Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 746309, 9 pages
http://dx.doi.org/10.1155/2014/746309

Research Article

p-Uniform Convexity and q-Uniform Smoothness of Absolute Normalized Norms on $\mathbb{C}^2$

Tomonari Suzuki

Department of Basic Sciences, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan

Correspondence should be addressed to Tomonari Suzuki; suzuki-t@mns.kyutech.ac.jp

Received 19 September 2013; Accepted 15 November 2013; Published 16 February 2014

Academic Editor: Henryk Hudzik

Copyright © 2014 Tomonari Suzuki. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We first prove characterizations of p-uniform convexity and q-uniform smoothness. We next give a formulation on absolute normalized norms on $\mathbb{C}^2$. Using these, we present some examples of Banach spaces. One of them is a uniformly convex Banach space which is not p-uniformly convex.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, real numbers, and complex numbers, respectively.

Let X be a *nontrivial* Banach space, which means a real Banach space with $\dim X \geq 2$ or a complex Banach space with $\dim X \geq 1$. The *modulus of convexity* of X is defined as

$$\delta(\varepsilon) = \inf\left(1 - \frac{\|x+y\|}{2}\right) \tag{1}$$

for $\varepsilon \in [0,2]$, where the infimum can be taken over all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$. The *modulus of smoothness* of X is defined as

$$\rho(\tau) = \sup\left(\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1\right) \tag{2}$$

for $\tau \in (0,\infty)$, where the supremum can be taken over all $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$. It is obvious that $\rho(\tau) \leq \tau$. We know that if X is a Hilbert space, then $\delta(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4}$ and $\rho(\tau) = \sqrt{1+\tau^2} - 1$.

We recall that X is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. Also, X is said to be *uniformly smooth* if $\lim_{\tau\to+0}\rho(\tau)/\tau = 0$.

For $p \in [2,\infty)$, X is called p-uniformly convex if there exists $C > 0$ satisfying

$$\delta(\varepsilon) \geq C\varepsilon^p \tag{3}$$

for all $\varepsilon \in [0,2]$. On the other hand, for $q \in (1,2]$, X is called q-uniformly smooth if there exists $K > 0$ satisfying

$$\rho(\tau) \leq K\tau^q \tag{4}$$

for all $\tau \in (0,\infty)$. It is obvious that p-uniformly convex Banach spaces are uniformly convex, and q-uniformly smooth Banach spaces are uniformly smooth. We also know that, for $p \in (1,\infty)$, L^p spaces are $\max\{2,p\}$-uniformly convex and $\min\{2,p\}$-uniformly smooth. See [1–6] and others.

A norm $\|\cdot\|$ on $\mathbb{C}^2$ is said to be *absolute* if

$$\|(x_1,x_2)\| = \|(|x_1|,|x_2|)\| \tag{5}$$

for all $(x_1,x_2) \in \mathbb{C}^2$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x_1,x_2)\|_p = \begin{cases} \left(|x_1|^p + |x_2|^p\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\{|x_1|,|x_2|\}, & \text{if } p = \infty. \end{cases} \tag{6}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$. We let Ψ_2 be the set of all convex functions ψ on $[0,1]$ satisfying

$$\max\{1-t,t\} \leq \psi(t) \leq 1 \tag{7}$$

for $t \in [0,1]$. Bonsall and Duncan in [7] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$. Namely, the set AN_2 of all absolute normalized norms on

$\mathbb{C}^2$ is in one-to-one correspondence with Ψ_2. The correspondence is given by

$$\psi(t) = \|(1-t, t)\| \quad \text{for } t \in [0,1]. \tag{8}$$

Indeed, for any $\psi \in \Psi_2$, the norm $\|\cdot\|_\psi$ on $\mathbb{C}^2$ defined as

$$\|(x_1, x_2)\|_\psi = \begin{cases} (|x_1| + |x_2|) \\ \quad \times \psi\left(\dfrac{|x_2|}{|x_1| + |x_2|}\right), & \text{if } (x_1, x_2) \neq (0,0), \\ 0, & \text{if } (x_1, x_2) = (0,0) \end{cases} \tag{9}$$

belongs to AN_2 and satisfies (8). Saito et al. in [8] extended this result to $\mathbb{C}^n$.

