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Research Article

Ulam's Type Stability of Involutional-Exponential Functional Equations

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Let S be a commutative semigroup, $f, g : S \to \mathbb{C}$ and $\sigma : S \to S$ an involution. In this paper we consider the stability of involution-exponential functional equations $|f(x+\sigma y) - g(x) f(y)| \le \phi(x)$ [resp., $\phi(y)$], $|f(x+\sigma y) - f(x)g(y)| \le \phi(x)$[resp., $\phi(y)$] for all $x, y \in S$, where $\phi : S \to \mathbb{R}^+$ satisfies the growth condition: there exists $C > 1$ such that $\lim_{k\to\infty} C^{-k}\phi(kx) = 0$ for each $x \in S$. We also consider the stability of L^∞-version $|f(x+\sigma y) - f(x)f(y)|_{L^\infty(\mathbb{R}^{2n})} \le \epsilon$, where $f : \mathbb{R}^n \to \mathbb{C}$ is a locally integrable function.

1. Introduction

Throughout this paper we denote by $S, \mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{R}^n$, a commutative semigroup with an identity element, the set of real numbers, nonnegative real numbers, complex numbers, and the n-dimensional Euclidean space, respectively, and $\mathbb{R}_0^n = \mathbb{R}^n \backslash \{0\}, \mathbb{C}_0 = \mathbb{C} \backslash \{0\}, \phi : S \to \mathbb{R}^+, \epsilon \ge 0$. A function $m : S \to \mathbb{C}$ is called *exponential* provided that $m(x+y) = m(x)m(y)$ for all $x, y \in S$, and $\sigma : S \to S$ is called *an involution* provided that $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in S$. An exponential function $m : S \to \mathbb{C}$ is called *σ-exponential* if m satisfies $m(\sigma x) = m(x)$ for all $x \in S$ and denote by m_σ a σ-exponential function.

In [1], the following functional inequalities with involution are investigated:

$$|f(x+\sigma y) - g(x) f(y)| \le \phi(x) [\text{resp. } \phi(y)], \quad \forall x, y \in S, \tag{1}$$

$$|f(x+\sigma y) - f(x) g(y)| \le \phi(x) [\text{resp. } \phi(y)], \quad \forall x, y \in S. \tag{2}$$

As a result, all unbounded functions f, g satisfying the inequalities (1) and (2) are exactly described only when ϕ is a constant function while only one of unbounded functions f, g satisfying each of (1) and (2) is exactly described when ϕ is an arbitrary unbounded function.

In this paper we investigate the functional inequalities (1) and (2) by imposing some growth conditions on ϕ, f, or g. First, we introduce the condition on $h : S \to \mathbb{C}$:

$$\inf_{x\in S} \frac{1+\phi(x)}{|h(x)|} = 0, \tag{3}$$

where h will stand for f and g.

Secondly, we introduce the condition on ϕ; there exists $C > 1$ such that

$$\lim_{k\to\infty} C^{-k}\phi(kx) = 0 \tag{4}$$

for all $x \in S$.

As a result, we completely determine f and g satisfying each of the inequalities (1) and (2): if g satisfies (3) [resp., f satisfies (3)] or ϕ satisfies (4), then (f, g) satisfying (1) [resp., (2)] are of the form

$$\begin{gathered} f(x) = f(0) m_\sigma(x), \\ g(x) = m_\sigma(x) \\ \times [\text{resp. } f(x) = f(0) m(x), g(x) = m(\sigma x)] \end{gathered} \tag{5}$$

for all $x \in S$, where m_σ is a σ-exponential function and m is an exponential function.

As an application of our result, we determine all unbounded functions $f, g : \mathbb{R}_0^2 \to \mathbb{R}$ satisfying the functional inequalities

$$\begin{aligned} &\left| f(ux + vy, uy - vx) - g(x, y) f(u, v) \right| \\ &\quad \le \phi(x, y)\,[\phi(u, v)], \end{aligned} \tag{6}$$

$$\begin{aligned} &\left| f(ux + vy, uy - vx) - f(x, y) g(u, v) \right| \\ &\quad \le \phi(x, y)\,[\phi(u, v)] \end{aligned} \tag{7}$$

for all $(x, y), (u, v) \in \mathbb{R}_0^2$, where f, g satisfy (3) or $\phi : \mathbb{R}_0^2 \to \mathbb{R}^+$ satisfies (4) (see [2–5] for related equations) and determine all unbounded functions $f, g : \mathbb{R}_0^4 \to \mathbb{R}$ satisfying the functional inequalities

$$\begin{aligned} &\left| f(x_1, y_1, u_1, v_1)\, g(x_2, y_2, u_2, v_2) \right. \\ &\quad - f(x_1 x_2 + y_1 y_2 + u_1 u_2 + v_1 v_2, x_1 y_2 - y_1 x_2 \\ &\qquad + u_1 v_2 - v_1 u_2, x_1 u_2 - y_1 v_2 - u_1 x_2 \\ &\qquad \left. + v_1 y_2, x_1 v_2 + y_1 u_2 - u_1 y_2 - v_1 x_2) \right| \\ &\le \psi(x_1, y_1, u_1, v_1)\,[\psi(x_2, y_2, u_2, v_2)], \end{aligned} \tag{8}$$

for all x_1, y_1, u_1, v_1, x_2, y_2, u_2, and $v_2 \in \mathbb{R}$, where f, g satisfy (3) or $\psi : \mathbb{R}_0^4 \to \mathbb{R}^+$ satisfies (4) (see [2, 4] for related equations). Finally, we consider the stability of L^∞-version

$$\left| f(x + \sigma y) - f(x) f(y) \right|_{L^\infty(\mathbb{R}^{2n})} \le \epsilon, \tag{9}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a locally integrable function. As a result, we prove that every unbounded solution f (i.e., $\|f\|_{L^\infty(\mathbb{R}^n)} = \infty$) of (9) satisfies

$$f(x) = m\left(\frac{x + \sigma x}{2}\right) \tag{10}$$

for almost every $x \in \mathbb{R}^n$, where $m : \mathbb{R}^n \to \mathbb{C}$ is an unbounded exponential function. Every bounded solution f (i.e., $\|f\|_{L^\infty(\mathbb{R}^n)} < \infty$) satisfies

