (20). The proof is completed. □

3. Existence of Periodic Solution

Suppose that (H) holds. We will discuss the existence of ω-periodic solution of (4)–(6).

Lemma 4. *Assume that there is a number $\alpha > 0$, such that $ABS_0\alpha \ge rS^0[\delta^0(1+\alpha) + B\alpha^2]$, where $\delta^0 = \max_{t\in[0,\omega]}\delta(t)$. Then the following differential equations*

$$\begin{aligned} \frac{dx(t)}{dt} &= -rx(t) + \frac{A}{S(t)}y(t), \\ \frac{dy(t)}{dt} &= -\delta(t)y(t) + B(S(t) - y(t))\frac{\alpha^2}{1+\alpha} \end{aligned} \tag{21}$$

admit a unique positive ω-periodic solution $(x^*(t), y^*(t))$, satisfying

$$x^*(t) \ge \alpha, \quad y^*(t) \ge \frac{BS_0\alpha^2}{\delta^0(1+\alpha) + B\alpha^2}, \quad t \in [0,\omega]. \tag{22}$$

Proof. Since the second equation of (21)

$$\begin{aligned}\frac{dy(t)}{dt} &= -\delta(t)y(t) + B(S(t)-y(t))\frac{\alpha^2}{1+\alpha}\\ &= -\left(\delta(t)+\frac{B\alpha^2}{1+\alpha}\right)y(t) + \frac{B\alpha^2}{1+\alpha}S(t)\end{aligned} \tag{23}$$

is a linear equation with respect to $y(t)$ and $\int_0^\omega (B\alpha^2/(1+\alpha)+\delta(t))dt > 0$, it admits a unique periodic solution denoted by $y^*(t)$. It is obvious that there holds

$$\begin{aligned}\frac{dy(t)}{dt} &= -\delta(t)y(t) + B(S(t)-y(t))\frac{\alpha^2}{1+\alpha}\\ &\geq -\delta^0 y(t) + B(S_0-y(t))\frac{\alpha^2}{1+\alpha},\end{aligned} \tag{24}$$

and that the following equation,

$$\frac{du(t)}{dt} = -\delta^0 u(t) + B(S_0-u(t))\frac{\alpha^2}{1+\alpha}, \tag{25}$$

has a globally stable equilibrium $u_* = BS_0\alpha^2/(\delta^0(1+\alpha)+B\alpha^2)$. Therefore, by comparison principle, the following inequality,

$$y^*(t) \geq u^*(t) \longrightarrow \frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}, \quad (t \longrightarrow +\infty), \tag{26}$$

is true for $t \geq 0$, where $u^*(t)$ is the solution of (25) with $u^*(0) = y^*(0)$. Noting that $y^*(t)$ is a periodic function, we obtain

$$y^*(t) \geq \frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}, \quad t \in [0,\omega]. \tag{27}$$

Substitute $y = y^*(t)$ into the first equation of (21); we get

$$\begin{aligned}x^*(t) &= x^*(0)e^{-rt} + \int_0^t \frac{A}{S(\theta)}y^*(\theta)e^{-r(t-\theta)}d\theta\\ &\geq x^*(0)e^{-rt} + \frac{A}{S^0}\frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}\frac{1}{r}(1-e^{-rt})\\ &= \frac{A}{S^0}\frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}\frac{1}{r}\\ &\quad + \left(x^*(0) - \frac{A}{S^0}\frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}\frac{1}{r}\right)e^{-rt},\end{aligned} \tag{28}$$

where

$$\begin{aligned}x^*(0) &= \frac{\int_0^\omega (Ay^*(\theta)e^{-r(\omega-\theta)}/S(\theta))d\theta}{1-e^{-r\omega}}\\ &\geq \frac{(A/S^0)(BS_0\alpha^2/(\delta^0(1+\alpha)+B\alpha^2))(1/r)(1-e^{-r\omega})}{1-e^{-r\omega}}\\ &= \frac{A}{S^0}\frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}\frac{1}{r}.\end{aligned} \tag{29}$$

Thus there holds that

$$x^*(t) \geq \frac{A}{S^0}\frac{BS_0\alpha^2}{\delta^0(1+\alpha)+B\alpha^2}\frac{1}{r} \geq \alpha \quad t \in [0,\omega]. \tag{30}$$

The conclusion of the lemma follows (27), (30). It completes the proof. ☐

Let $C_\omega(R) = \{x \in C(R) : x(t+\omega) = x(t), t \in R\}$ be the set of all continuous functions in $[0,\omega]$ with the distance norm $\|\mu(t)-\nu(t)\| = \max_{t\in[0,\omega]}|\mu(t)-\nu(t)|$; then $C_\omega(R)$ is a Banach space.