In this paper, we first prove characterizations of p-uniform convexity and q-uniform smoothness. We next give another formulation on absolute normalized norms on $\mathbb{C}^2$. Using these, we present some examples, one of which is a uniformly convex Banach space which is not p-uniformly convex.

2. Characterizations

In this section, we prove characterizations of p-uniform convexity and q-uniform smoothness.

Proposition 1. *Let X be a Banach space and let $p \in [2, \infty)$. Then the following are equivalent:*

(i) *X is p-uniformly convex,*

(ii) $\liminf_{\varepsilon \to +0} \delta(\varepsilon)/\varepsilon^p > 0$.

Proof. We first assume that $\liminf_{\varepsilon \to +0} \delta(\varepsilon)/\varepsilon^p = 0$. Then for every $C > 0$, there exists a small $\varepsilon > 0$ such that $\delta(\varepsilon)/\varepsilon^p < C$. That is, X is not p-uniformly convex. Conversely, we next assume that X is not p-uniformly convex. That is, for every $C > 0$, there exists $\varepsilon \in (0,2]$ such that $\delta(\varepsilon) < C\varepsilon^p$. Putting $C = 1/n$, we can define a sequence $\{\varepsilon_n\}$ in $(0,2]$ such that $\delta(\varepsilon_n)/\varepsilon_n^p < 1/n$. In the case of $\liminf_n \varepsilon_n = 0$, without loss of generality, we may assume $\lim_n \varepsilon_n = 0$. We have

$$0 \le \liminf_{\varepsilon \to +0} \frac{\delta(\varepsilon)}{\varepsilon^p} \le \liminf_{n \to \infty} \frac{\delta(\varepsilon_n)}{\varepsilon_n^p} \le \lim_{n \to \infty} \frac{1}{n} = 0 \tag{10}$$

and hence $\liminf_{\varepsilon \to +0} \delta(\varepsilon)/\varepsilon^p = 0$. In the other case, there exists $\varepsilon_0 > 0$ such that $\varepsilon_0 < \varepsilon_n$ for all $n \in \mathbb{N}$. Then since δ is nondecreasing, we have

$$0 \le \frac{\delta(\varepsilon_0)}{2^p} \le \frac{\delta(\varepsilon_n)}{\varepsilon_n^p} < \frac{1}{n} \tag{11}$$

for $n \in \mathbb{N}$ and hence $\delta(\varepsilon_0) = 0$. Therefore, $\delta(\varepsilon) = 0$ for $\varepsilon \in [0, \varepsilon_0]$. This implies $\liminf_{\varepsilon \to +0} \delta(\varepsilon)/\varepsilon^p = 0$. □

Proposition 2. *Let X be a Banach space and let $q \in (1, 2]$. Then the following are equivalent:*

(i) *X is q-uniformly smooth,*

(ii) $\limsup_{\tau \to +0} \rho(\tau)/\tau^q < \infty$.

Proof. We first assume that $\limsup_{\tau \to +0} \rho(\tau)/\tau^q = \infty$. Then for every $K > 0$, there exists a small $\tau > 0$ such that $\rho(\tau)/\tau^q > K$. That is, X is not q-uniformly smooth. Conversely, we next assume that X is not q-uniformly smooth. That is, for every $K > 0$, there exists $\tau > 0$ such that $\rho(\tau) > K\tau^q$. Putting $K = n$, we can define a sequence $\{\tau_n\}$ in $(0, \infty)$ such that $\rho(\tau_n)/\tau_n^q > n$. Then we have

$$n < \frac{\rho(\tau_n)}{\tau_n^q} \le \frac{\tau_n}{\tau_n^q} = \frac{1}{\tau_n^{q-1}}. \tag{12}$$

Hence, $\lim_n \tau_n = 0$ because $q - 1 > 0$. Therefore, we obtain

$$\limsup_{\tau \to +0} \frac{\rho(\tau)}{\tau^q} \ge \limsup_{n \to \infty} \frac{\rho(\tau_n)}{\tau_n^q} \ge \lim_{n \to \infty} n = \infty. \tag{13}$$

This completes the proof. □

We know that Hilbert spaces are 2-uniformly convex and 2-uniformly smooth Banach spaces. We can easily check this thing by Propositions 1 and 2.