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right); \tag{11}$$

If $\epsilon < 1/4$, then f satisfies either

$$\frac{1}{2}\left(1 + \sqrt{1 - 4\epsilon}\right) \le \|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right) \tag{12}$$

or

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1 - \sqrt{1 - 4\epsilon}\right). \tag{13}$$

We refer the reader to [1, 6–16] for related functional equations and their stabilities. We also refer the reader to [17–19] for some recent developments on the issues of stability and superstability for functional equations.

2. Stability of (1) and (2)

In this section we investigate unbounded functions f, g satisfying (1) and (2) when some of f and g satisfy (3) or ϕ satisfies (4). For bounded solutions of (1) and (2) we refer the reader to [1].

Lemma 1. *Assume that $m : S \to \mathbb{C}$ is an unbounded exponential function and $\phi : S \to \mathbb{R}^+$ satisfies (4). Then m satisfies (3).*

Proof. Since m is unbounded, we can choose a $x_0 \in S$ such that $|m(x_0)| \ge C$, where $C > 1$ is the constant in (4). Since ϕ satisfies (4) we have

$$\begin{aligned} \lim_{k \to \infty} \frac{1 + \phi(kx_0)}{|m(kx_0)|} &= \lim_{k \to \infty} \frac{1 + \phi(kx_0)}{|m(x_0)|^k} \\ &\le \lim_{k \to \infty} \left(C^{-k} + C^{-k}\phi(kx_0)\right) = 0. \end{aligned} \tag{14}$$

This completes the proof. □

Theorem 2. *Let $f, g : S \to \mathbb{C}$ be unbounded functions satisfying*

$$\left| f(x + \sigma y) - g(x) f(y) \right| \le \phi(x) \tag{15}$$

for all $x, y \in S$. Then g is a σ-exponential function. In particular if g satisfies (3) or ϕ satisfies (4), then there exists a σ-exponential function $m_\sigma : S \to \mathbb{C}$ such that

$$f(x) = f(0)\, m_\sigma(x), \qquad g(x) = m_\sigma(x) \tag{16}$$

for all $x \in S$.

Proof. Choosing a sequence $y_n \in S$, $n = 1, 2, 3, \ldots$, such that $|f(y_n)| \to \infty$ as $n \to \infty$, putting $y = y_n$, $n = 1, 2, 3, \ldots$, in (15), dividing the result by $|f(y_n)|$, and letting $n \to \infty$ we have

$$g(x) = \lim_{n \to \infty} \frac{f(x + \sigma y_n)}{f(y_n)} \tag{17}$$

for all $x \in S$. Putting $x = 0$ in (15) we have

$$\left| f(\sigma y) - g(0) f(y) \right| \le \phi(0) \tag{18}$$

for all $y \in S$. Multiplying both sides of (17) by $g(y)$ and using (15), (17), and (18) we have

$$\begin{aligned} g(y)\, g(x) &= \lim_{n \to \infty} \frac{g(y) f(x + \sigma y_n)}{f(y_n)} = \lim_{n \to \infty} \frac{f(y + \sigma x + y_n)}{f(y_n)} \\ &= \lim_{n \to \infty} \frac{g(0) f(x + \sigma y + \sigma y_n)}{f(y_n)} = g(0)\, g(x + \sigma y) \end{aligned} \tag{19}$$

for all $x, y \in S$. Dividing (19) by $g(0)^2$ we have

$$g_0(x)\, g_0(y) = g_0(x + \sigma y) \tag{20}$$

for all $x, y \in S$, where $g_0(x) = g(x)/g(0)$. From (20) we have

$$g(x) = g(0)\, m_\sigma(x) \tag{21}$$

for some σ-exponential m_σ. If g satisfies (3) or ϕ satisfies (4), then, by Lemma 1, we can choose a sequence $x_n \in S$, $n = 1, 2, 3, \ldots$, such that $(1 + \phi(x_n))/|g(x_n)| \to 0$ as $n \to \infty$. Putting $x = x_n$, $n = 1, 2, 3, \ldots$, in (15), dividing the result by $|g(x_n)|$, and letting $n \to \infty$ we have

$$f(y) = \lim_{n\to\infty} \frac{f(x_n + \sigma y)}{g(x_n)} \quad (22)$$

for all $y \in S$. Multiplying both sides of (22) by $g(x)$ and using (15), (18), and (22) we have

$$\begin{aligned} g(x) f(y) &= \lim_{n\to\infty} \frac{g(x) f(x_n + \sigma y)}{g(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x + \sigma x_n + y)}{g(x_n)} \\ &= \lim_{n\to\infty} \frac{g(0) f(\sigma x + \sigma y + x_n)}{g(x_n)} = g(0) f(x + y) \end{aligned} \quad (23)$$

for all $x, y \in S$. Putting $y = 0$ in (23) and dividing the result by $g(0)$ we have

$$f(x) = f(0) g_0(x) = f(0) m_\sigma(x) \quad (24)$$

for all $x \in S$. Putting $x = 0$ in (15) and using (24) we have

$$|f(0)(1 - g(0))| |m_\sigma(y)| \le \phi(0) \quad (25)$$

for all $y \in S$. Since m_σ is unbounded, from (25) we have $g(0) = 1$. Now, from (21) and (24) we get (16). This completes the proof. □

We denote by $c\cdot x$ the inner product of $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ which is defined as $c \cdot x = \sum_{j=1}^n c_j x_j$, and $\mathfrak{R}c = (\mathfrak{R}c_1, \ldots, \mathfrak{R}c_n)$, where $\mathfrak{R}c_j$ are the real parts of c_j, $j = 1, 2, \ldots, n$. It is easy to see that if S is uniquely 2-divisible (i.e., for each $x \in S$ there exists a unique $y \in S$ such that $2y = x$), then m_σ is σ-exponential if and only if

$$m_\sigma(x) = m\left(\frac{x + \sigma x}{2}\right), \quad x \in S \quad (26)$$

for some exponential function $m : S \to \mathbb{C}$.

