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Review Article

A Class of Logarithmically Completely Monotonic Functions and Their Applications

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We study the recent investigations on a class of functions which are logarithmically completely monotonic. Two open problems are also presented.

1. Introduction

Recall [1] that a positive function f is said to be logarithmically completely monotonic (LCM) on an open interval I if f has derivatives of all orders on I and for all $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$,

$$(-1)^n [\ln f(x)]^{(n)} \geq 0. \tag{1}$$

LCM functions are related to completely monotonic (CM) functions [2], strongly logarithmically completely monotonic (SLCM) functions [3], almost strongly completely monotonic (ASCM) functions [3], almost completely monotonic (ACM) functions [4], Laplace transforms, and Stieltjes transforms and have wide applications. It is evident that the set of SLCM functions is a nontrivial subset of the set of LCM functions, which is a nontrivial subset of the set of CM functions, and that the set of CM functions is a nontrivial subset of the set of ACM functions. It was established [3] that the set of SLCM functions is a nontrivial subset of the set of ASCM functions and that the set of SLCM functions on the interval $(0, \infty)$ is disjoint with the set of strongly completely monotonic (SCM) functions (see [5] for its definition) on the interval $(0, \infty)$.

It is well known that the classical Euler gamma function is defined for $x > 0$ by

$$\Gamma(z) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{2}$$

The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{3}$$

is called psi function, and $\psi^{(k)}$ for $k \in \mathbb{N}$ are called polygamma functions.

For $\alpha, \gamma \in \mathbb{R}$ and $\beta \geq 0$, define

$$f_{\alpha,\beta,\gamma}(x) := \left[\frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right]^\gamma, \quad x \in (0, \infty), \tag{4}$$

which is encountered in probability and statistics.

Since $f_{\alpha,\beta,\gamma}(x)$ ($\gamma > 0$) is logarithmically completely monotonic if and only if $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic and $f_{\alpha,\beta,\gamma}(x)$ ($\gamma < 0$) is logarithmically completely monotonic if and only if $f_{\alpha,\beta,-1}(x)$ is logarithmically completely monotonic, we only need to study the logarithmically complete monotonicity of the function

$$f_{\alpha,\beta,\pm 1}(x) = \left[\frac{e^x \Gamma(x+\beta)}{x^{x+\beta-\alpha}}\right]^{\pm 1}, \quad x \in (0, \infty). \tag{5}$$

In [6, Theorem 3.2], it was proved that the function $f_{1/2,0,1}(x)$ is decreasing and logarithmically convex from $(0, \infty)$ onto $(\sqrt{2\pi}, \infty)$ and that the function $f_{1,0,1}(x)$ is increasing and logarithmically concave from $(0, \infty)$ onto $(1, \infty)$.

In [7, Theorem 1], for showing

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b} \tag{6}$$

for

$$b > a > 1, \tag{7}$$

monotonic properties of the functions $\ln f_{\alpha,0,1}(x)$ and $\ln f_{\alpha,0,1}(x)$ on the interval $(1,\infty)$ were obtained.

In [8, Theorem 2], it was presented that the function $f_{\alpha,0,1}(x)$ is decreasing on the interval (c,∞) for $c \ge 0$ if and only if

$$\alpha \le \frac{1}{2} \tag{8}$$

and increasing on the interval (c,∞) if and only if

$$\alpha \ge \begin{cases} c\left[\ln c - \psi(c)\right] & \text{if } c > 0, \\ 1 & \text{if } c = 0. \end{cases} \tag{9}$$

In [9], after proving the logarithmically completely monotonic property of the functions $f_{1/2,0,1}(x)$ and $f_{1,0,-1}(x)$, in virtue of Jensen's inequality for convex functions, the upper and lower bounds for the Gurland's ratio were established: for positive numbers x and y, the inequality

$$\frac{x^{x-1/2}y^{y-1/2}}{\left[(x+y)/2\right]^{x+y-1}} \le \frac{\Gamma(x)\Gamma(y)}{\left[\Gamma((x+y)/2)\right]^2} \le \frac{x^{x-1}y^{y-1}}{\left[(x+y)/2\right]^{x+y-2}} \tag{10}$$

holds true, where the middle term in (10) is called Gurland's ratio [10].