Let $\mu(t)$ be a continuous ω-periodic function satisfying $\alpha \leq \mu(t) \leq \beta, t \in [0,\omega]$, where $\beta = AS^0/S_0 r$. Denote the set of such functions by D, clearly, $D \subseteq C[0,\omega]$. For any $\mu \in D$, the following differential equations,

$$\begin{aligned}\frac{dx(t)}{dt} &= -rx(t) + \frac{A}{S(t)}y(t),\\ \frac{dy(t)}{dt} &= -\delta(t)y(t) + B(S(t)-y(t))\frac{\mu^2(t-T)}{1+\mu(t-T)},\end{aligned} \tag{31}$$

admit a unique ω-periodic solution:

$$\begin{aligned}y(t) &= y^0\exp\left[-\int_0^t\left(\delta(\theta)+\frac{B\mu^2(\theta-T)}{1+\mu(\theta-T)}\right)d\theta\right]\\ &\quad + \int_0^t \exp\left[-\int_\theta^t\left(\delta(\xi)+\frac{B\mu^2(\xi-T)}{1+\mu(\xi-T)}\right)d\xi\right]\\ &\quad \times \frac{B\mu^2(\theta-T)}{1+\mu(\theta-T)}S(\theta)d\theta,\end{aligned} \tag{32}$$

$$\begin{aligned}x(t) &= x^0 e^{-rt} + \int_0^t \frac{A}{S(\theta)}y(\theta)e^{-r(t-\theta)}du,\\ x^0 &= \frac{1}{1-e^{-r\omega}}\int_0^\omega \frac{A}{S(\theta)}y(\theta)e^{-r(\omega-\theta)}d\theta,\end{aligned} \tag{33}$$

where

$$\begin{aligned}y^0 = &\left(\int_0^\omega \exp\left[-\int_\theta^\omega\left(\delta(\xi)+\frac{B\mu^2(\xi-T)}{1+h(\xi-T)}\right)d\xi\right]\right.\\ &\left.\times\frac{B\mu^2(\theta-T)}{1+\mu(\theta-T)}S(\theta)d\theta\right)\\ &\times\left(1-\exp\left[-\int_0^\omega\left(\delta(\theta)+\frac{B\mu^2(\theta-T)}{1+\mu(\theta-T)}\right)d\theta\right]\right)^{-1}.\end{aligned} \tag{34}$$

Obviously, $x(t) \in C_\omega(R)$ is a periodic function. By the proof of Theorem 3, one knows that $\limsup_{t\to+\infty} x(t) \leq A_0 S^0/rS_0$, and so $x(t) \leq \beta, t \in [0,\omega]$.

Denote by T the mapping from D into $C_\omega(R)$ defined by (32)–(34); that is, $T: D \to C_\omega(R)$, $x = T\mu$.

Lemma 5. *The mapping T is monotonic; that is, if $\mu_1, \mu_2 \in D$ and $\mu_1(t) \le \mu_2(t)$ for $t \in [0,\omega]$, then $(T\mu_2)(t) \ge (T\mu_1)(t)$, for $t \in [0,\omega]$.*

Proof. Let $\mu_1, \mu_2 \in D$ and $\mu_1 \le \mu_2$. Substitute μ_1, μ_2 into the second equation of (21); one gets

$$\frac{dy_j(t)}{dt} = -\delta(t) y_j(t) + B\left(S(t) - y_j(t)\right) \times \frac{\mu_j^2(t-T)}{1+\mu_j(t-T)}, \quad j = 1,2. \tag{35}$$

Therefore,

$$\begin{aligned}
&\frac{d\left(y_2(t) - y_1(t)\right)}{dt} \\
&= -\delta(t)\left(y_2(t) - y_1(t)\right) + B\left(S(t) - y_2(t)\right)\frac{\mu_2^2(t-T)}{1+\mu_2(t-T)} \\
&\quad - B\left(S(t) - y_1(t)\right)\frac{\mu_1^2(t-T)}{1+\mu_1(t-T)} \\
&\ge -\left(y_2(t) - y_1(t)\right)\left(\delta(t) + B\frac{\mu_1^2(t-T)}{1+\mu_1(t-T)}\right).
\end{aligned} \tag{36}$$

By comparison principle, the solution $u(t)$ of the following initial value problem,

$$\begin{gathered}
\frac{du(t)}{dt} = -\delta(t) u(t) - B\frac{\mu_1^2(t-T)}{1+\mu_1(t-T)} u(t), \\
u(0) = y_2(0) - y_1(0),
\end{gathered} \tag{37}$$

for $t \ge 0$ satisfies

$$\begin{aligned}
y_2(t) - y_1(t) &\ge u(t) \\
&= u(0)\exp\left(-\int_0^t \left[\frac{\mu_1^2(\theta - T)}{1+\mu_1(\theta-T)} + \delta(\theta)\right] d\theta\right) \longrightarrow 0, \\
&\qquad (t \longrightarrow +\infty).
\end{aligned} \tag{38}$$