Corollary 3. *Let $P(x)$, $x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded function satisfying*

$$|f(x + \sigma y) - g(x) f(y)| \le |P(x)| \quad (27)$$

for all $x, y \in \mathbb{R}^n$. Then there exists a σ-exponential function $m_\sigma : \mathbb{R}^n \to \mathbb{C}$ such that

$$f(x) = f(0) m_\sigma(x), \qquad g(x) = m_\sigma(x) \quad (28)$$

for all $x \in \mathbb{R}^n$. In particular if g is continuous, then there exists $c \in \mathbb{C}^n$, $\mathfrak{R}(c + c\sigma) \ne 0$ such that

$$f(x) = f(0) e^{(1/2)(c+c\sigma)\cdot x}, \qquad g(x) = e^{(1/2)(c+c\sigma)\cdot x} \quad (29)$$

for all $x \in \mathbb{R}^n$.

Proof. It is easy to see that $|P(x)|$ satisfies (4). Thus, by Theorem 2 we get (28). Assume that g is continuous. It is well known that every continuous exponential functional $m : \mathbb{R}^n \to \mathbb{C}$ is given by $m(x) = e^{c\cdot x}$ for some $c \in \mathbb{C}^n$. Thus, from (26) we have $m_\sigma(x) = e^{(1/2)(c+c\sigma)\cdot x}$ for all $x \in \mathbb{R}^n$, where $c\sigma$ denotes matrix multiplication. Thus, we get (29). This completes the proof. □

Remark 4. Let $a, b \in \mathbb{R}^n$ be two nonzero vectors that are not parallel; that is, $b \ne ra$ for all $r \in \mathbb{R}$. Then, the hyperplane $b \cdot x = 0$ is not parallel to $(b-a)\cdot x = 0$ and hence there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b - a) \cdot x_0 > 0$. If $b = ta$ for some $t \in \mathbb{R}$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b - a) \cdot x_0 > 0$ if and only if $t > 1$. Thus, if $b \ne ta$ for all $t \le 1$, then there exists $x_0 \in \mathbb{R}^n$ such that $b \cdot x_0 > 0$ and $(b - a) \cdot x_0 > 0$.

Corollary 5. *Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded continuous function satisfying*

$$|f(x + \sigma y) - g(x) f(y)| \le e^{\gamma\cdot x} \quad (30)$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n$, $\mathfrak{R}(c + c\sigma) \ne 0$ such that $g(x) = e^{(1/2)(c+c\sigma)\cdot x}$ for all $x \in \mathbb{R}^n$. If $(1/2)\mathfrak{R}(c + c\sigma) \ne t\gamma$ for all $t \le 1$, then one has

$$f(x) = f(0) e^{(1/2)(c+c\sigma)\cdot x}, \qquad g(x) = e^{(1/2)(c+c\sigma)\cdot x} \quad (31)$$

for all $x \in \mathbb{R}^n$.

Proof. Recall that every continuous σ-exponential functional $m_\sigma : \mathbb{R}^n \to \mathbb{C}$ is given by

$$m_\sigma(x) = e^{(1/2)(c+c\sigma)\cdot x} \quad (32)$$

for all $x \in \mathbb{R}^n$, where $c\sigma$ denotes matrix multiplication. If $(1/2)\mathfrak{R}(c + c\sigma) \ne t\gamma$ for all $t \le 1$, then by Remark 4 there exists $x_0 \in \mathbb{R}^n$ such that

$$\frac{1}{2}(c + c\sigma) \cdot x_0 > 0, \qquad \frac{1}{2}(c + c\sigma) \cdot x_0 > \gamma \cdot x_0. \quad (33)$$

From (32) and (33) we have

$$\lim_{k\to\infty} \frac{1 + e^{\gamma\cdot kx_0}}{g(kx_0)} = \frac{1 + e^{\gamma\cdot kx_0}}{e^{(1/2)(c+c\sigma)\cdot kx_0}} = 0, \quad (34)$$

which implies that g satisfies the condition (3). Thus, we get (31). This completes the proof. □

Theorem 6. *Let $f, g : S \to \mathbb{C}$ be unbounded functions satisfying*

$$|f(x + \sigma y) - g(x) f(y)| \le \phi(y) \quad (35)$$

for all $x, y \in S$. Then there exists an unbounded σ-exponential function $m_\sigma : S \to \mathbb{C}$ such that $f(x) = f(0)m_\sigma(x)$ for all $x \in S$. In particular if f satisfies (3) or ϕ satisfies (4), then one has

$$f(x) = f(0) m_\sigma(x), \qquad g(x) = m_\sigma(x) \quad (36)$$

for all $x \in S$.

