

Research Article

Exit Problems for Jump Processes Having Double-Sided Jumps with Rational Laplace Transforms

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We consider the two-sided first-exit problem for a jump process having jumps with rational Laplace transform. We derive the joint distribution of the first passage time to two-sided barriers and the value of process at the first passage time. As applications, we present explicit expressions of the dividend formulae for barrier strategy and threshold strategy.

1. Introduction

Let $X = \{X(t) : t \geq 0\}$ be a jump process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \mathcal{F}_t : t \geq 0, P)$ and

$$X(t) = x + \mu t + \sum_{k=1}^{N(t)} Y_k \equiv x + \mu t + S(t), \quad t \geq 0, \quad (1)$$

where x is the starting point of $X(t)$, $\{N_t; t \geq 0\}$ is a Poisson process with rate λ , constant $\mu \in R$ represents the drift, and the jump sizes $\{Y_k; k \geq 1\}$ are independent and identically distributed random variables whose probability density function is given by $p(y)$. Moreover, it is assumed that N_t and Y_k are independent. Recently, one-sided and two-sided first-exit problems for processes with two-sided jumps have attracted a lot of attention in applied probability. For example, Perry et al. [1] studied two-sided first-exit time for processes with two-sided exponential jumps; Kou and Wang [2] studied the one-sided first passage times for a jump diffusion process with exponential positive and negative jumps. Cai [3] investigated the first passage time of a hyperexponential jump diffusion process. Chi [4] discussed the first passage time to two barriers of a hyperexponential jump diffusion process. Closed form expressions are obtained in [5] for the integral transforms of the joint distribution of the first-exit time from an interval and the value of the overshoot through boundaries at the exit time for the Poisson process with an exponential component. For some related works, see Perry et al. [6], Lewis and Mordecki [7], Kuznetsov [8],

Perry et al. [1], and the references therein. On the other hand, the jump processes with two-sided jumps have been studied widely in dividend problems. The problem of first-exit from a half-line is of fundamental interest with regard to the expected total discounted dividends up to ruin. See, for example, Chi [4], Chi and Lin [9], and Yin et al. [10].

Motivated by some related work mentioned above, we will assume $\{Y_k : k = 1, 2, \dots\}$ is a sequence of iid random variables with a density $p(x)$ given by

$$p(x) = \sum_{j=1}^J \sum_{i=1}^{m_j} p_{ij} \frac{\rho_j^i x^{i-1}}{(i-1)!} e^{-\rho_j x} \mathbf{1}_{\{x>0\}} + \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} \hat{p}_{ij} \frac{\hat{\rho}_j^i |x|^{i-1}}{(i-1)!} e^{\hat{\rho}_j x} \mathbf{1}_{\{x<0\}}. \quad (2)$$

We assume that $J, \hat{J} \in N$, $m_j, \hat{m}_j \in N$, $p_{ij}, \hat{p}_{ij} \in R^+$, and $\sum_{j=1}^J \sum_{i=1}^{m_j} p_{ij} + \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} \hat{p}_{ij} = 1$, $\text{Re}(\rho_j) > 0$ and $\text{Re}(\hat{\rho}_j) > 0$, and that $\rho_i \neq \rho_j$ and $\hat{\rho}_i \neq \hat{\rho}_j$ for all $i \neq j$.

Define τ to be the first-exit time of X_t to two flat barriers a and b ($a < b$); that is,

$$\tau := \inf \{t \geq 0 : X_t \geq b \text{ or } X_t \leq a\}, \quad (3)$$

and the infimum of an empty set is defined as $+\infty$.

The rest of the paper is organized as follows. In Section 2, we study the roots of the generalized Lundberg equation.

Section 3 concentrates on deriving the joint distribution of the first-exit time and a nonnegative measurable function of the process value at the first-exit time to two flat barriers. In Section 4, we will study two kinds of dividend payment strategies by using the results in Section 3.

2. Preliminary Results

The infinitesimal generator of the jump process (1) is given by

$$(Lu)(x) = \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] p(y) dy \quad (4)$$

for any continuously differentiable function $u(x)$ and the Laplace exponent $G(z) = \ln E[\exp(zX_1)]$ is a rational function, which has the following partial fraction decomposition:

$$G(z) = z\mu + \lambda \left(\sum_{j=1}^J \sum_{i=1}^{m_j} \frac{p_{ij}(\rho_j)^i}{(\rho_j - z)^i} + \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} \frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j + z)^i} - 1 \right), \quad z \in C. \quad (5)$$

We see that $G(z)$ has poles at ρ_j ($1 \leq j \leq J$) and $-\hat{\rho}_j$ ($1 \leq j \leq \hat{J}$) with the corresponding multiplicities m_j and \hat{m}_j . Note that $G(z)$ has $\sum_{j=1}^J m_j$ poles (counting with multiplicity) in the half-plane $\text{Re}(z) > 0$ and $\sum_{j=1}^{\hat{J}} \hat{m}_j$ poles in the half-plane $\text{Re}(z) < 0$. We denote four marks:

- (i) $M \{\widehat{M}\}$ is the number of poles of $G(z)$ in the half-plane $\text{Re}(z) > 0$ $\{\text{Re}(z) < 0\}$,
- (ii) $K \{\widehat{K}\}$ is the number of zeros of $G(z) - \theta$ in the half-plane $\text{Re}(z) > 0$ $\{\text{Re}(z) < 0\}$.

In [8], the author has given the solutions of the Cramér-Lundberg equation $G(z) = \theta$. However, for this simple Lévy process X , we will give another simple proof for the zeros of this equation.

Lemma 1. Assume that $\theta > 0$.

- (1) If $\mu > 0$, then $K = M + 1$ and $\widehat{K} = \widehat{M}$.
- (2) If $\mu < 0$, then $K = M$ and $\widehat{K} = \widehat{M} + 1$.
- (3) If $\mu = 0$, then $K = M$ and $\widehat{K} = \widehat{M}$.

Proof. We denote

$$G_1(z) = z\mu - \lambda - \theta, \quad z \in C, \quad (6)$$

$$G_2(z) = \lambda \left(\sum_{j=1}^J \sum_{i=1}^{m_j} \frac{p_{ij}(\rho_j)^i}{(\rho_j - z)^i} + \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} \frac{\hat{p}_{ij}(\hat{\rho}_j)^i}{(\hat{\rho}_j + z)^i} \right), \quad z \in C.$$

Then $(\prod_{j=1}^{\hat{J}} (\hat{\rho}_j + z)^{\hat{m}_j})G_1(z)$ and $(\prod_{j=1}^J (\rho_j - z)^{m_j})G_2(z)$ are analytic in C^- . Define a half circle $|x| = r, x \in C^-$, where $r >$

$\varepsilon + \max_{1 \leq j \leq \hat{J}} \{|\hat{\rho}_j|\}$, where ε is an arbitrary positive constant. Note that $|G_1(z)| \rightarrow \infty$ for $\text{Re}(z) \rightarrow -\infty$ and $|G_2(z)| \leq \lambda \sum_{j=1}^J \sum_{i=1}^{m_j} (|\rho_j|^i / \varepsilon^i) + \lambda \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} (|\hat{\rho}_j|^i / \varepsilon^i)$ is bounded for $\text{Re}(z) \rightarrow -\infty$. When r is large enough,

$$\left| \left(\prod_{j=1}^{\hat{J}} (\hat{\rho}_j + z)^{\hat{m}_j} \right) G_1(z) \right| > \left| \left(\prod_{j=1}^J (\rho_j - z)^{m_j} \right) G_2(z) \right| \quad (7)$$

on the boundary of the half circle in C^- . On the imaginary axis, we observe that

$$|G_1(ia)| = |\lambda + \theta + ia\mu| > \lambda = |G_2(ia)|, \quad a \in R. \quad (8)$$