In [11] the authors proved the following result.

Theorem 1 (see [11]). *If*

$$2\alpha \le 1 \le \beta, \tag{11}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

The necessary and sufficient conditions for the functions $f_{\alpha,0,1}(x)$ and $f_{\alpha,0,-1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ were also given in [11].

Using monotonic properties of the functions $f_{1/2,0,1}(x)$ and $f_{1,0,-1}(x)$, the inequality (6) was extended (see [11, Remark 1]) from

$$b > a > 1 \tag{12}$$

to

$$b > a > 0. \tag{13}$$

In [12] the authors proved the following results.

Theorem 2 (see [12]). *If $\beta > 0$ and $\alpha \le 0$, then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval* $(0,\infty)$.

Theorem 3 (see [12]). *For $\beta > 0$, a necessary condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that*

$$\alpha \le \min\left\{\beta, \frac{1}{2}\right\}. \tag{14}$$

Theorem 4 (see [12]). *For $\beta \ge 1$, a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0,\infty)$ is that*

$$\alpha \le \frac{1}{2}. \tag{15}$$

As direct consequences of the above results, the following Kečkić-Vasić-type inequality is deduced.

Theorem 5 (see [12]). *Let x and y be positive numbers with $x \ne y$.*

(1) *For $\beta \ge 1$, the following inequality*

$$I(x,y) > \left[\left(\frac{x}{y}\right)^{\alpha-\beta}\frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{16}$$

holds true if and only if $\alpha \le 1/2$, where

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \quad (a > 0,\ b > 0,\ a \ne b) \tag{17}$$

is the identric or exponential mean.

(2) *For $\beta > 0$, the inequality* (16) *holds true also if $\alpha \le 0$.*

In [13], the following result was established.

Theorem 6 (see [13]). (1) *For $\beta \in [0, 1/2)$, if*

$$\alpha \le \beta - e^{-4}(1-\beta)^2\exp\left(\frac{2}{1-\beta}\right), \tag{18}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

(2) *For $\beta \in [1/2, 1]$, if*

$$\alpha \le \min\left\{3\beta^2 - 3\beta + 1, \frac{1}{2}\right\}, \tag{19}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

From Theorem 6 we can directly obtain the following new result.

Corollary 7. (1) *For $\beta \in [1/4, 1/2]$, if*

$$\alpha \le \beta - \frac{1}{4}, \tag{20}$$

then the function $f_{\alpha,\beta,1}(x)$ is logarithmically completely monotonic on the interval $(0,\infty)$.

(2) *For* $\beta \in (1/2, 3/4]$, *if*

$$\alpha \leq \beta - \frac{1}{3}, \tag{21}$$

then the function $f_{\alpha,\beta,1}(x)$ *is logarithmically completely monotonic on the interval* $(0, \infty)$.

(3) *For* $\beta \in (3/4, 1]$, *if*

$$\alpha \leq \beta - \frac{1}{2}, \tag{22}$$

then the function $f_{\alpha,\beta,1}(x)$ *is logarithmically completely monotonic on the interval* $(0, \infty)$.

A necessary and sufficient condition is obtained in [13] as follows.

Theorem 8 (see [13]). *For*

$$\beta \in \{0\} \cup \left[\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right), \tag{23}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ *to be logarithmically completely monotonic on the interval* $(0, \infty)$ *is that*

$$\alpha \leq \frac{1}{2}. \tag{24}$$

Regarding the logarithmically complete monotonicity for the function $f_{\alpha,\beta,-1}(x)$ and their applications. In [14], the authors proved the following results.