Proof. Putting $y = 0$ in (35) we have

$$|f(x) - f(0) g(x)| \leq \phi(0) \tag{37}$$

for all $x \in S$. Choose a sequence $x_n \in S$, $n = 1, 2, 3, \ldots$, such that $|g(x_n)| \to \infty$ as $n \to \infty$. Putting $x = x_n$, $n = 1, 2, 3, \ldots$, in (35), dividing the result by $|g(x_n)|$, letting $n \to \infty$, and using (37) we have

$$f(y) = \lim_{n\to\infty} \frac{f(x_n + \sigma y)}{g(x_n)} = \lim_{n\to\infty} \frac{f(0) g(x_n + \sigma y)}{g(x_n)} \tag{38}$$

for all $y \in S$. Multiplying both sides of (38) by $f(x)$ and using (35) and (38) we have

$$\begin{aligned} f(y) f(x) &= \lim_{n\to\infty} \frac{f(0) g(x_n + \sigma y) f(x)}{g(x_n)} \\ &= \lim_{n\to\infty} \frac{f(0) f(x_n + \sigma y + \sigma x)}{g(x_n)} = f(0) f(x+y) \end{aligned} \tag{39}$$

for all $x, y \in S$. From (39) we have $f_0(x) := f(x)/f(0)$ is an exponential function, say m. Now, from (37) we can write

$$f(x) = f(0) m(x), \qquad g(x) = m(x) + r(x) \tag{40}$$

for all $x \in S$, where $|r(x)| \leq \phi(0)/|f(0)|$ for all $x \in S$. Putting (40) in (35) and using the triangle inequality we have

$$\begin{aligned} |f(0) m(x)| |m(\sigma y) - m(y)| &\leq \phi(y) + r(x) |f(y)| \\ &\leq \phi(y) + \frac{\phi(0) |f(y)|}{|f(0)|} \end{aligned} \tag{41}$$

for all $x, y \in S$. Since m is unbounded, from (41) we have $m(\sigma y) = m(y)$ for all $y \in S$. Assume that f satisfies (3) or ϕ satisfies (4). Choose a sequence $y_n \in S$, $n = 1, 2, 3, \ldots$, such that $(1 + \phi(y_n))/|f(y_n)| \to 0$ as $n \to 0$. Putting $y = y_n$, $n = 1, 2, 3, \ldots$, in (35), dividing the result by $|f(y_n)|$, letting $n \to 0$, and using (37) we have

$$g(x) = \lim_{n\to\infty} \frac{f(x + \sigma y_n)}{f(y_n)} = \lim_{n\to\infty} \frac{f(0) g(x + \sigma y_n)}{f(y_n)} \tag{42}$$

for all $x \in S$. Multiplying both sides of (42) by $f(y)$ and using (35) and (42) we have

$$\begin{aligned} g(x) f(y) &= \lim_{n\to\infty} \frac{f(0) g(x + \sigma y_n) f(y)}{f(y_n)} \\ &= \lim_{n\to\infty} \frac{f(0) f(x + \sigma y + \sigma y_n)}{f(y_n)} = f(0) g(x + \sigma y) \end{aligned} \tag{43}$$

for all $x, y \in S$. Putting $x = 0$ in (43), replacing σy by x, and dividing the result by $f(0)$ we have

$$g(x) = g(0) f_0(\sigma x) = g(0) m_\sigma(\sigma x) = g(0) m_\sigma(x) \tag{44}$$

for all $x \in S$. Putting $y = 0$ in (35) and using (40) and (44) we get $g(0) = 1$. This completes the proof. □

Using Theorem 6 and applying the same method as in the proof of Corollary 3 we have the following.

Corollary 7. *Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded function satisfying*

$$|f(x + \sigma y) - g(x) f(y)| \leq |P(y)| \tag{45}$$

for all $x, y \in \mathbb{R}^n$. Then there exists a σ-exponential function $m_\sigma : \mathbb{R}^n \to \mathbb{C}$ such that

$$f(x) = f(0) m_\sigma(x), \qquad g(x) = m_\sigma(x) \tag{46}$$

for all $x \in \mathbb{R}^n$.

Using Theorem 6 and applying the same method as in the proof of Corollary 5 we have the following.

Corollary 8. *Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded continuous function satisfying*

$$|f(x + \sigma y) - g(x) f(y)| \leq e^{\gamma \cdot y} \tag{47}$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n$, $\mathfrak{R}(c + c\sigma) \neq 0$ such that $f(x) = f(0) e^{(1/2)(c+c\sigma)\cdot x}$ for all $x \in \mathbb{R}^n$. If $(1/2)\mathfrak{R}(c+c\sigma) \neq t\gamma$ for all $t \leq 1$, then one has

$$f(x) = f(0) e^{(1/2)(c+c\sigma)\cdot x}, \qquad g(x) = e^{(1/2)(c+c\sigma)\cdot x} \tag{48}$$

for all $x \in \mathbb{R}^n$.

Theorem 9. *Let $f, g : S \to \mathbb{C}$ be unbounded functions satisfying*

$$|f(x + \sigma y) - f(x) g(y)| \leq \phi(x) \tag{49}$$

for all $x, y \in S$. Then there exists an unbounded exponential $m : S \to \mathbb{C}$ such that $f(x) = f(0) m(x)$ for all $x \in S$. In particular if f satisfies (3) or ϕ satisfies (4), then one has

$$f(x) = f(0) m(x), \qquad g(x) = m(\sigma x) \tag{50}$$

for all $x \in S$.