Since $(\prod_{j=1}^{\hat{J}} (\hat{\rho}_j + z)^{\hat{m}_j})G_1(z)$ has \widehat{M} roots with negative real part, so the equation $G(z) = \theta, \theta > 0$, has $\widehat{M} + 1$ roots with negative real part according to Rouché's theorem. Similarly, we can prove that $G(z) = \theta, \theta > 0$ has M roots with positive real part. The proofs of (2) and (3) are similar to the proof of (1). \square

Remark 2. In [8], the author states that it is very unlikely for the equation $G(z) = \theta$ to have multiple solutions, so we assume all the roots of this equation are distinct. We denote $\gamma_i(\theta) = \gamma_i$ ($i = 1, 2, \dots, M + 1$) and $\hat{\gamma}_i(\theta) = \hat{\gamma}_i$ ($i = 1, 2, \dots, \widehat{M} + 1$) for notational simplicity.

3. Distribution of the First-Exit Time to Two Flat Barriers

In this section, we consider the distribution of the first-exit time to two flat barriers. Let $k, J, m_j, \hat{J}, \hat{m}_j$ be positive integers. Suppose $I_{1k} = (1, 1, \dots, 1)_{1 \times k}, k = M + 1, \widehat{M} + 1, E_{1\widehat{M}} = (e^{\hat{\gamma}_1(b-a)}, \dots, e^{\hat{\gamma}_{\widehat{M}}(b-a)})_{1 \times \widehat{M}}$, and $E_{1M} = (e^{\gamma_1(b-a)}, \dots, e^{\gamma_M(b-a)})_{1 \times M}$. For $i = M, M + 1$ and $j = \widehat{M}, \widehat{M} + 1$, define

$$A_{11i} = \begin{pmatrix} \frac{\rho_1}{\rho_1 - \gamma_1} & \dots & \frac{\rho_1}{\rho_1 - \gamma_i} \\ \vdots & \ddots & \vdots \\ \left(\frac{\rho_1}{\rho_1 - \gamma_1} \right)^{m_1} & \dots & \left(\frac{\rho_1}{\rho_1 - \gamma_i} \right)^{m_1} \\ \vdots & \ddots & \vdots \\ \frac{\rho_J}{\rho_J - \gamma_1} & \dots & \frac{\rho_J}{\rho_J - \gamma_i} \\ \vdots & \ddots & \vdots \\ \left(\frac{\rho_J}{\rho_J - \gamma_1} \right)^{m_J} & \dots & \left(\frac{\rho_J}{\rho_J - \gamma_i} \right)^{m_J} \end{pmatrix},$$

$$\begin{aligned}
 A_{12j} &= \begin{pmatrix} \frac{e^{\widehat{r}_1(b-a)} \rho_1}{\rho_1 - \widehat{\gamma}_1} & \cdots & \frac{e^{\widehat{r}_{\widehat{M}}(b-a)} \rho_1}{\rho_1 - \widehat{\gamma}_j} \\ \vdots & \ddots & \vdots \\ \frac{e^{\widehat{r}_1(b-a)} \rho_1^{m_1}}{(\rho_1 - \widehat{\gamma}_1)^{m_1}} & \cdots & \frac{e^{\widehat{r}_{\widehat{M}}(b-a)} \rho_1^{m_1}}{(\rho_1 - \widehat{\gamma}_j)^{m_1}} \\ \vdots & \ddots & \vdots \\ \frac{e^{\widehat{r}_1(b-a)} \rho_j}{\rho_j - \widehat{\gamma}_1} & \cdots & \frac{e^{\widehat{r}_{\widehat{M}}(b-a)} \rho_j}{\rho_j - \widehat{\gamma}_j} \\ \vdots & \ddots & \vdots \\ \frac{e^{\widehat{r}_1(b-a)} \rho_j^{m_j}}{(\rho_j - \widehat{\gamma}_1)^{m_j}} & \cdots & \frac{e^{\widehat{r}_{\widehat{M}}(b-a)} \rho_j^{m_j}}{(\rho_j - \widehat{\gamma}_j)^{m_j}} \end{pmatrix} \\
 A_{21i} &= \begin{pmatrix} \frac{e^{\gamma_1(a-b)} \widehat{\rho}_1}{\widehat{\rho}_1 + \gamma_1} & \cdots & \frac{e^{\gamma_{M+1}(a-b)} \widehat{\rho}_1}{\widehat{\rho}_1 + \gamma_i} \\ \vdots & \ddots & \vdots \\ \frac{e^{\gamma_1(a-b)} \widehat{\rho}_1^{\widehat{m}_1}}{(\widehat{\rho}_1 + \gamma_1)^{\widehat{m}_1}} & \cdots & \frac{e^{\gamma_{M+1}(a-b)} \widehat{\rho}_1^{\widehat{m}_1}}{(\widehat{\rho}_1 + \gamma_i)^{\widehat{m}_1}} \\ \vdots & \ddots & \vdots \\ \frac{e^{\gamma_1(a-b)} \widehat{\rho}_j}{\widehat{\rho}_j + \gamma_1} & \cdots & \frac{e^{\gamma_{M+1}(a-b)} \widehat{\rho}_j}{\widehat{\rho}_j + \gamma_i} \\ \vdots & \ddots & \vdots \\ \frac{e^{\gamma_1(a-b)} \widehat{\rho}_j^{\widehat{m}_j}}{(\widehat{\rho}_j + \gamma_1)^{\widehat{m}_j}} & \cdots & \frac{e^{\gamma_{M+1}(a-b)} \widehat{\rho}_j^{\widehat{m}_j}}{(\widehat{\rho}_j + \gamma_i)^{\widehat{m}_j}} \end{pmatrix}, \\
 A_{22j} &= \begin{pmatrix} \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1} & \cdots & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_j} \\ \vdots & \ddots & \vdots \\ \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1}\right)^{\widehat{m}_1} & \cdots & \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_j}\right)^{\widehat{m}_1} \\ \vdots & \ddots & \vdots \\ \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1} & \cdots & \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_j} \\ \vdots & \ddots & \vdots \\ \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1}\right)^{\widehat{m}_j} & \cdots & \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_j}\right)^{\widehat{m}_j} \end{pmatrix} \\
 A_1 &= \begin{pmatrix} I_{1M+1} & E_{1 \times \widehat{M}} \\ A_{11M+1} & A_{12\widehat{M}} \\ A_{21M+1} & A_{22\widehat{M}} \end{pmatrix}, \quad A_2 = \begin{pmatrix} A_{11M} & A_{12\widehat{M}+1} \\ E_{1M} & I_{1\widehat{M}+1} \\ A_{21M} & A_{22\widehat{M}+1} \end{pmatrix}, \\
 A_3 &= \begin{pmatrix} A_{11M} & A_{12\widehat{M}} \\ A_{21M} & A_{22\widehat{M}} \end{pmatrix}. \tag{9}
 \end{aligned}$$

Note that A_1 and A_2 are $(M + \widehat{M} + 1) \times (M + \widehat{M} + 1)$ matrices and A_3 is an $(M + \widehat{M}) \times (M + \widehat{M})$ matrix.