Theorem 9 (see [14]). *If the function* $f_{\alpha,\beta,-1}(x)$ *is logarithmically completely monotonic on the interval* $(0, \infty)$, *then either*

$$\beta > 0, \qquad \alpha \geq \max\left\{\beta, \frac{1}{2}\right\} \tag{25}$$

or

$$\beta = 0, \qquad \alpha \geq 1. \tag{26}$$

Theorem 10 (see [14]). *For*

$$\beta \geq \frac{1}{2}, \tag{27}$$

the necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ *to be logarithmically completely monotonic on the interval* $(0, \infty)$ *is that*

$$\alpha \geq \beta. \tag{28}$$

As first application, the following inequalities are derived by using logarithmically completely monotonic properties of the function $f_{\alpha,\beta,\pm 1}(x)$ on the interval $(0, \infty)$.

Theorem 11 (see [14]). (1) *For* $k \in \mathbb{N}$, *double inequalities*

$$\begin{gathered} \ln x - \frac{1}{x} \leq \psi(x) \leq \ln x - \frac{1}{2x}, \\ \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} \leq (-1)^{k+1} \psi^{(k)}(x) \leq \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \end{gathered} \tag{29}$$

hold true on the interval $(0, \infty)$.

(2) *When* $\beta > 0$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \leq \ln x + \frac{\beta}{x}, \\ (-1)^k \psi^{(k-1)}(x+\beta) \geq \frac{(k-2)!}{x^{k-1}} - \frac{\beta(k-1)!}{x^k} \end{gathered} \tag{30}$$

hold true on the interval $(0, \infty)$ *for* $k \geq 2$.

(3) *When* $\beta \geq 1/2$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \geq \ln x, \\ (-1)^k \psi^{(k-1)}(x+\beta) \leq \frac{(k-2)!}{x^{k-1}} \end{gathered} \tag{31}$$

hold true on the interval $(0, \infty)$ *for* $k \geq 2$.

(4) *When* $\beta \geq 1$, *inequalities*

$$\begin{gathered} \psi(x+\beta) \leq \ln x + \frac{\beta - 1/2}{x}, \\ (-1)^k \psi^{(k-1)}(x+\beta) \geq \frac{(k-2)!}{x^{k-1}} - \frac{(\beta - 1/2)(k-1)!}{x^k} \end{gathered} \tag{32}$$

hold true on the interval $(0, \infty)$ *for* $k \geq 2$.

As second application, the following inequalities are derived by using logarithmically convex properties of the function $f_{\alpha,\beta,\pm 1}(x)$ on $(0, \infty)$.

Theorem 12 (see [14]). *Let* $n \in \mathbb{N}$ *and*

$$x_k > 0 \quad (1 \leq k \leq n). \tag{33}$$

Suppose also that

$$\sum_{k=1}^{n} p_k = 1 \quad (p_k \geq 0). \tag{34}$$

If either

$$\beta > 0, \qquad \alpha \leq 0 \tag{35}$$

or

$$\beta \geq 1, \qquad \alpha \leq \frac{1}{2}, \tag{36}$$

then

$$\frac{\prod_{k=1}^{n} [\Gamma(x_k + \beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)} \geq \frac{\prod_{k=1}^{n} x_k^{p_k (x_k + \beta - \alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}. \tag{37}$$

If

$$\alpha \geq \beta \geq \frac{1}{2}, \tag{38}$$

then the inequality (37) *reverses.*

As final application, the following inequality can be derived by using the decreasingly monotonic property of the function $f_{\alpha,\beta,-1}(x)$ on $(0, \infty)$.

Theorem 13 (see [14]). *If*

$$\alpha \geq \beta \geq \frac{1}{2}, \tag{39}$$

then

$$I(x, y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta} \frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \tag{40}$$

holds true for $x, y \in (0, \infty)$ with $x \neq y$, where $I(x, y)$, defined by (17)*, is the identric or exponential mean.*

The following results were shown in [15].