Proof. Putting $x = 0$ in (49) we have

$$|f(\sigma y) - f(0) g(y)| \leq \phi(0) \tag{51}$$

for all $y \in S$. Choose a sequence $y_n \in S, n = 1, 2, 3, \ldots$, such that $|g(y_n)| \to \infty$ as $n \to \infty$. Putting $y = y_n, n = 1, 2, 3, \ldots$, in (49), dividing the result by $|g(y_n)|$, letting $n \to \infty$, and using (51) we have

$$f(x) = \lim_{n\to\infty} \frac{f(x + \sigma y_n)}{g(y_n)} = \lim_{n\to\infty} \frac{f(0) g(\sigma x + y_n)}{g(y_n)} \tag{52}$$

for all $x \in S$. Multiplying both sides of (52) by $f(y)$ and using (49) and (52) we have

$$\begin{aligned} f(y) f(x) &= \lim_{n\to\infty} \frac{f(0) f(y) g(\sigma x + y_n)}{g(y_n)} \\ &= \lim_{n\to\infty} \frac{f(0) f(y + x + \sigma y_n)}{g(y_n)} = f(0) f(x+y) \end{aligned} \tag{53}$$

for all $x, y \in S$. From (53) we have $f_0(x) := f(x)/f(0)$ is an exponential function, say m. Assume that g satisfies (3) or ϕ satisfies (4). Choose a sequence $x_n \in S$, $n = 1, 2, 3, \ldots$, such that $(1 + \phi(x_n))/|f(x_n)| \to 0$ as $n \to 0$. Putting $x = x_n, n = 1, 2, 3, \ldots$, in (49), dividing the result by $|f(x_n)|$, letting $n \to 0$, and using (51) we have

$$g(y) = \lim_{n\to\infty} \frac{f(x_n + \sigma y)}{f(x_n)} = \lim_{n\to\infty} \frac{f(0)\, g(y + \sigma x_n)}{f(x_n)} \tag{54}$$

for all $y \in S$. Multiplying both sides of (54) by $f(x)$ and using (49) and (54) we have

$$\begin{aligned} f(x)\, g(y) &= \lim_{n\to\infty} \frac{f(0)\, f(x)\, g(y + \sigma x_n)}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(0)\, f(x + \sigma y + x_n)}{f(x_n)} = f(0)\, g(\sigma x + y) \end{aligned} \tag{55}$$

for all $x, y \in S$. Putting $y = 0$ in (55), replacing σx by x, and dividing the result by $f(0)$ we have

$$g(x) = g(0)\, f_0(\sigma x) = g(0)\, m(\sigma x) \tag{56}$$

for all $x \in S$. Putting $x = 0$ in (49) and using (56) we get $g(0) = 1$. This completes the proof. □

Using Theorem 9 we have the following.

Corollary 10. *Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded function satisfying*

$$|f(x + \sigma y) - f(x)\, g(y)| \le |P(x)| \tag{57}$$

for all $x, y \in \mathbb{R}^n$. Then there exists an exponential function $m : \mathbb{R}^n \to \mathbb{C}$ such that

$$f(x) = f(0)\, m(x), \qquad g(x) = m(\sigma x) \tag{58}$$

for all $x \in \mathbb{R}^n$.

Corollary 11. *Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded continuous function satisfying*

$$|f(x + \sigma y) - f(x)\, g(y)| \le e^{\gamma \cdot x} \tag{59}$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n, \Re c \ne 0$ such that $f(x) = f(0)e^{c\cdot x}$ for all $x \in \mathbb{R}^n$. If $\Re c \ne t\gamma$ for all $t \le 1$, then we have

$$f(x) = f(0)\, e^{c\cdot x}, \qquad g(x) = e^{c\sigma\cdot x} \tag{60}$$

for all $x \in \mathbb{R}^n$.

Theorem 12. *Let $f, g : S \to \mathbb{C}$ be unbounded functions satisfying*

$$|f(x + \sigma y) - f(x)\, g(y)| \le \phi(y) \tag{61}$$

for all $x, y \in S$. Then g is an exponential function. In particular, if g satisfies the condition (3) or ϕ satisfies (4), then there exists an unbounded exponential $m : S \to \mathbb{C}$ such that

$$f(x) = f(0)\, m(x), \qquad g(x) = m(\sigma x) \tag{62}$$

for all $x \in S$.

Proof. Choose a sequence $x_n \in S, n = 1, 2, 3, \ldots$, such that $|f(x_n)| \to \infty$ as $n \to \infty$. Putting $x = x_n, n = 1, 2, 3, \ldots$, in (61), dividing the result by $|f(x_n)|$, and letting $n \to \infty$ we have

$$g(y) = \lim_{n\to\infty} \frac{f(x_n + \sigma y)}{f(x_n)} \tag{63}$$

for all $y \in S$. Multiplying both sides of (63) by $g(x)$ and using (61) and (63) we have

$$\begin{aligned} g(y)\, g(x) &= \lim_{n\to\infty} \frac{f(x_n + \sigma y)\, g(x)}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n + \sigma y + \sigma x)}{f(x_n)} = g(x + y) \end{aligned} \tag{64}$$

for all $x, y \in S$. Therefore, g is an exponential function, say m. Assume that g satisfies (3) or ϕ satisfies (4). Choose a sequence $y_n \in S, n = 1, 2, 3, \ldots$, such that $(1 + \phi(y_n))/|g(y_n)| \to 0$ as $n \to \infty$. Putting $y = y_n, n = 1, 2, 3, \ldots$, in (61), dividing the result by $|g(y_n)|$, and letting $n \to \infty$ we have

$$f(x) = \lim_{n\to\infty} \frac{f(x + \sigma y_n)}{g(y_n)} \tag{65}$$

for all $x \in S$. Multiplying both sides of (65) by $g(y)$ and using (61) and (65) we have

$$\begin{aligned} f(x)\, g(y) &= \lim_{n\to\infty} \frac{f(x + \sigma y_n)\, g(y)}{g(y_n)} \\ &= \lim_{n\to\infty} \frac{f(x + \sigma y_n + \sigma y)}{g(y_n)} = f(x + \sigma y) \end{aligned} \tag{66}$$

for all $x, y \in S$. Putting $x = 0$ and replacing y by σx in (66) we have $f(x) = f(0)m(\sigma x)$ for all $x \in S$. Replacing $m(x)$ by $m(\sigma x)$, we get (62). This completes the proof. □

Using Theorem 12 we have the following.