Because of the periodicity of $y_2 - y_1$, one has $y_2(t) \ge y_1(t)$ for $t \in [0,\omega]$. By (33) we obtain that, for $t \in [0,\omega]$,

$$\begin{aligned}
&x_2(t) - x_1(t) \\
&= \left(x_2^0 - x_1^0\right)e^{-rt} + \int_0^t \frac{A}{S(\theta)}\left(y_2(\theta) - y_1(\theta)\right)e^{-r(t-\theta)} du \\
&= \frac{e^{-rt}}{1-e^{-r\omega}} \int_0^\omega \frac{A}{S(\theta)}\left(y_2(\theta) - y_1(\theta)\right)e^{-r(\omega-\theta)} d\theta \\
&\quad + \int_0^t \frac{A}{S(\theta)}\left(y_2(\theta) - y_1(\theta)\right)e^{-r(t-\theta)} du \ge 0,
\end{aligned} \tag{39}$$

which implies that $(T\mu_2)(t) \ge (T\mu_1)(t)$, for $t \in [0,\omega]$. It completes the proof of Lemma 5. □

By Lemma 5, we know that the mapping T maps D into D. Based on the results above, we can give the existence of periodic solution.

Theorem 6. *If (H) is satisfied and if there is a positive number α, such that $ABS_0\alpha \ge rS^0[\delta^0(1+\alpha) + B\alpha^2]$, then (4) admits a positive periodic solution.*

Proof. For fixed α satisfying $ABS_0\alpha \ge rS^0[\delta^0(1+\alpha) + B\alpha^2]$, consider the following iterative sequence:

$$x_n = Tx_{n-1}, \quad x_0 = \alpha \quad (n = 1,2,\ldots). \tag{40}$$

From Lemmas 4-5, for any positive integer n, one has $x_n \in D$ and that the $\{x_n\}$ is an increasing sequence. Let $D_0 = \{x_n(t) \mid \text{defined by (40)}, n = 1,2,\ldots\}$. It is obvious that $D_0 \subset D \subset C[0,\omega]$, and so for any n, we have $|x_n(t)| \le \beta$, where β is independent on D_0. We claim that $D_0 = \{x_n\}$ is equicontinuous on D. In fact, for arbitrary $\varepsilon > 0$ and any $u_n \in D_0$, by (32)-(33), we get that the following equation holds for any $t_1, t_2 \in [0,\omega]$,

$$\begin{aligned}
&u_n(t_2) - u_n(t_1) \\
&\quad = \left(e^{-rt_2} - e^{-rt_1}\right)u^0 + \int_{t_1}^{t_2} e^{-r(t_2-s)}\frac{A}{S(\theta)} y_u(\theta)\, d\theta,
\end{aligned} \tag{41}$$

where y_u is determined by (32) with $h(\cdot) = u_{n-1}(\cdot)$ and u^0 is given by the second equation of (33) with $y(\cdot) = y_u(\cdot)$. By differential mean theorem, there is a τ between t_1 and t_2 such that

$$\begin{aligned}
&u_n(t_2) - u_n(t_1) \\
&\quad = -re^{-r\tau}u^0(t_2 - t_1) \\
&\qquad + \frac{d}{dt}\left[\int_0^t e^{-r(t-s)}\frac{A}{S(s)} y_u(s)\, ds\right]_{t=\tau}(t_2 - t_1) \\
&\quad = -re^{-r\tau}u^0(t_2 - t_1) \\
&\qquad + \left[\frac{A}{S(\tau)} y_u(\tau) + \int_0^\tau e^{-r(t-s)}\frac{A}{S(s)} y_u(s)\, ds\right](t_2 - t_1).
\end{aligned} \tag{42}$$

Therefore

$$\begin{aligned}
&\left|u_n(t_2) - u_n(t_1)\right| \\
&\quad \le \left|-re^{-r\tau}\right|u^0\left|(t_2 - t_1)\right| + \left[\frac{A}{S(\tau)} y_u(\tau)\right] \\
&\qquad + \int_0^\omega \left|re^{-r(\tau-s)}\frac{A}{S(s)} y_u(s)\right| ds\left|t_2 - t_1\right| \\
&\quad \le \left[ru^0 + M_1(1 + r\omega)\right]\left|t_2 - t_1\right|,
\end{aligned} \tag{43}$$