Theorem 3. Consider any nonnegative measurable function f such that

$$f_{jk}^u = \int_0^\infty f(y+b) \frac{(\rho_j y)^{k-1} \rho_j e^{-\rho_j y}}{(k-1)!} dy < \infty, \tag{10}$$

$$f_{j'k'}^d = \int_{-\infty}^0 f(y+a) \frac{(-\widehat{\rho}_j y)^{k'-1} \widehat{\rho}_j e^{\widehat{\rho}_j y}}{(k'-1)!} dy < \infty$$

for $j = 1, 2, \dots, J, k = 1, \dots, m_j, j' = 1, \dots, \widehat{J}, k' = 1, \dots, \widehat{m}_{j'}$. For any $\theta > 0$ and $u \in (a, b)$, one has the following.

(1) If $\mu > 0$, then

$$E^u [e^{-\theta\tau} f(X_\tau)] = \pi_1 \mathbf{f}_1. \tag{11}$$

Here

$$\begin{aligned}
 \pi_1 &= \left(E^u [e^{-\theta\tau} \mathbf{1}_{K_0}], E^u [e^{-\theta\tau} \mathbf{1}_{K_{11}}], \dots, \right. \\
 & E^u [e^{-\theta\tau} \mathbf{1}_{K_{j,m_j}}], E^u [e^{-\theta\tau} \mathbf{1}_{G_{11}}], \dots, \\
 & \left. E^u [e^{-\theta\tau} \mathbf{1}_{G_{j,\widehat{m}_j}}] \right)^T,
 \end{aligned}$$

$$f_1 = \left(f(b), f_{11}^u, \dots, f_{jk}^u, \dots, f_{j,m_j}^u, f_{11}^d, \dots, f_{j'k'}^d, \dots, f_{j,\widehat{m}_j}^d \right)^T \tag{12}$$

are both $M + \widehat{M} + 1$ -dimensional vectors, and π_1 satisfies

$$\pi_1 A_1 = \omega_1, \tag{13}$$

where $\omega_1 = (e^{\gamma_1(u-b)}, e^{\gamma_2(u-b)}, \dots, e^{\gamma_{M+1}(u-b)}, e^{\widehat{\gamma}_1(u-a)}, e^{\widehat{\gamma}_2(u-a)}, \dots, e^{\widehat{\gamma}_{\widehat{M}+1}(u-a)})$ is an $M + \widehat{M} + 1$ -dimensional vector. Moreover, when A_1 is a nonsingular matrix, π_1 is the unique solution of (13); that is,

$$\pi_1 = \omega_1 \cdot A_1^{-1}. \tag{14}$$

(2) If $\mu < 0$, then

$$E^u [e^{-\theta\tau} f(X_\tau)] = \pi_2 \mathbf{f}_2. \tag{15}$$

Here $\pi_2 = (E^u [e^{-\theta\tau} \mathbf{1}_{K_{11}}], \dots, E^u [e^{-\theta\tau} \mathbf{1}_{K_{j,m_j}}], E^u [e^{-\theta\tau} \mathbf{1}_{G_0}], E^u [e^{-\theta\tau} \mathbf{1}_{G_{11}}], \dots, E^u [e^{-\theta\tau} \mathbf{1}_{G_{j,\widehat{m}_j}}])^T$ and $f_2 = (f_{11}^u, \dots, f_{jk}^u, \dots, f_{j,m_j}^u, f(a), f_{11}^d, \dots, f_{j'k'}^d, \dots, f_{j,\widehat{m}_j}^d)^T$ are both $M + \widehat{M} + 1$ -dimensional vectors, and π_2 satisfies

$$\pi_2 A_2 = \omega_2, \tag{16}$$

where $\omega_2 = (e^{\gamma_1(u-b)}, e^{\gamma_2(u-b)}, \dots, e^{\gamma_M(u-b)}, e^{\widehat{\gamma}_1(u-a)}, e^{\widehat{\gamma}_2(u-a)}, \dots, e^{\widehat{\gamma}_{\widehat{M}+1}(u-a)})$ is an $M + \widehat{M} + 1$ -dimensional vector. Moreover, when A_2 is a nonsingular matrix, π_2 is the unique solution of (16); that is,

$$\pi_2 = \omega_2 \cdot A_2^{-1}. \tag{17}$$

(3) If $\mu = 0$, then

$$E^u \left[e^{-\theta\tau} f(X_\tau) \right] = \pi_3 \mathbf{f}_3. \quad (18)$$

Here $\pi_3 = (E^u[e^{-\theta\tau} \mathbf{1}_{K_{j1}}], \dots, E^u[e^{-\theta\tau} \mathbf{1}_{K_{jm_j}}], E^u[e^{-\theta\tau} \mathbf{1}_{G_{j1}}], \dots, E^u[e^{-\theta\tau} \mathbf{1}_{G_{j\widehat{m}_j}}])^T$ and $\mathbf{f}_3 = (f_{11}^u, \dots, f_{jk}^u, \dots, f_{jm_j}^u, f_{11}^d, \dots, f_{j'k'}^d, \dots, f_{j\widehat{m}_j}^d)^T$ are both $M + \widehat{M}$ -dimensional vectors, and π_3 satisfies

$$\pi_3 A_3 = \widehat{\omega}_3, \quad (19)$$

where $\widehat{\omega}_3 = (e^{\gamma_1(u-b)}, e^{\gamma_2(u-b)}, \dots, e^{\gamma_M(u-b)}, e^{\widehat{\gamma}_1(u-a)}, e^{\widehat{\gamma}_2(u-a)}, \dots, e^{\widehat{\gamma}_{\widehat{M}}(u-a)})$ is an $M + \widehat{M}$ -dimensional vector. Moreover, when A_3 is a nonsingular matrix, π_3 is the unique solution of (19); that is,

$$\pi_3 = \widehat{\omega}_3 \cdot A_3^{-1}. \quad (20)$$

Proof. When the jump process X exits the band (a, b) , sometimes it hits the boundary exactly and sometimes it jumps across the boundaries when leaving. We introduce a sequence of events:

$$K_0 := \{\omega : X_\tau = b\}, G_0 := \{\omega : X_\tau = a\},$$

$$K_{jil} := \{\omega : X_{(\cdot)} \text{ crosses } b \text{ at time } \tau \text{ by the } l\text{th phase of } i\text{th positive jump whose parameter is } \rho_j\},$$

$$G_{j'l'l'} := \{\omega : X_{(\cdot)} \text{ crosses } a \text{ at time } \tau \text{ by the } l'\text{th phase of } i'\text{th negative jump whose parameter is } \widehat{\rho}_{j'}\}$$

$$K_{jk} = K_{jk1} \cup K_{j(k+1)2} \cup \dots \cup K_{jm_j(m_j-k+1)},$$

$$G_{j'k'} = G_{j'k'1} \cup G_{j'(k'+1)2} \cup \dots \cup G_{j'm'_j(m'_j-k'+1)}$$

$$\text{for } j = 1, 2, \dots, J, j' = 1, 2, \dots, \widehat{J}, i = 1, \dots, m_i, i' = 1, \dots, \widehat{m}_i, l = 1, \dots, i, \text{ and } l' = 1, \dots, i'.$$

It is easy to check that

$$M_t = e^{-\theta t + \beta X_t} - e^{\beta X_0} - (G(z) - \theta) \int_0^t e^{-\theta s + \beta X_s} ds, \quad t \geq 0 \quad (21)$$

is a zero-mean martingale for every $\theta > 0$ and every $\beta \in C$. We apply the sampling theorem to the martingale $M = (M_t | t \geq 0)$ and get $E^u[M_\tau] = E^u[M_0] = 0$; that is,

$$E^u \left[e^{-\theta\tau + \beta X_\tau} \right] - e^{\beta X_0} = (G(z) - \theta) E^u \left[\int_0^\tau e^{-\theta s + \beta X_s} ds \right]. \quad (22)$$

Inserting the roots of $G(z) = \theta$ in (22), we get that the right side of (22) becomes zero.