Corollary 13. *Let $P(x), x \in \mathbb{R}^n$, be a polynomial. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded function satisfying*

$$|f(x + \sigma y) - f(x)\, g(y)| \le |P(y)| \tag{67}$$

for all $x, y \in \mathbb{R}^n$. Then there exists an exponential function $m : \mathbb{R}^n \to \mathbb{C}$ such that

$$f(x) = f(0)\, m(x), \qquad g(x) = m(\sigma x) \tag{68}$$

for all $x \in \mathbb{R}^n$.

Corollary 14. *Let $\gamma \in \mathbb{R}^n$ be fixed. Suppose that $f, g : \mathbb{R}^n \to \mathbb{C}$ are unbounded continuous function satisfying*

$$|f(x + \sigma y) - f(x)\, g(y)| \le e^{\gamma \cdot y} \tag{69}$$

for all $x, y \in \mathbb{R}^n$. Then there exists $c \in \mathbb{C}^n, \Re c \ne 0$ such that $g(x) = e^{c\sigma\cdot x}$ for all $x \in S$. If $\Re c \ne t\gamma$ for all $t \le 1$, then we have

$$f(x) = f(0)\, e^{c\cdot x}, \qquad g(x) = e^{c\sigma\cdot x} \tag{70}$$

for all $x \in S$.

3. Applications

In this section we consider the stability of (6)~(8). A function $M : (0, \infty) \to \mathbb{R}$ is called *multiplicative function* provided that $M(xy) = M(x)M(y)$ for all $x, y > 0$. Let $F(x + yi) = f(x, y)$, $G(x + iy) = g(x, y)$, and $\Phi(x + yi) = \phi(x, y)$ for all $(x, y) \in \mathbb{R}_0^2$. Then the functional inequalities (6) and (7) are converted to

$$|F(z\overline{w}) - G(z)F(w)| \le \Phi(z)\,[\text{resp., } \Phi(w)], \quad (71)$$

$$|F(z\overline{w}) - F(z)G(w)| \le \Phi(z)\,[\text{resp., } \Phi(w)] \quad (72)$$

for all $z, w \in \mathbb{C}_0$.

Viewing $\mathbb{C}_0$ as a multiplicative group, letting $\sigma(z) = \overline{z}$, and applying Theorems 2 and 6 to the inequalities (71) we have the following.

Theorem 15. *Let $f, g : \mathbb{R}_0^2 \to \mathbb{R}$ be unbounded functions satisfying* (6). *Then f, g are of the form*

$$f(x, y) = f(1, 0)\,M\left(\sqrt{x^2 + y^2}\right), \qquad g(x, y) = M\left(\sqrt{x^2 + y^2}\right) \quad (73)$$

for all $x, y \in \mathbb{R}$, where $M : (0, \infty) \to \mathbb{R}$ is a multiplicative function.

Applying Theorems 9 and 12 to the inequalities (72) we have the following.

Theorem 16. *Let $f, g : \mathbb{R}_0^2 \to \mathbb{R}$ be unbounded functions satisfying* (7). *Then f, g are of the form*

$$f(x, y) = f(1, 0)\,M\left(\sqrt{x^2 + y^2}\right)E\left(\tan^{-1}\left(\frac{y}{x}\right)\right), \qquad g(x, y) = M\left(\sqrt{x^2 + y^2}\right)E\left(-\tan^{-1}\left(\frac{y}{x}\right)\right) \quad (74)$$

for all $(x, y) \in \mathbb{R}_0^2$, where $M : (0, \infty) \to \mathbb{R}$ is a multiplicative function and $E : \mathbb{R} \to \mathbb{R}$ is an exponential function satisfying $E(2\pi) = 1$.

Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the set of quaternions. Recall that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$ and the conjugate of $q = a + bi + cj + dk \in \mathbb{H}$ is given by $q^* = a - bi - cj - dk$. We denote $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$. Let $F(x + yi + uj + vk) = f(x, y, u, v)$, $G(x + iy + uj + vk) = g(x, y, u, v)$, and $\Phi(x + yi + uj + vk) = \phi(x, y, u, v)$ for all $(x, y, u, v) \in \mathbb{R}_0^4$. Then the functional inequalities (8) are converted to

$$\left|F(qp^*) - G(q)F(p)\right| \le \Phi(q)\,[\text{resp., } \Phi(p)] \quad (75)$$

for all $p, q \in \mathbb{H} \setminus \{0\}$.

Applying Theorems 2 and 6 to the inequalities (75) we have the following.

Theorem 17. *Let $f, g : \mathbb{R}_0^4 \to \mathbb{R}$ be unbounded functions satisfying* (8). *Then f, g are of the form*

$$f(x, y, u, v) = f(1, 0, 0, 0)\,M\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right), \qquad g(x, y, u, v) = M\left(\sqrt{x^2 + y^2 + u^2 + v^2}\right) \quad (76)$$

for all $(x, y, u, v) \in \mathbb{R}_0^4$, where $M : (0, \infty) \to \mathbb{R}$ is a multiplicative function.

4. Stability in L^∞-Version

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a locally integrable function and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ an involution. In this section, we consider an L^∞-version of the stability of functional equation

$$f(x + \sigma y) = f(x)\,f(y) \quad (77)$$

for almost every $(x, y) \in \mathbb{R}^{2n}$. More precisely, we study the functional inequality

$$\|f(x + \sigma y) - f(x)\,f(y)\|_{L^\infty(\mathbb{R}^{2n})} \le \epsilon. \quad (78)$$

As is well known, inequality (78) implies

$$\left|\iint \left(f(x + \sigma y) - f(x)\,f(y)\right)\varphi(x, y)\,dx\,dy\right| \le \epsilon\|\varphi\|_{L^1(\mathbb{R}^{2n})} \quad (79)$$

for all $\varphi \in L^1(\mathbb{R}^4)$.