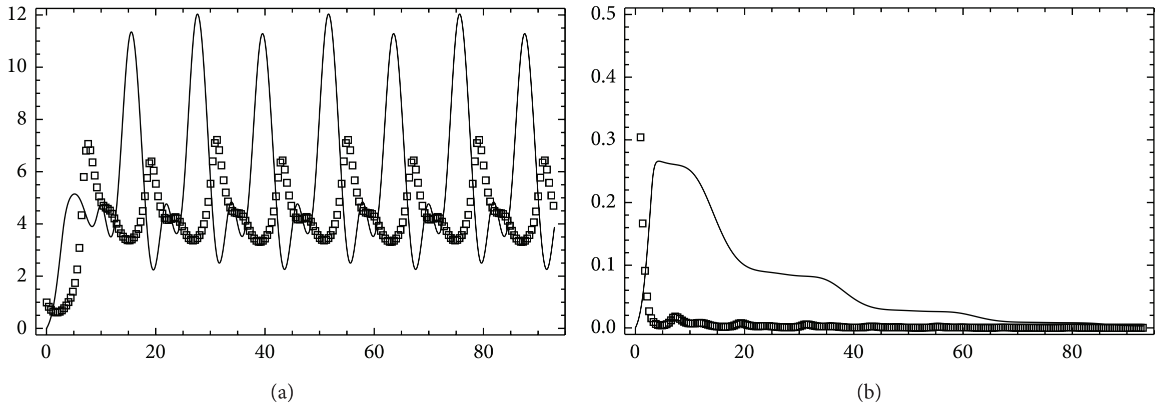


Figure 1: A periodic solution ((a) $A = 1$, $B = 2.4$, and $r = 0.2$ satisfy the conditions of Theorem 6) and a solution tending to zero ((b) $A = 0.2$, $B = 0.02$, and $r = 1.5$ satisfy the conditions of Theorem 7), where square is $x(t)$, line is $y(t)$, $y_0 = 0$, and $x_0(\theta) = \cos(\pi\theta/10)$.

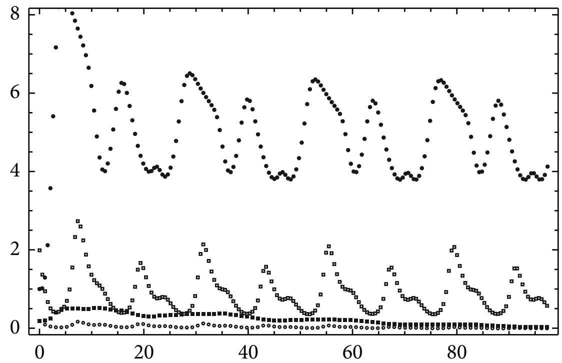


Figure 2: Coexistence of a stable zero solution (solid squares and circles, $y_0 = 0.2$, $x_0(\theta) = 0.2$) and a periodic solutions (dots and hollow squares, $y_0 = 2, x_0(\theta) = 2$) where $A = 0.6$, $B = 0.4$, and $r = 0.8$.

where

$$M_1 = \max \max_{t\in[0,\omega]} \left| \frac{A}{S(t)} y_u(t) \right|,$$
$$x_0 = \frac{1}{1-e^{-r\omega}} \int_0^{\omega} e^{-r(\omega-s)} \frac{A}{S(s)} y(s)\, ds \le \frac{M_1 \omega}{1-e^{-r\omega}}. \tag{44}$$

Since $y_u(t)$ is a solution of the second equation of (31) with $h(\cdot) = u_{n-1}(\cdot)$, by the proof of Theorem 3, $y_u(t)$ has a bound of S^0. Consequently $M_1 \le AS^0/S_0$, which is independent on u_n. Therefore D_0 is equicontinuous on D. By Arzela-Ascoli's theorem, one has a subsequence $\{u_{n_k}\} \subset D_0$ such that $u_{n_k} \to x^*$. Obviously, for any $n = 1, 2, \ldots$, inequality $x^* \ge u_n$ holds. For all $t \in [0, \omega]$, one has $0 \le x^* - u_m \le x^* - u_{n_k}$ as $m > n_k$. On the other hand, the cone $P = \{\varphi \mid \varphi \in C_\omega(R), \varphi(t) \ge 0\}$ is normal, and in cone P, there holds that if $x_n, y_n, z_n \in P$ with $x_n \le z_n \le y_n, x_n \to x$, and $y_n \to x$, then $z_n \to x$ (see Guo 2001 [13]). Thus as $m \to +\infty$, it follows $0 \le x^* - u_m \le x^* - u_{n_k} \to 0$. Replace μ in (32) and (34) by the limit x^*; we get y^*. Obviously (x^*, y^*) is the periodic solution of (4). It completes the proof. □