From Perry et al. [1], we know that for $l = 1, 2, \dots, i, i = 1, 2, \dots, m_j$, and $j = 1, 2, \dots, J$, given K_{jil} , the overshoot $X_\tau - b$ is independent of τ and Erlang($i + j - l$) distributed with scale parameter ρ_j and for $l' = 1, \dots, i', i' = 1, \dots, \widehat{m}_i$, and $j' = 1, 2, \dots, \widehat{J}$, given $G_{j'l'l'}$, the excess $a - X_\tau$ is independent of τ and Erlang($i' + j' - l'$) distributed with scale parameter $\widehat{\rho}_{j'}$. Therefore, we know that, for $j = 1, 2, \dots, J$ and $k =$

$1, 2, \dots, m_j$, given K_{jk} , the overshoot $X_\tau - b$ is independent of τ and Erlang(k) distributed with scale parameter ρ_j and for $j' = 1, 2, \dots, \widehat{J}$ and $k' = 1, 2, \dots, \widehat{m}_{j'}$, given $G_{j'k'}$, the excess $a - X_\tau$ is independent of τ and Erlang(k') distributed with scale parameter $\widehat{\rho}_{j'}$.

If $\mu > 0$, by the law of total probability, we have

$$\begin{aligned} E^u \left[e^{-\theta\tau} f(X_\tau) \right] &= E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{K_0} \right] + \sum_{j=1}^J \sum_{i=1}^{m_j} \sum_{l=1}^i E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{K_{jil}} \right] \\ &\quad + \sum_{j'=1}^{\widehat{J}} \sum_{i'=1}^{\widehat{m}_{j'}} \sum_{l'=1}^{i'} E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{G_{j'l'l'}} \right] \\ &= E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{K_0} \right] + \sum_{j=1}^J \sum_{k=1}^{m_j} E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{K_{jk}} \right] \\ &\quad + \sum_{j'=1}^{\widehat{J}} \sum_{k'=1}^{\widehat{m}_{j'}} E^u \left[e^{-\theta\tau} f(X_\tau) \mathbf{1}_{G_{j'k'}} \right] \\ &= E^u \left[e^{-\theta\tau} \mathbf{1}_{K_0} \right] f(b) + \sum_{j=1}^J \sum_{k=1}^{m_j} E^u \left[e^{-\theta\tau} \mathbf{1}_{K_{jk}} \right] f_{jk}^u \\ &\quad + \sum_{j'=1}^{\widehat{J}} \sum_{k'=1}^{\widehat{m}_{j'}} E^u \left[e^{-\theta\tau} \mathbf{1}_{G_{j'k'}} \right] f_{j'k'}^d. \end{aligned} \quad (23)$$

Inserting $\gamma_i, i = 1, 2, \dots, M + 1$ and $\widehat{\gamma}_i, i = 1, 2, \dots, \widehat{M}$, in (22), it follows that, for $i = 1, 2, \dots, M + 1$,

$$\begin{aligned} e^{\gamma_i u} &= E^u \left[e^{-\theta\tau + \gamma_i X_\tau} \right] \\ &= E^u \left[e^{-\theta\tau} \mathbf{1}_{K_0} \right] e^{\gamma_i b} \\ &\quad + \sum_{j=1}^J \sum_{k=1}^{m_j} E^u \left[e^{-\theta\tau} \mathbf{1}_{K_{jk}} \right] \left(\frac{\rho_j}{\rho_j - \gamma_i} \right)^k e^{\gamma_i b} \\ &\quad + \sum_{j'=1}^{\widehat{J}} \sum_{k'=1}^{\widehat{m}_{j'}} E^u \left[e^{-\widehat{\rho}_{j'}\tau} \mathbf{1}_{G_{j'k'}} \right] \left(\frac{\widehat{\rho}_{j'}}{\widehat{\rho}_{j'} + \gamma_i} \right)^{k'} e^{\gamma_i a} \end{aligned} \quad (24)$$

and for $i = 1, 2, \dots, \widehat{M}$

$$\begin{aligned} e^{\widehat{\gamma}_i u} &= E^u \left[e^{-\theta\tau + \widehat{\gamma}_i X_\tau} \right] \\ &= E^u \left[e^{-\theta\tau} \mathbf{1}_{K_0} \right] e^{\widehat{\gamma}_i b} \\ &\quad + \sum_{j=1}^J \sum_{k=1}^{m_j} E^u \left[e^{-\theta\tau} \mathbf{1}_{K_{jk}} \right] \left(\frac{\rho_j}{\rho_j - \widehat{\gamma}_i} \right)^k e^{\widehat{\gamma}_i b} \\ &\quad + \sum_{j'=1}^{\widehat{J}} \sum_{k'=1}^{\widehat{m}_{j'}} E^u \left[e^{-\widehat{\rho}_{j'}\tau} \mathbf{1}_{G_{j'k'}} \right] \left(\frac{\widehat{\rho}_{j'}}{\widehat{\rho}_{j'} + \widehat{\gamma}_i} \right)^{k'} e^{\widehat{\gamma}_i a}. \end{aligned} \quad (25)$$

Then the vector π satisfies $\pi_1 \cdot A_1 = \bar{\omega}_1$ and we have (11). If A_1 is nonsingular, we have $\pi_1 = \bar{\omega}_1 A_1^{-1}$.

The proof of (2) and (3) is similar and we omit their proofs. \square

Corollary 4. *If the matrix D is nonsingular, then for any $\delta \in (-\min(|\hat{\rho}_1|, |\hat{\rho}_2|, \dots, |\hat{\rho}_J|), \min(|\rho_1|, |\rho_2|, \dots, |\rho_J|))$, one has the following.*

(1) *If $\mu > 0$, then*

$$E^u [e^{-\theta\tau + \delta X_\tau}] = e^{\delta b} \cdot \left(\sum_{i=1}^{M+1} \omega_i e^{\gamma_i(u-b)} + \sum_{j=1}^{\bar{M}} \nu_j e^{\hat{\gamma}_j(u-a)} \right), \quad (26)$$

$$a < u < b,$$

where

$$A_1 \cdot (\omega_1, \omega_2, \dots, \omega_{M+1}, \nu_1, \nu_2, \dots, \nu_{\bar{M}})^T = J_1(\delta),$$

$$J_1(\delta) = \left(1, \left(\frac{\rho_1}{\rho_1 - \delta} \right)^{m_1}, \dots, \frac{\rho_1}{\rho_1 - \delta}, \dots, \left(\frac{\rho_J}{\rho_J - \delta} \right)^{m_J}, \dots, \right. \\ \left. \frac{\rho_J}{\rho_J - \delta}, \left(\frac{\hat{\rho}_1}{\hat{\rho}_1 + \delta} \right)^{\hat{m}_1} e^{\delta(a-b)}, \dots, \frac{\hat{\rho}_J}{\hat{\rho}_J + \delta} e^{\delta(a-b)} \right)^T. \quad (27)$$