Theorem 14 (see [15]). *For*

$$\beta \geq 0, \tag{41}$$

a sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$ is that

$$\alpha \geq \max\left\{\frac{1}{2}, \beta, 3\beta^2 - 3\beta + 1\right\}. \tag{42}$$

Remark 15. From Theorems 9 and 14 we see that the necessary and sufficient condition for the function $f_{\alpha,0,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$ is that

$$\alpha \geq 1. \tag{43}$$

This result is Theorem 2 in [11]. Here we recovered it.

Theorem 16 (see [15]). *Let*

$$\beta \in \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right]. \tag{44}$$

Then the necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$ is that

$$\alpha \geq \frac{1}{2}. \tag{45}$$

The following results are applications of the above theorems.

Theorem 17 (see [15]). *When*

$$\frac{1}{2} - \frac{\sqrt{3}}{6} \leq \beta \leq \frac{1}{2}, \tag{46}$$

the following inequalities

$$\begin{gathered} \psi(x+\beta) \geq \ln x - \frac{1/2 - \beta}{x}, \\ (-1)^k \psi^{(k-1)}(x+\beta) \leq \frac{(k-2)!}{x^{k-1}} + \frac{(1/2-\beta)(k-1)!}{x^k} \\ (k \geq 2) \end{gathered} \tag{47}$$

hold true on the interval $(0, \infty)$.

Theorem 18 (see [15]). *Let $n \in \mathbb{N}$ and*

$$x_k > 0 \quad (1 \leq k \leq n). \tag{48}$$

Suppose also that

$$\sum_{k=1}^{n} p_k = 1 \quad (p_k \geq 0). \tag{49}$$

If

$$\begin{gathered} 0 \leq \beta \leq \frac{1}{2}, \\ \alpha \geq \max\left\{\frac{1}{2}, 3\beta^2 - 3\beta + 1\right\}, \end{gathered} \tag{50}$$

then

$$\frac{\prod_{k=1}^{n}[\Gamma(x_k+\beta)]^{p_k}}{\Gamma\left(\sum_{k=1}^{n} p_k x_k + \beta\right)} \leq \frac{\prod_{k=1}^{n} x_k^{p_k(x_k+\beta-\alpha)}}{\left(\sum_{k=1}^{n} p_k x_k\right)^{\sum_{k=1}^{n} p_k x_k + \beta - \alpha}}. \tag{51}$$

Theorem 19 (see [15]). *If*

$$\begin{gathered} 0 \leq \beta \leq \frac{1}{2}, \\ \alpha \geq \max\left\{\frac{1}{2}, 3\beta^2 - 3\beta + 1\right\}, \end{gathered} \tag{52}$$

then

$$\begin{gathered} I(x, y) < \left[\left(\frac{x}{y}\right)^{\alpha-\beta} \frac{\Gamma(x+\beta)}{\Gamma(y+\beta)}\right]^{1/(x-y)} \\ (x > 0;\ y > 0;\ x \neq y), \end{gathered} \tag{53}$$

where in (53) *$I(x, y)$, defined by* (17)*, is the identric or exponential mean.*

2. Open Problems

2.1. Open Problem 1. From Theorem 8 we have already known, for

$$\beta \in \{0\} \cup \left[\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right), \tag{54}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$.

For

$$\beta \in \left(0, \frac{1}{2} + \frac{\sqrt{3}}{6}\right), \tag{55}$$

what is a necessary and sufficient condition for the function $f_{\alpha,\beta,1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$?

Already Known. Theorem 3 gave a necessary condition; Theorem 6 provided a sufficient condition.

2.2. Open Problem 2. From Remark 15, Theorems 10 and 16 we have already known, for

$$\beta \in \{0\} \cup \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \infty\right), \tag{56}$$

a necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$.

For

$$\beta \in \left(0, \frac{1}{2} - \frac{\sqrt{3}}{6}\right), \tag{57}$$

what is a necessary and sufficient condition for the function $f_{\alpha,\beta,-1}(x)$ to be logarithmically completely monotonic on the interval $(0, \infty)$?

Already Known. Theorem 9 gave a necessary condition; Theorem 14 provided a sufficient condition.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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