4. Stability of Zero Solution

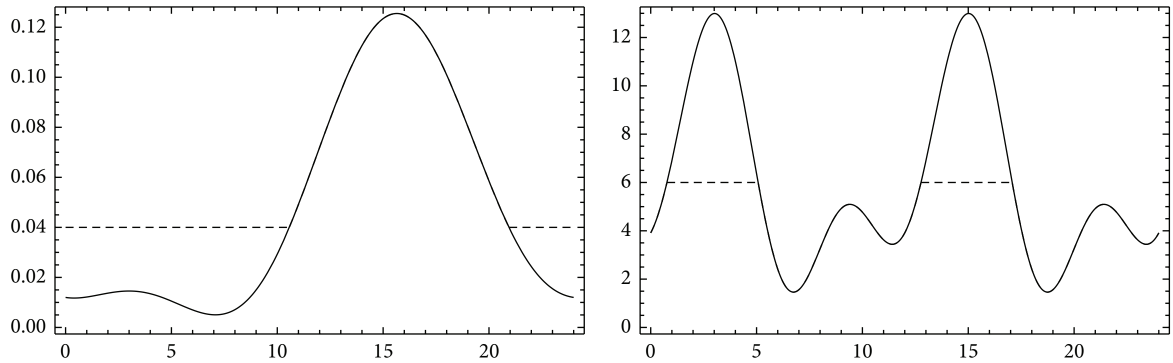
Theorem 7. *The zero solution of* (4) *is uniformly stable, if the following inequality holds:*

$$\max_{t\in[0,\omega]} \left[\left(\frac{A}{S(t)} + BS(t) \right)^2 - 4r\delta(t) \right] < 0. \tag{45}$$

Proof. Take $V(x, y) = (x^2 + y^2)/2$; we get that

$$\begin{aligned} \frac{dV}{dt}\Big|_{(4)} &= x\left(-rx + \frac{A}{S(t)} y\right) + y\left[-\delta(t) y + B(S(t) - y) \times \frac{x^2(t-T)}{1+x(t-T)} \right] \\ &= -rx^2 - \delta(t) y^2 + \frac{A}{S(t)} xy + BS(t) y \frac{x^2(t-T)}{1+x(t-T)} - By^2 \frac{x^2(t-T)}{1+x(t-T)} \\ &\le -rx^2 - \delta(t) y^2 + \frac{A}{S(t)} xy + BS(t) yx(t-T) \\ &\le -rx^2 - \delta(t) y^2 + \left[\frac{A}{S(t)} + BS(t) \right] xy, \end{aligned} \tag{46}$$

if $x(t) \ge x(t-T)$. Because (45) holds, there are two positive numbers γ_1 and γ_2 small enough such that $(A/S(t) + BS(t))^2 - 4r\delta(t) + 4\gamma_1\delta(t) + 4\gamma_2(r-\alpha) < 0$ hold for all $t \in [0, \omega]$. It follows



(a) (b)

Figure 3: The graphs (solid lines) and a control strategy (dash line) of $\delta(t)$ (a) and $S(t)$ (b).

that $-rx^2 - (A/S(t) + BS(t))xy - \delta(t)y^2 \le -(\gamma_1 x^2 + \gamma_2 y^2)$. Consequently,

$$\frac{dV}{dt}\Big|_{(4)} \le -\left(\gamma_1 x^2 + \gamma_2 y^2\right) \tag{47}$$

if $x(t) \ge x(t-T)$ and (45). As $V(x, y) = (x^2 + y^2)/2$, by the stability theorems (Hale and Verduyn Lunel, 1993 [14]), the conclusions of the theorem are true. The proof is completed. □

Theorem 8. *If $r > BS^0$ and $A < \delta_0 S_0$ hold, then the zero solution of* (4) *is globally asymptotically stable.*

Proof. Define

$$V(t) = |x(t)| + |y(t)| + BS^0 \int_{t-T}^{t} |x(s)|\, ds. \tag{48}$$

Now we calculate and estimate the upper-right derivative of $V(t)$ along the solutions of model (4):

$$\begin{aligned}
D^+V(t) &= \dot{x}(t)\operatorname{sgn} x(t) + \dot{y}(t)\operatorname{sgn} y(t) + BS^0|x(t)| \\
&\quad - BS^0|x(t-T)| \\
&= -r|x(t)| + \frac{A}{S(t)} y(t)\operatorname{sgn} x(t) - \delta(t)|y(t)| \\
&\quad + B(S(t) - y(t))\frac{x^2(t-T)}{1 + x(t-T)}\operatorname{sgn} y(t) \\
&\quad + BS^0|x(t)| - BS^0|x(t-T)| \\
&\le -r|x(t)| + \frac{A}{S_0}|y(t)| - \delta(t)|y(t)| + BS^0|x(t-T)| \\
&\quad - B|y(t)|\frac{x^2(t-T)}{1 + x(t-T)} + BS^0|x(t)| - BS^0|x(t-T)| \\
&\le -\left(r - BS^0\right)|x(t)| + \left(\frac{A}{S_0} - \delta_0\right)|y(t)|.
\end{aligned} \tag{49}$$