(2) *If $\mu < 0$, then*

$$E^u [e^{-\theta\tau + \delta X_\tau}] = e^{\delta b} \cdot \left(\sum_{i=1}^M \bar{\omega}_i e^{\gamma_i(u-b)} + \sum_{j=1}^{\bar{M}+1} \bar{\nu}_j e^{\hat{\gamma}_j(u-a)} \right), \\ a < u < b, \quad (28)$$

where

$$A_2 \cdot (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_M, \bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{\bar{M}+1})^T = J_2(\delta), \\ J_2(\delta) = \left(\left(\frac{\rho_1}{\rho_1 - \delta} \right)^{m_1}, \dots, \frac{\rho_1}{\rho_1 - \delta}, \dots, \left(\frac{\rho_J}{\rho_J - \delta} \right)^{m_J}, \dots, \right. \\ \left. \frac{\rho_J}{\rho_J - \delta}, e^{\delta(a-b)}, \dots, \left(\frac{\hat{\rho}_1}{\hat{\rho}_1 + \delta} \right)^{\hat{m}_1} e^{\delta(a-b)}, \dots, \frac{\hat{\rho}_J}{\hat{\rho}_J + \delta} e^{\delta(a-b)} \right)^T. \quad (29)$$

(3) *If $\mu = 0$, then*

$$E^u [e^{-\theta\tau + \delta X_\tau}] = e^{\delta b} \cdot \left(\sum_{i=1}^{M+1} \bar{\omega}_i e^{\gamma_i(u-b)} + \sum_{j=1}^{\bar{M}+1} \bar{\nu}_j e^{\hat{\gamma}_j(u-a)} \right), \quad (30) \\ a < u < b,$$

where

$$A_3 \cdot (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_{M+1}, \bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{\bar{M}+1})^T = J_3(\delta), \\ J_3(\delta) = \left(\left(\frac{\rho_1}{\rho_1 - \delta} \right)^{m_1}, \dots, \frac{\rho_1}{\rho_1 - \delta}, \dots, \left(\frac{\rho_J}{\rho_J - \delta} \right)^{m_J}, \dots, \right. \\ \left. \frac{\rho_J}{\rho_J - \delta}, \left(\frac{\hat{\rho}_1}{\hat{\rho}_1 + \delta} \right)^{\hat{m}_1} e^{\delta(a-b)}, \dots, \frac{\hat{\rho}_J}{\hat{\rho}_J + \delta} e^{\delta(a-b)} \right)^T. \quad (31)$$

Corollary 5. *For any $\theta > 0$, $u, b \in \mathbb{R}$, one has the following.*

(1) *If $\mu > 0$, then*

$$E^u [e^{-\theta\tau^b} f(X_{\tau^b})] = \begin{cases} f(u), & u \geq b; \\ \sum_{i=1}^{M+1} \omega_i e^{\gamma_i(u-b)}, & u < b. \end{cases} \quad (32)$$

Here $\omega = (\omega_1, \omega_2, \dots, \omega_{M+1})^T$ is a vector uniquely determined by the following linear system:

$$D_1 \omega = \bar{f}_1, \quad (33)$$

where $\bar{f}_1 = (f(b), f_{11}^u, \dots, f_{1m_1}^u, \dots, f_{J1}^u, \dots, f_{Jm_J}^u)$ is an $(M + 1)$ -dimensional vector and D_1 is an $(M + 1) \times (M + 1)$ nonsingular matrix:

$$D_1 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{\rho_1}{\rho_1 - \gamma_1} & \frac{\rho_1}{\rho_1 - \gamma_2} & \dots & \frac{\rho_1}{\rho_1 - \gamma_{M+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\rho_1}{\rho_1 - \gamma_1} \right)^{m_1} & \left(\frac{\rho_1}{\rho_1 - \gamma_2} \right)^{m_1} & \dots & \left(\frac{\rho_1}{\rho_1 - \gamma_{M+1}} \right)^{m_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\rho_J}{\rho_J - \gamma_1} & \frac{\rho_J}{\rho_J - \gamma_2} & \dots & \frac{\rho_J}{\rho_J - \gamma_{M+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\rho_J}{\rho_J - \gamma_1} \right)^{m_J} & \left(\frac{\rho_J}{\rho_J - \gamma_2} \right)^{m_J} & \dots & \left(\frac{\rho_J}{\rho_J - \gamma_{M+1}} \right)^{m_J} \end{pmatrix} = \dots \quad (34)$$

(2) *If $\mu \leq 0$, then*

$$E^u [e^{-\theta\tau^b} f(X_{\tau^b})] = \begin{cases} f(u), & u \geq b; \\ \sum_{i=1}^M \bar{\omega}_i e^{\gamma_i(u-b)}, & u < b. \end{cases} \quad (35)$$

Here $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_M)^T$ is a vector uniquely determined by the following linear system:

$$D_2 \bar{\omega} = \bar{f}_2, \quad (36)$$

where $\bar{f}_2 = (f_{11}^u, \dots, f_{1m_1}^u, \dots, f_{j1}^u, \dots, f_{jm_j}^u)$ is an M -dimensional vector and D_2 is an $M \times M$ nonsingular matrix:

$$D_2 = \begin{pmatrix} \frac{\rho_1}{\rho_1 - \gamma_1} & \frac{\rho_1}{\rho_1 - \gamma_2} & \dots & \frac{\rho_1}{\rho_1 - \gamma_{M+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\rho_1}{\rho_1 - \gamma_1}\right)^{m_1} & \left(\frac{\rho_1}{\rho_1 - \gamma_2}\right)^{m_1} & \dots & \left(\frac{\rho_1}{\rho_1 - \gamma_{M+1}}\right)^{m_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\rho_j}{\rho_j - \gamma_1} & \frac{\rho_j}{\rho_j - \gamma_2} & \dots & \frac{\rho_j}{\rho_j - \gamma_{M+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\rho_j}{\rho_j - \gamma_1}\right)^{m_j} & \left(\frac{\rho_j}{\rho_j - \gamma_2}\right)^{m_j} & \dots & \left(\frac{\rho_j}{\rho_j - \gamma_{M+1}}\right)^{m_j} \end{pmatrix}. \tag{37}$$

Corollary 6. For any $\theta > 0, u, a \in \mathbb{R}$, one has the following.

(1) If $\mu > 0$, then

$$E^u \left[e^{-\theta \tau_a} f(X_{\tau_a}) \right] = \begin{cases} f(u), & u \leq a; \\ \sum_{l=1}^{\widehat{M}} \omega_l e^{\gamma_l(u-a)}, & u > a. \end{cases} \tag{38}$$

Here $v = (v_1, v_2, \dots, v_{\widehat{M}})^T$ is a vector uniquely determined by the following linear system:

$$C_1 v = \bar{f}_1, \tag{39}$$

where $\bar{f}_1 = (f_{11}^d, \dots, f_{1\widehat{m}_1}^d, \dots, f_{j1}^d, \dots, f_{j\widehat{m}_j}^d)$ is an \widehat{M} -dimensional vector and C_1 is an $(\widehat{M}) \times (\widehat{M})$ nonsingular matrix:

$$C_1 = \begin{pmatrix} \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1} & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_2} & \dots & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_{\widehat{M}}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1}\right)^{\widehat{m}_1} & \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_2}\right)^{\widehat{m}_1} & \dots & \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_{\widehat{M}}}\right)^{\widehat{m}_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1} & \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_2} & \dots & \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_{\widehat{M}}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1}\right)^{\widehat{m}_j} & \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_2}\right)^{\widehat{m}_j} & \dots & \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_{\widehat{M}}}\right)^{\widehat{m}_j} \end{pmatrix}. \tag{40}$$