We first employ $\delta : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta(x) = \begin{cases} qe^{-(1-|x|^2)^{-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1, \end{cases} \quad (80)$$

where $q = (\int_{|x|<1} e^{-(1-|x|^2)^{-1}}dx)^{-1}$. It is easy to see that δ is an infinitely differentiable function with support $\{x : |x| \le 1\}$.

Let f be a locally integrable function on $\mathbb{R}^n$ and $\delta_t(x) := t^{-n}\delta(x/t)$, $t > 0$. Then for each $t > 0$,

$$f * \delta_t(x) = \int_{\mathbb{R}^n} f(\xi)\,\delta_t(x - \xi)\,d\xi \quad (81)$$

is a smooth function and $f * \delta_t(x) \to f(x)$ for almost every $x \in \mathbb{R}^n$ as $t \to 0^+$.

In the following, we exclude the case when $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

Theorem 18. *Let $f : \mathbb{R}^n \to \mathbb{C}$ satisfy* (78). *Then either there exists an unbounded exponential function $m : \mathbb{R}^n \to \mathbb{C}$ such that*

$$f(x) = m\left(\frac{x + \sigma x}{2}\right) \quad (82)$$

for almost every $x \in \mathbb{R}^n$, or else

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right). \quad (83)$$

If $\epsilon < 1/4$, *then either*

$$\frac{1}{2}\left(1+\sqrt{1-4\epsilon}\right) \le \|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1+\sqrt{1+4\epsilon}\right) \tag{84}$$

or

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{2}\left(1-\sqrt{1-4\epsilon}\right). \tag{85}$$

Proof. Applying $\varphi(x', y') = \delta_t(x-x')\delta_s(y-y')$ in (79) we have

$$\begin{aligned}
&\iint f\left(x'+\sigma y'\right)\delta_t\left(x-x'\right)\delta_s\left(y-y'\right)dx'dy' \\
&= \int f\left(z'\right)\left(\int \delta_t\left(x-z'+\sigma y'\right)\delta_s\left(y-y'\right)dy'\right)dz' \\
&= \int f\left(z'\right)\left(\int \delta_t\left(x-z'+\sigma y'\right)\left(\delta_s\circ\sigma\right) \times \left(\sigma y-\sigma y'\right)dy'\right)dz' \\
&= \int f\left(z'\right)\left(\int \delta_t\left(x-z'+y'\right)\left(\delta_s\circ\sigma\right)\left(\sigma y-y'\right)dy'\right)dz' \\
&= \int f\left(z'\right)\left(\int \delta_t\left(y'\right)\left(\delta_s\circ\sigma\right)\left(x+\sigma y-z'-y'\right)dy'\right)dz' \\
&= \int f\left(z'\right)\delta_t * \left(\delta\circ\sigma\right)_s\left(x+\sigma y-z'\right)dz' \\
&= \left(f*\delta_t*\left(\delta\circ\sigma\right)_s\right)\left(x+\sigma y\right).
\end{aligned} \tag{86}$$

We also have

$$\begin{aligned}
&\iint f\left(x'\right)f\left(y'\right)\delta_t\left(x-x'\right)\delta_s\left(y-y'\right)dx'dy' \\
&= \int f\left(y'\right)\left(\int f\left(x'\right)\delta_t\left(x-x'\right)dx'\right)\delta_s\left(y-y'\right)dy' \\
&= \left(f*\delta_t\right)(x)\int f\left(y'\right)\left(f*\delta_s\right)\left(y-y'\right)dy \\
&= \left(f*\delta_t\right)(x)\left(f*\delta_s\right)(y).
\end{aligned} \tag{87}$$

Thus, the inequality (78) is converted to the classical functional inequality

$$\left|\left(f*\delta_t*\delta_s^\sigma\right)(x+\sigma y)-\left(f*\delta_t\right)(x)\left(f*\delta_s\right)(y)\right| \le \epsilon \tag{88}$$

for all $x, y \in \mathbb{R}^n$, where $\delta^\sigma = \delta\circ\sigma$.

Choosing $y_0 \in \mathbb{R}$, $s_0 > 0$ such that $(f*\delta_{s_0})(y_0) \ne 0$, putting $y = y_0$, $s = s_0$ in (88), using the triangle inequality, and dividing the result by $|(f*\delta_{s_0})(y_0)|$ we have

$$\left|\left(f*\delta_t\right)(x)\right| \le \frac{\left|\left(f*\delta_t*\delta_{s_0}^\sigma\right)\left(x+\sigma y_0\right)\right|+\epsilon}{\left|\left(f*\delta_{s_0}\right)\left(y_0\right)\right|} \tag{89}$$

for all $x \in \mathbb{R}^n$. Since $(f*\delta_t*\delta_{s_0}^\sigma)(x+\sigma y_0) \to (f*\delta_{s_0}^\sigma)(x+\sigma y_0)$ as $t \to 0^+$, it follows that

$$F(x) := \limsup_{t\to 0^+}\left(f*\delta_t\right)(x) \tag{90}$$

exists for all $x \in \mathbb{R}^n$. Since $(f*\delta_t)(x) \to f(x)$ for almost every $x \in \mathbb{R}^n$, it follows from (90) that

$$F(x) = f(x) \tag{91}$$

for almost every $x \in \mathbb{R}^n$.