According to the assumption $r > BS^0$ and $A < \delta_0 S_0$, there exists $\varepsilon > 0$ such that

$$D^+V(t) \le -\varepsilon\left(|x(t)| + |y(t)|\right). \tag{50}$$

Integrating both sides of (50) from a sufficiently large number T^* to t, we get

$$\begin{aligned}
V(t) + \varepsilon\int_{T^*}^{t}\left[|x(s)| + |y(s)|\right] ds &\le V\left(T^*\right) < +\infty, \\
\left(T^*\right) &\text{ for } t > T^*,
\end{aligned} \tag{51}$$

which implies that $V(t)$ is bounded on $[T^*, +\infty)$ and

$$\int_{T^*}^{+\infty} |x(s)|\, ds < +\infty, \qquad \int_{T^*}^{+\infty} |y(s)|\, ds < +\infty. \tag{52}$$

By Theorem 3, $|x(t)|$ and $|y(t)|$ are bounded on $[T^*, +\infty)$. It is obvious that $\dot{x}(t)$ and $\dot{y}(t)$ are bounded for $t \ge T^*$. Therefore, $|x(t)|$ and $|y(t)|$ are uniformly continuous on $[T^*, +\infty)$. By Barbalat's lemma (Lemmas 1.2.2 and 1.2.3, Gopalsamy [15]), one can conclude that

$$\lim |x(t)| = 0, \qquad \lim |y(t)| = 0. \tag{53}$$

It follows that the zero solution is globally asymptotically stable. □

5. Numerical Simulations

In order to exemplify the results presented above to show some possible behaviors of the solutions of the model, we performed some simulations.

For the following model

$$\begin{aligned}
\frac{dx}{dt} &= f(t, x(t), y(t)), \\
\frac{dy}{dt} &= g(t, x(t), y(t), x(t-T)), \\
y(0) &= y_0, \quad x(s) = x_0(s), \quad s \in [-T, 0],
\end{aligned} \tag{54}$$