(2) If $\mu \leq 0$, then

$$E^u \left[e^{-\theta \tau_a} f(X_{\tau_a}) \right] = \begin{cases} f(u), & u \leq a; \\ \sum_{l=1}^{\widehat{M}+1} \bar{\omega}_l e^{\bar{\gamma}_l(u-a)}, & u < a. \end{cases} \tag{41}$$

Here $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{\widehat{M}+1})^T$ is a vector uniquely determined by the following linear system:

$$C_2 \bar{v} = \bar{f}_2, \tag{42}$$

where $\bar{f}_2 = (f(a), f_{11}^d, \dots, f_{1\widehat{m}_1}^d, \dots, f_{j1}^d, \dots, f_{j\widehat{m}_j}^d)$ is an $\widehat{M}+1$ -dimensional vector and C_2 is an $(\widehat{M}+1) \times (\widehat{M}+1)$ nonsingular matrix:

$$C_2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1} & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_2} & \dots & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_{\widehat{M}+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1}\right)^{\widehat{m}_1} & \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_2}\right)^{\widehat{m}_1} & \dots & \left(\frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_{\widehat{M}+1}}\right)^{\widehat{m}_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1} & \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_2} & \dots & \frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_{\widehat{M}+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_1}\right)^{\widehat{m}_j} & \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_2}\right)^{\widehat{m}_j} & \dots & \left(\frac{\widehat{\rho}_j}{\widehat{\rho}_j + \widehat{\gamma}_{\widehat{M}+1}}\right)^{\widehat{m}_j} \end{pmatrix}. \tag{43}$$

Example 7. When $J = m_1 = \widehat{J} = \widehat{m}_1 = 1, p(x) = p_{11}\rho_1 e^{-\rho_1 x} I_{(x>0)} + \widehat{p}_{11}\widehat{\rho}_1 e^{\widehat{\rho}_1 x} I_{(x<0)}$, and $p_{11} + \widehat{p}_{11} = 1, \mu > 0$, then the equation $G(z) = \theta$ has 3 real roots: γ_1, γ_2 , and $\widehat{\gamma}_1$ ($-\infty < \widehat{\gamma}_1 < 0 < \gamma_1 < \gamma_2 < \infty$). Let

$$A_1 = \begin{pmatrix} 1 & 1 & e^{\widehat{\gamma}_1(b-a)} \\ \frac{\rho_1}{\rho_1 - \gamma_1} & \frac{\rho_1}{\rho_1 - \gamma_2} & \frac{\rho_1 e^{\widehat{\gamma}_1(b-a)}}{\rho_1 - \widehat{\gamma}_1} \\ \frac{\widehat{\rho}_1 e^{\gamma_1(a-b)}}{\widehat{\rho}_1 + \gamma_1} & \frac{\widehat{\rho}_1 e^{\gamma_2(a-b)}}{\widehat{\rho}_1 + \gamma_2} & \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1} \end{pmatrix}. \tag{44}$$

Denote A_1^{-1} by

$$A_1^{-1} = \frac{1}{|A_1|} \begin{pmatrix} a_{11} & -a_{21} & a_{31} \\ -a_{12} & a_{22} & -a_{32} \\ a_{13} & -a_{23} & a_{33} \end{pmatrix}. \tag{45}$$

Then we have

$$E^u \left[e^{-\theta \tau} f(X_\tau) \right] = \omega_1 e^{\gamma_1(u-b)} + \omega_2 e^{\gamma_2(u-b)} + \omega_3 e^{\widehat{\gamma}_1(u-a)}, \tag{46}$$

where

$$\omega_1 = \frac{a_{11}f(b) - a_{21}f_{11}^u + a_{31}f_{11}^d}{|A_1|},$$

$$\omega_2 = \frac{-a_{12}f(b) + a_{22}f_{11}^u - a_{32}f_{11}^d}{|A_1|},$$

$$\omega_3 = \frac{a_{13}f(b) - a_{23}f_{11}^u + a_{33}f_{11}^d}{|A_1|},$$

$$f_{11}^u = \int_0^\infty f(y+b) \rho_1 e^{-\rho_1 y} dy,$$

$$\begin{aligned}
 f_{11}^d &= \int_{-\infty}^0 f(y+a) \widehat{\rho}_1 e^{\widehat{\rho}_1 y} dy, \\
 |A_1| &= \frac{\rho_1 \widehat{\rho}_1 (\gamma_2 - \gamma_1)}{(\rho_1 - \gamma_1)(\rho_1 - \gamma_2)(\widehat{\rho}_1 + \widehat{\gamma}_1)} \\
 &\quad + \frac{\rho_1 \widehat{\rho}_1 (\gamma_1 - \widehat{\gamma}_1) e^{(\gamma_2 - \widehat{\gamma}_1)(a-b)}}{(\rho_1 - \gamma_1)(\rho_1 - \widehat{\gamma}_1)(\widehat{\rho}_1 + \gamma_2)} \\
 &\quad + \frac{\rho_1 \widehat{\rho}_1 (\gamma_1 - \widehat{\gamma}_2) e^{(\gamma_1 - \widehat{\gamma}_1)(a-b)}}{(\rho_1 - \gamma_2)((\rho_1 - \widehat{\gamma}_1))(\widehat{\rho}_1 + \gamma_1)}, \\
 a_{11} &= \frac{\rho_1 \widehat{\rho}_1}{(\rho_1 - \gamma_2)(\widehat{\rho}_1 + \widehat{\gamma}_1)} - \frac{\rho_1 \widehat{\rho}_1 e^{(\gamma_2 - \widehat{\gamma}_1)(a-b)}}{(\rho_1 - \widehat{\gamma}_1)(\widehat{\rho}_1 + \gamma_2)}, \\
 a_{21} &= \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \widehat{\gamma}_1} - \frac{\widehat{\rho}_1 e^{(\gamma_2 - \widehat{\gamma}_1)(a-b)}}{\widehat{\rho}_1 + \gamma_2}, \\
 a_{13} &= \frac{\rho_1 \widehat{\rho}_1 e^{\gamma_2(a-b)}}{(\rho_1 - \gamma_1)(\widehat{\rho}_1 + \gamma_2)} - \frac{\rho_1 \widehat{\rho}_1 e^{\gamma_1(a-b)}}{(\rho_1 - \gamma_2)(\widehat{\rho}_1 + \gamma_1)}, \\
 a_{23} &= \frac{\rho_1 e^{\gamma_2(a-b)}}{\widehat{\rho}_1 + \gamma_2} - \frac{\widehat{\rho}_1 e^{\gamma_1(a-b)}}{\widehat{\rho}_1 + \gamma_1}, \quad a_{31} = \frac{\rho(\widehat{\gamma}_1 - \gamma_2) e^{\widehat{\gamma}(b-a)}}{(\rho_1 - \widehat{\gamma}_1)(\widehat{\rho}_1 - \gamma_2)}, \\
 a_{33} &= \frac{\rho(\gamma_1 - \gamma_2)}{(\rho_1 - \gamma_1)(\widehat{\rho}_1 - \gamma_2)}.
 \end{aligned} \tag{47}$$

We define $a_{12}(a_{22}, a_{32})$ as follows: let $a_{12}(a_{22}, a_{32})$ be obtained from $a_{11}(a_{21}, a_{31})$ by changing γ_2 to γ_1 in $a_{11}(a_{21}, a_{31})$.