Fixing $y \in \mathbb{R}^n$ and letting $s \to 0^+$ so that $(f*\delta_s)(y) \to F(y)$ in (88), we have

$$\left|\left(f*\delta_t\right)(x+\sigma y)-\left(f*\delta_t\right)(x)F(y)\right| \le \epsilon \tag{92}$$

for all $x, y \in \mathbb{R}^n$. We first consider the case when F is unbounded. Let $y_n \in \mathbb{R}^n$, $n = 1, 2, 3, \ldots$, be a sequence such that $|F(y_n)| \to \infty$. Putting $y = y_n$ in (92), dividing the result by $|F(y_n)|$, and letting $n \to \infty$ we have

$$\left(f*\delta_t\right)(x) = \lim_{n\to\infty}\frac{\left(f*\delta_t\right)\left(x+\sigma y_n\right)}{F\left(y_n\right)} \tag{93}$$

for all $(x, y) \in \mathbb{R}^n$. Multiplying $F(y)$ in (93) and using (92) and (93) we have

$$\begin{aligned}
\left(f*\delta_t\right)(x)F(y) &= \lim_{n\to\infty}\frac{\left(f*\delta_t\right)\left(x+\sigma y_n\right)F(y)}{F\left(y_n\right)} \\
&= \lim_{n\to\infty}\frac{\left(f*\delta_t\right)\left(x+\sigma y+\sigma y_n\right)}{F\left(y_n\right)} \\
&= \left(f*\delta_t\right)(x+\sigma y)
\end{aligned} \tag{94}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Putting $x = 0$ in (94) we have

$$\left(f*\delta_t\right)(0)F(y) = \left(f*\delta_t\right)(\sigma y) \tag{95}$$

for all $y \in \mathbb{R}^n$, $t > 0$. From (95) we have $(f*\delta_t)(0) \ne 0$ for some $t > 0$. Putting (95) in (94) we have

$$F(\sigma x)F(y) = F(y+\sigma x) \tag{96}$$

for all $x, y \in \mathbb{R}^n$. From (96) F is an exponential function. Now, we prove that

$$F(x) = F(\sigma x) \tag{97}$$

for all $x \in \mathbb{R}^n$. In view of (94), replacing $(f*\delta_t)(x)$ by $(f*\delta_t)(0)F(\sigma x)$ and $(f*\delta_s)(y)$ by $(f*\delta_s)(0)F(\sigma y)$ in (88) and letting $s \to 0^+$ so that $(f*\delta_s)(0) \to F(0)$ we have

$$\left|\left(f*\delta_t\right)(x+\sigma y)-\left(f*\delta_t\right)(0)F(\sigma x)F(\sigma y)\right| \le \epsilon \tag{98}$$

for all $x, y \in \mathbb{R}^n$, $t > 0$. Using (95) and (98) we have

$$\left|\left(f*\delta_t\right)(0)F(y+\sigma x)-\left(f*\delta_t\right)(0)F(\sigma x)F(\sigma y)\right| \le \epsilon. \tag{99}$$

Letting $t \to 0^+$ in (99) so that $(f * \delta_t)(0) \to F(0)$ we have

$$|F(y+\sigma x) - F(\sigma x) F(\sigma y)| \le \epsilon \tag{100}$$

for all $x, y \in \mathbb{R}$. Since F is an exponential function, it follows from (100) that

$$|F(\sigma x)| |F(y) - F(\sigma y)| \le \epsilon \tag{101}$$

for all $x, y \in \mathbb{R}^n$. Since F is unbounded, from (101) we have $F(y) = F(\sigma y)$ for all $y \in \mathbb{R}^n$. Now, F is written in the form

$$\begin{aligned} F(x) &= F\left(\frac{x}{2} + \frac{x}{2}\right) = F\left(\frac{x}{2}\right) F\left(\frac{x}{2}\right) \\ &= F\left(\frac{x}{2}\right) F\left(\frac{\sigma x}{2}\right) = F\left(\frac{x + \sigma x}{2}\right) \end{aligned} \tag{102}$$

for all $x, y \in \mathbb{R}^n$. Conversely, let $F(x) = m((x+\sigma x)/2)$, where $m : \mathbb{R}^n \to \mathbb{C}$ is an arbitrary exponential function. Then F is an exponential function satisfying $F(x) = F(\sigma x)$ for all $x \in \mathbb{R}^n$. Thus, we get (82). From now on, we assume that F is bounded, say $|F(x)| \le M$ for all $x \in \mathbb{R}^n$. Then, it follows from (91) that $\|f\|_{L^\infty(\mathbb{R}^n)} \le M$. Thus, we have

$$\begin{aligned} |(f * \delta_t)(x)| &= \left| \int f(x') \delta(x - x') \, dx' \right| \\ &\le M \int |\delta(x - x')| \, dx' = M \end{aligned} \tag{103}$$

for all $x \in \mathbb{R}^n$. From the inequality (92), using the method in Theorem 10 of [1] we have

$$|(f * \delta_t)(x)(|F(y)| - 1)| \le \epsilon \tag{104}$$

for all $x, y \in \mathbb{R}^n, t > 0$. Fixing $x \in \mathbb{R}^n$ and letting $t \to 0^+$ in (104) we have

$$|F(x)(|F(y)| - 1)| \le \epsilon \tag{105}$$

for all $x, y \in \mathbb{R}$. From (105), using the method in Theorem 10 of [1] we have

$$|F(x)| \le \frac{1}{2}\left(1 + \sqrt{1 - 4\epsilon}\right) \tag{106}$$

for all $x \in \mathbb{R}$, and if $\epsilon < 1/4$, then we have either

$$\frac{1}{2}\left(1 - \sqrt{1 + 4\epsilon}\right) \le |F(x)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right) \tag{107}$$

for all $x \in \mathbb{R}$ or

$$|F(x)| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\epsilon}\right) \tag{108}$$

for all $x \in \mathbb{R}$. Since $f(x) = F(x)$ almost every $x \in \mathbb{R}^n$, we get (83), (84), and (85). This completes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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