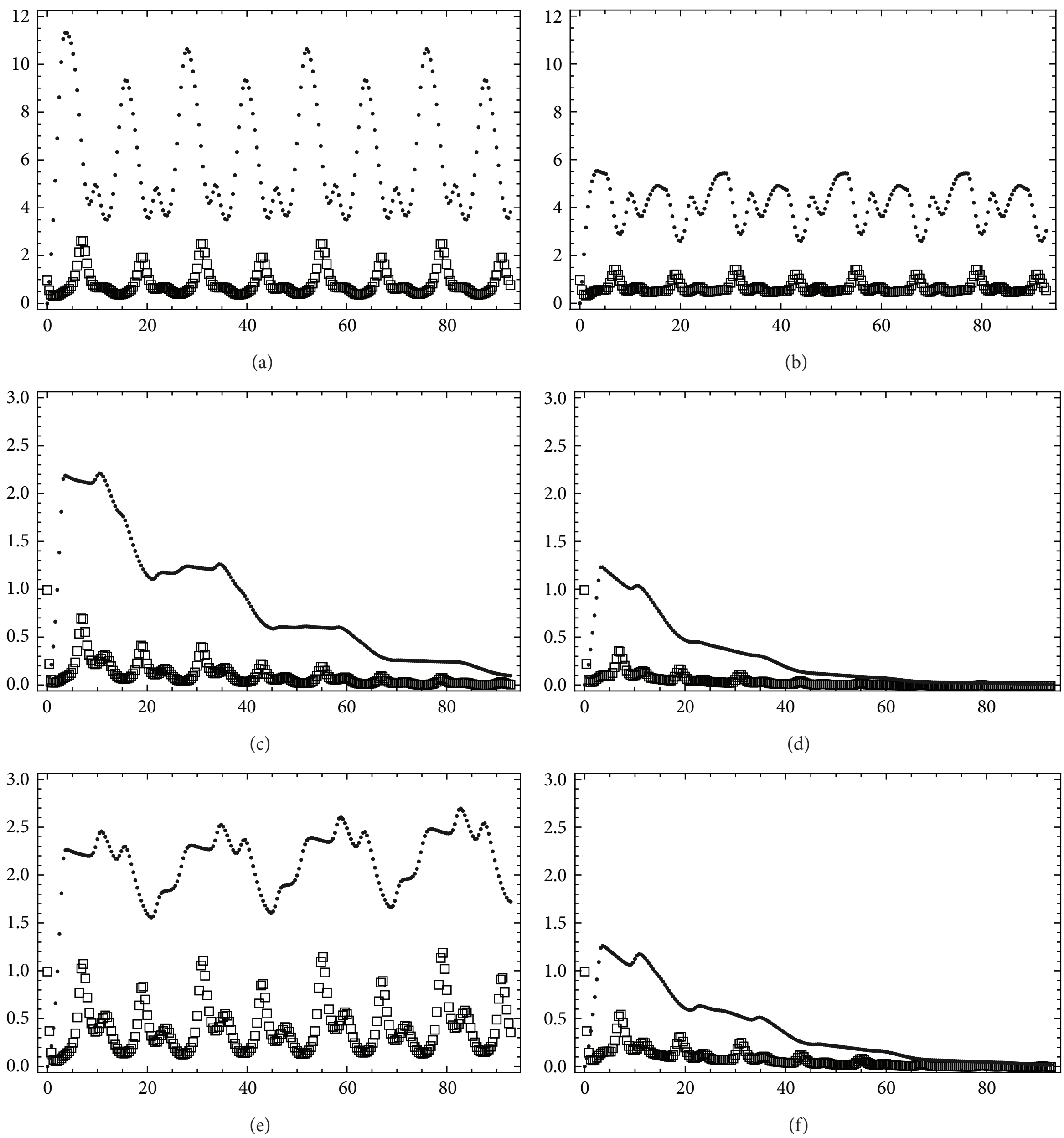


Figure 4: Comparison between uncontrolled and the controlled snails: a periodic solution ((a-b), $A = 1$, $B = 2.4$, and $r = 2.5$), a solution tending to zero ((c-d), $A = 2$, $B = 0.2$, and $r = 3.8$), and a changed case ((e-f), $A = 2$, $B = 0.2$, and $r = 2.5$), where square is $x(t)$, dot is $y(t)$, $y_0 = 0$, and $x_0(\theta) = \cos(\pi\theta/10)$.

based on the fourth-order Runge-Kutta's formula for a ordinary differential system, we applied the following scheme:

$$x_{n+1} = x_n + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right),$$

$$y_{n+1} = y_n + \frac{h}{6}\left(S_1 + 2S_2 + 2S_3 + S_4\right),$$

$$K_1 = f\left(t_n, x_n, y_n\right),$$

$$S_1 = g\left(t_n, x_n, y_n, x_{n-m}\right), \quad m = \frac{T}{h},$$

$$K_2 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_1, y_n + \frac{h}{2}S_1\right),$$

$$S_2 = g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_1, y_n + \frac{h}{2}S_1, x_{n-m}\right),$$

$$K_3 = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_2, y_n + \frac{h}{2}S_2\right),$$

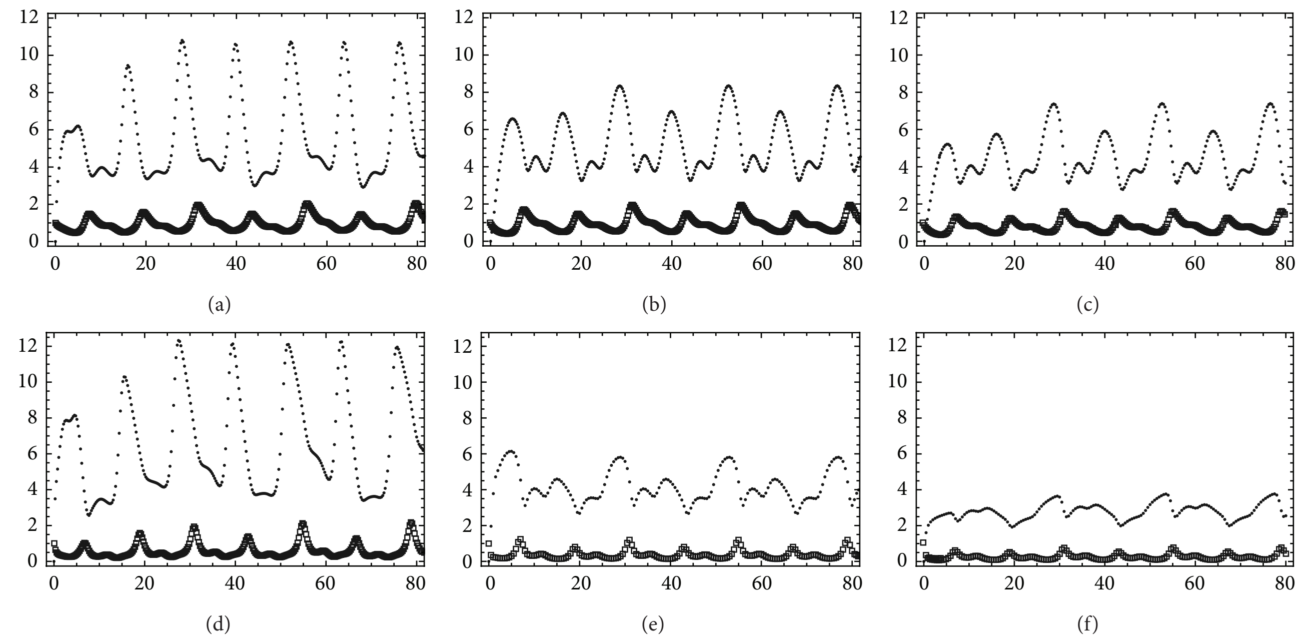


Figure 5: The effects of delays on the behavior of solutions: (1) $A = 0.4$, $B = 0.98$, and $r = 0.5$ (a–c) and (2) $A = 1$, $B = 2.4$, and $r = 2.5$ (d–f) with initial data of $y_0 = 0$, $x_0(\theta) = \cos(\pi\theta/10)$.