(i) If $f(X_\tau) = e^{\delta X_\tau}$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau + \delta X_\tau}] &= e^{\delta b} (\omega'_1 e^{\gamma_1(u-b)} + \omega'_2 e^{\gamma_2(u-b)} + \omega'_3 e^{\widehat{\gamma}_1(u-a)}), \\
 a < u < b,
 \end{aligned} \tag{48}$$

where

$$\begin{aligned}
 f(b) &= e^{\delta b}, \quad f_{11}^u = e^{\delta b} \frac{\rho_1}{\rho_1 - \delta}, \quad f_{11}^d = e^{\delta a} \frac{\widehat{\rho}_1}{\widehat{\rho}_1 + \delta}, \\
 \omega'_1 &= \frac{a_{11} - a_{21}(\rho_1/(\rho_1 - \delta)) + a_{31}(\widehat{\rho}_1 e^{\delta(a-b)}/(\widehat{\rho}_1 + \delta))}{|A_1|}, \\
 \omega'_2 &= \frac{-a_{12} + a_{22}(\rho_1/(\rho_1 - \delta)) - a_{32}(\widehat{\rho}_1 e^{\delta(a-b)}/(\widehat{\rho}_1 + \delta))}{|A_1|}, \\
 \omega'_3 &= \frac{a_{13} - a_{23}(\rho_1/(\rho_1 - \delta)) + a_{33}(\widehat{\rho}_1 e^{\delta(a-b)}/(\widehat{\rho}_1 + \delta))}{|A_1|}.
 \end{aligned} \tag{49}$$

(ii) If $f(X_\tau) = I_{(X_\tau - b \geq 0)}$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau} I_{(X_\tau - b \geq 0)}] &= \widehat{\omega}_1 e^{\gamma_1(u-b)} + \widehat{\omega}_2 e^{\gamma_2(u-b)} + \widehat{\omega}_3 e^{\widehat{\gamma}_1(u-a)}, \\
 a < u < b,
 \end{aligned} \tag{50}$$

where

$$\begin{aligned}
 f(b) &= 1, \quad f_{11}^u = 1, \quad f_{11}^d = 0, \\
 \widehat{\omega}_1 &= \frac{a_{11} - a_{21}}{|A_1|}, \quad \widehat{\omega}_2 = \frac{-a_{12} + a_{22}}{|A_1|}, \quad \widehat{\omega}_3 = \frac{a_{13} - a_{23}}{|A_1|}.
 \end{aligned} \tag{51}$$

(iii) If $f(X_\tau) = I_{(X_\tau - a \leq 0)}$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau} I_{(X_\tau - a \leq 0)}] &= \widehat{\omega}_1 e^{\gamma_1(u-b)} + \widehat{\omega}_2 e^{\gamma_2(u-b)} + \widehat{\omega}_3 e^{\widehat{\gamma}_1(u-a)}, \\
 a < u < b,
 \end{aligned} \tag{52}$$

where

$$\begin{aligned}
 f(b) &= 0, \quad f_{11}^u = 0, \quad f_{11}^d = 1, \\
 \widehat{\omega}_1 &= \frac{a_{31}}{|A_1|}, \quad \widehat{\omega}_2 = \frac{-a_{32}}{|A_1|}, \quad \widehat{\omega}_3 = \frac{a_{33}}{|A_1|}.
 \end{aligned} \tag{53}$$

(iv) If $f(X_\tau) = I_{(X_\tau - b > y)}$, $y > 0$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau} I_{(X_\tau - b > y)}] &= \check{\omega}_1 e^{\gamma_1(u-b)} + \check{\omega}_2 e^{\gamma_2(u-b)} + \check{\omega}_3 e^{\widehat{\gamma}_1(u-a)}, \\
 a < u < b,
 \end{aligned} \tag{54}$$

where

$$\begin{aligned}
 f(b) &= 0, \quad f_{11}^u = e^{-\rho_1 y}, \quad f_{11}^d = 0, \\
 \check{\omega}_1 &= -\frac{a_{21} e^{-\rho_1 y}}{|A_1|}, \quad \check{\omega}_2 = \frac{a_{22} e^{-\rho_1 y}}{|A_1|}, \quad \check{\omega}_3 = \frac{-a_{23} e^{-\rho_1 y}}{|A_1|}.
 \end{aligned} \tag{55}$$

(v) If $f(X_\tau) = I_{(X_\tau - a < -y)}$, $y > 0$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau} I_{(X_\tau - a < -y)}] &= \check{\check{\omega}}_1 e^{\gamma_1(u-b)} + \check{\check{\omega}}_2 e^{\gamma_2(u-b)} + \check{\check{\omega}}_3 e^{\widehat{\gamma}_1(u-a)}, \\
 a < u < b,
 \end{aligned} \tag{56}$$

where

$$\begin{aligned}
 f(b) &= 0, \quad f_{11}^u = 0, \quad f_{11}^d = e^{-\widehat{\rho}_1 y}, \\
 \check{\check{\omega}}_1 &= \frac{-a_{31} e^{-\widehat{\rho}_1 y}}{|A_1|}, \quad \check{\check{\omega}}_2 = \frac{a_{32} e^{-\widehat{\rho}_1 y}}{|A_1|}, \quad \check{\check{\omega}}_3 = \frac{-a_{33} e^{-\widehat{\rho}_1 y}}{|A_1|}.
 \end{aligned} \tag{57}$$

(vi) If $f(X_\tau) = I_{(X_\tau = b)}$, then we have

$$\begin{aligned}
 E^u [e^{-\theta\tau} I_{(X_\tau = b)}] &= \widetilde{\omega}_1 e^{\gamma_1(u-b)} + \widetilde{\omega}_2 e^{\gamma_2(u-b)} + \widetilde{\omega}_3 e^{\widehat{\gamma}_1(u-a)}, \\
 a < u < b,
 \end{aligned} \tag{58}$$

where

$$f(b) = 1, \quad f_{11}^u = 0, \quad f_{11}^d = 0, \quad \tilde{\omega}_1 = \frac{a_{11}}{|A_1|},$$

$$\tilde{\omega}_2 = \frac{-a_{12}}{|A_1|}, \quad \tilde{\omega}_3 = \frac{a_{13}}{|A_1|}. \tag{59}$$

4. Dividend Problems

In this section, we are going to derive dividend formulae for barrier strategy and threshold strategy, based on the results obtained in Section 3.

For barrier strategy, it is assumed that dividends are paid according to a barrier strategy b . Such a strategy has a level of the barrier $b > 0$; when the surplus exceeds the barrier, the excess is paid out immediately as the dividend. The modified surplus at time t is given by

$$U(t) = X(t) - D(t), \quad t \geq 0, \tag{60}$$

where $D(t)$ denote the aggregate dividends paid between time 0 and time t ; that is,

$$D(t) = \left(\max_{0 \leq s \leq t} X(s) - b \right)^+. \tag{61}$$

Let

$$T = \inf \{t \geq 0 : U(t) < 0\} \tag{62}$$

be the time of ruin for the modified surplus U and

$$D = \int_0^T e^{-\theta t} dD(t) \tag{63}$$

the present value of all dividends until ruin, where $\theta > 0$ is the discount factor.

Denote by $V(u; b)$ the expectation of the discounted dividends until ruin, if the barrier strategy with parameter b is applied; that is,

$$V(u; b) = E[D | X(0) = u], \quad 0 \leq u \leq b. \tag{64}$$

Note that we have

$$V(u; b) = u - b + V(b; b), \quad u > b. \tag{65}$$

Theorem 8. Let $\tau_b^+ = \inf\{t \geq 0 : X(t) \geq b\}$ and $\tau = T \wedge \tau_b^+$. Then for $\theta > 0$, one has the following.

(1) for $0 < u < b$,

$$V(u; b) = \sum_{j=1}^J \sum_{i=1}^{m_j} \frac{m_j + 1 - i}{\eta_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ V(b; b) \left(E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_0} \right] \right. \tag{66}$$

$$\left. + \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right] \right);$$

(2) if $\mu > 0$, then

$$V(b; b) = \frac{\sum_{j=1}^J \sum_{i=1}^{m_j} \left((m_j + 1 - i) / \eta_j \right) \left(E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right] \right) \Big|_{u=b}}{\left(E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_0} \right] + \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right] \right) \Big|_{u=b}}; \tag{67}$$

(3) if $\mu \leq 0$, then

$$V(b; b) = \frac{\sum_{j=1}^J \sum_{i=1}^{m_j} \left((m_j + 1 - i) / \eta_j \right) E^b \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]}{1 - \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]}. \tag{68}$$

Proof. It follows from the conditional memoryless property, the conditional independence, and the strong Markov property that, for $0 \leq u \leq b$, we have

$$V(u; b) = E^u \left[\int_0^\tau e^{-\theta t} dD(t) \right] = E^u \left[\int_0^\tau e^{-\theta t} dD(t) \mathbf{1}_{\{\tau_b^+ < T\}} \right]$$

$$= E^u \left[(X_\tau - b + V(b; b)) e^{-\theta \tau_b^+} \mathbf{1}_{\{\tau_b^+ < T\}} \mathbf{1}_{\{X_\tau > b\}} \right]$$

$$+ E^u \left[\int_{\tau_b^+}^T e^{-\theta t} dD(t) \mathbf{1}_{\{\tau_b^+ < T\}} \mathbf{1}_{\{X_\tau = b\}} \right]$$

$$= \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[(X_{\tau_b^+} - b) e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ V(b; b) \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{\{\tau_b^+ < T\}} \mathbf{1}_{\{X_\tau = b\}} E \right.$$

$$\left. \times \left(\int_0^T e^{-\theta t} dD(t) \cdot \theta_{\tau_b^+} \mid \mathcal{F}_{\tau_b^+}^X \right) \right]$$

$$= \sum_{j=1}^J \sum_{i=1}^{m_j} \frac{m_j + 1 - i}{\eta_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ V(b; b) \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{\{\tau_b^+ < \tau_u\}} \mathbf{1}_{\{X_\tau = b\}} E^{X_{\tau_b^+}} \right.$$

$$\left. \times \left(\int_0^T e^{-\theta t} dD(t) \cdot \theta_{\tau_b^+} \right) \right]$$

$$= \sum_{j=1}^J \sum_{i=1}^{m_j} \frac{m_j + 1 - i}{\eta_i} E^u \left[e^{-\theta \tau_b^+} \mathbf{1}_{K_{ji}} \right]$$

$$+ V(b, b) \left(E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_0} \right] + \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_{ji}} \right] \right). \tag{69}$$

Assertion (2) follows from (1) and $V'(b; b) = 1$, and assertion (3) follows from (1) and $P(K_0) = 0$. This ends the proof. \square

For threshold strategy, we assume that dividends are paid according to the following strategy governed by parameters $b > 0$ and $\alpha > 0$. Whenever the modified surplus is below the threshold level b , no dividends are paid. However, when the surplus is above this threshold level, dividends are paid at a constant rate α that does not exceed the premium rate $\mu > 0$. We define the modified risk process $U_b = \{U_b(t) : t \geq 0\}$ by $U_b(t) = X(t) - D_b(t)$, where

$$D_b(t) = \alpha \int_0^t \mathbf{1}(U_b(s) > b) ds. \tag{70}$$

Let D_b denote the present value of all dividends until the time of ruin \tilde{T}_2 ,

$$D_b = \alpha \int_0^{\tilde{T}_2} e^{-\theta t} dD_b(t), \tag{71}$$

where $\tilde{T}_2 = \inf\{t > 0 : U_b(t) < 0\}$ with $\tilde{T}_2 = \infty$ if $U_b(t) = 0$ for all $t \geq 0$. Here $\theta > 0$ is the discount factor. Denote by $V(u, b)$ the expected discounted value of dividend payments; that is, $V(u, b) = E[D_b | U_b(0) = u]$.

Similar to the proof of Theorem 8, we have the following result.

Theorem 9. Let $\tau_b^- = \inf\{t \geq 0 : X(t) \leq b\}$, $\tau_b^+ = \inf\{t \geq 0 : X(t) \geq b\}$, and $\tau = T \wedge \tau_b^+$. Then, for $\theta > 0$, one has the following.

(1) for $u > b$,

$$V(u; b) = \frac{\alpha}{\theta} - \frac{\alpha}{\theta} E^u \left[e^{-\theta\tau_b^-} \mathbf{1}_{F_0} \right] - \frac{\alpha}{\theta} \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} E^u \left[e^{-\theta\tau_b^-} \mathbf{1}_{F_{ji}} \right] + V(b, b) E^u \left[e^{-\theta\tau_b^-} \mathbf{1}_{F_0} \right] + \sum_{j=1}^{\hat{J}} \sum_{i=1}^{\hat{m}_j} E^u \left[e^{-\theta\tau_b^-} \mathbf{1}_{F_{ji}} \right] f_{ji}; \tag{72}$$

(2) for $b > u > 0$, if $\mu > 0$, then

$$V(u; b) = \alpha \sum_{j=1}^J \sum_{i=1}^{m_j} \frac{m_j + 1 - i}{\eta_j} E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_{ji}} \right] + V(b, b) E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_0} \right] + V(b, b) \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_{ji}} \right]; \tag{73}$$

(3) for $b > u > 0$, if $\mu \leq 0$, then

$$V(u; b) = \alpha \sum_{j=1}^J \sum_{i=1}^{m_j} \frac{m_j + 1 - i}{\eta_j} E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_{ji}} \right] + V(b, b) \sum_{j=1}^J \sum_{i=1}^{m_j} E^u \left[e^{-\theta\tau_b^+} \mathbf{1}_{K_{ji}} \right], \tag{74}$$

where $F_0 = \{\omega : U_b(\tau_b^-) = b\}$, $F_{ji} = \{\omega : U_b(\tau_b^-) < b, -Y_{N_{\tau_b^-}} \sim \text{Erlang}(\hat{\rho}_j, i)\}$, $j = 1, 2, \dots, \hat{J}$, $i = 1, 2, \dots, \hat{m}_j$, and $f_{ji} = \int_{-\infty}^0 V(y + b; b)((-\hat{\rho}_j y)^{i-1} \hat{\rho}_j e^{\hat{\rho}_j y} / (i-1)!) dy$.

Remark 10. If $m_j = 1$, $j = 1, 2, \dots, J$, $\hat{m}_j = 1$, and $\hat{J} = 1, 2, \dots, \hat{J}$, that is, the process X has the hyperexponential downward jumps and upward jumps, then we have the known results (Theorems 3.2 and 3.4, of Yin et al. [10]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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