$$S_3 = g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}K_2, y_n + \frac{h}{2}S_2, x_{n-m}\right),$$
$$K_4 = f\left(t_n + h, x_n + hK_3, y_n + hS_3\right),$$
$$S_1 = g\left(t_n + h, x_n + hK_3, y_n + hS_3, x_{n-m}\right), \tag{55}$$

where the step size h is equal to T/N_1, where cN_1 (c is about 20–400) is the total number of iterative steps. In computation the virtual range of parameters was $A \in [0.1, 2]$, $B \in [0.02, 3]$, $r \in [0.5, 4]$, and $T \in [0, 8]$; both of $S(t)$ and $\delta(t)$ were given by (3).

If one takes $A = 1$, $B = 2.4$, and $r = 0.2$, then the conditions of Theorem 6 are satisfied and a periodic solution follows (Figure 1(a)). If one takes $A = 0.2$, $B = 0.02$, and $r = 1.5$, then the condition (45) is satisfied and it can be found that the zero is uniformly stable (Figure 1(b)). Furthermore, the sufficient conditions given in Theorems 6 and 7 are far from being sharp. In some cases where the initial value is small enough, the solution tends to zero (solid squares and circles in Figure 2, $y_0 = 0.2$, $x_0(\theta) = 0.2$), meanwhile in other cases where the initial value is larger both the solution tends to a periodic solution (dots and hollow squares in Figure 2, $y_0 = 2$, $x_0(\theta) = 2$), where $A = 0.6$, $B = 0.4$, and $r = 0.8$. Further simulations indicate that the periodic solution, if it exists, is locally stable.

Concerning control strategies, many methods have been applied in order to reduce the number of snails, that is, $S(t)$, and at the same time to enlarge the death ratio of snails, $\delta(t)$. If a control approach was applied such that $S(t) \leq 6$ and $\delta(t) \geq 0.04$ (dash lines in Figure 3), we obtain some results shown in Figure 4. A periodic solution is shown in Figures 4(a)-4(b) and its amplitude is reduced by half by the control strategy. A solution tending to zero is shown in Figures 4(c)-4(d) and the time infected snails go extinct is reduced by a quarter by the control strategy. A periodic solution is changed to a solution tending to zero by the control strategy (Figures 4(e)-4(f)). Therefore it is an effective treatment by simply reducing the population of snails and enlarging the death ratio of snails.

In order to study the effects of the delay on the behavior of solutions, a simulation was performed in two cases: (1) $A = 0.4$, $B = 0.98$, and $r = 0.5$ (Figures 5(a)–5(c)) and (2) $A = 1$, $B = 2.4$, and $r = 2.5$ (Figures 5(d)–5(f)) with initial data of $y_0 = 0$, $x_0(\theta) = \cos(\pi\theta/10)$. It can be seen that the amplitude and period of the periodic solution $y(t)$ decrease, but no obvious changes were observed on the periodic solution $x(t)$ when the delay is changed from 8 to 1. Furthermore when the delay is changed from 1 to 0.2, the similar changes could not be found.

6. Discussion and Conclusion

In the preceding sections we modified MacDonald's models in schistosomiasis. It may be more reasonable in natural conditions that we consider a periodic model and introduce a delay to simulate the development period of helminthes.

The existence of periodic solution shows that the infected snails and infecting worms in the host coexist in a periodic pattern. The stability of the zero solution shows that the population of infected snails and infecting worms in a host will extinct eventually. Although much to our expectation, the condition of existence of periodic solution is easier to be satisfied, rather than that of global stability of zero solution.

It is needed to add that the condition of Theorem 6 can be improved, for example, the periodic solution shown in Figure 4(a) with the parameters satisfying $ABS_0\alpha < rS^0[\delta^0(1+\alpha)+B\alpha^2]$ for all $\alpha > 0$ rather than the condition of Theorem 6. Also, the condition of Theorem 7 may be improved, for example, the stable zero solution shown in Figure 4(b) with the parameters not satisfying (45).

From the simulations and Theorem 7, while the practical meaning of the parameter values chosen in numerical simulations has not been clear yet, we can see that an increase in r and decrease in A, B would enable (45) and thus lead to the stability of zero solution. In other words, to annihilate worms, it is required to increase the death probability of worms, while reducing the penetration of worms into snails and the disease transmission of snails to final hosts. This implication is in line with other published literature (Williams et al., 2002 [4], Liang et al., 2005 [5]).

In conclusion, we present the conditions under which the model admits a periodic solution and the conditions under which the zero solution is uniformly stable and globally stable, respectively. We show the conditional coexistence of locally stable zero solution and periodic solutions in numerical method and show that it is an effective treatment of simply reducing the population of snails and enlarging the death ratio of snails for the control of schistosomiasis.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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