3. Convex Functions

In this section, we discuss properties of convex functions belonging to Ψ_2. We first note that functions ψ belonging to Ψ_2 are continuous and satisfy $\psi(0) = \psi(1) = 1$ and $\psi(t) \ge 1/2$ for all $t \in [0,1]$.

Let $\psi \in \Psi_2$. Then we define ψ'_-, ψ'_+, and $\partial\psi$ as follows:

$$\psi'_-(s) = \lim_{t \to s-0} \frac{\psi(t) - \psi(s)}{t - s} \tag{14}$$

for $s \in (0, 1]$,

$$\psi'_+(s) = \lim_{t \to s+0} \frac{\psi(t) - \psi(s)}{t - s} \tag{15}$$

for $s \in [0, 1)$, and

$$\partial\psi(s) = \{a \in \mathbb{R} : \psi(t) \ge \psi(s) + a(t - s) \ \ \forall t \in [0,1]\} \tag{16}$$

for $s \in [0,1]$. See [9] and others.

We know the following.

Lemma 3 (see [9, 10]). *Let $\psi \in \Psi_2$. Then the following hold:*

(i) *For $s, t, u \in [0,1]$ with $0 \le s < t < u \le 1$,*

$$\frac{\psi(t) - \psi(s)}{t - s} \le \frac{\psi(u) - \psi(s)}{u - s} \le \frac{\psi(u) - \psi(t)}{u - t} \tag{17}$$

holds.

(ii) *For* $s, t, u \in [0,1]$ *with* $0 \le s < t < u \le 1$,

$$\psi_+'(s) \le \frac{\psi(t)-\psi(s)}{t-s} \le \psi_-'(t) \le \psi_+'(t) \le \frac{\psi(u)-\psi(t)}{u-t} \le \psi_-'(u) \tag{18}$$

holds.

(iii) *For* $t \in [0,1]$,

$$\partial\psi(t) = \begin{cases} \left(-\infty, \psi_+'(0)\right], & \text{if } t = 0, \\ \left[\psi_-'(t), \psi_+'(t)\right], & \text{if } 0 < t < 1, \\ \left[\psi_-'(1), +\infty\right), & \text{if } t = 1 \end{cases} \tag{19}$$

holds.

(iv) $\bigcup\{\partial\psi(t) : t \in [0,1]\} = \mathbb{R}$ *holds.*

(v) $-1 \le \psi_+'(0)$ *and* $\psi_-'(1) \le 1$ *hold.*

Remark 4. (i)–(iii) are stated in [9]. (iv) follows from Theorem 24.1 in [9]. (v) is proved in [10].

Using Lemma 3, we can easily prove the following.

Lemma 5. *Let* $\psi \in \Psi_2$. *Then the following hold:*

(i) $\psi_+'(t) \le (1-\psi(t))/(1-t)$ *for every* $t \in [0,1)$,

(ii) $\psi_-'(t) \ge (\psi(t)-1)/t$ *for every* $t \in (0,1]$.

Lemma 6. *Let* $\psi \in \Psi_2$ *and* $s, t, u \in [0,1]$ *with* $s < t < u$. *Then*

$$\begin{gathered} -1 \le \frac{\psi(t)-\psi(s)}{t-s} \le \frac{1-\psi(t)}{1-t}, \\ \frac{\psi(t)-1}{t} \le \frac{\psi(u)-\psi(t)}{u-t} \le 1 \end{gathered} \tag{20}$$

hold.

The following lemma is used in Section 5.

Lemma 7. *Let* $\psi \in \Psi_2$ *and* $s, u \in [0,1]$ *with* $s < u$. *Then*

$$u - s \le \psi(u)(1-2s) + \psi(s)(2u-1) \le 2(u-s) \tag{21}$$

holds.

Proof. In the case of $s \le 1/2 \le u$, we have

$$\begin{aligned} u - s &= \frac{1}{2}(1-2s) + \frac{1}{2}(2u-1) \\ &\le \psi(u)(1-2s) + \psi(s)(2u-1) \\ &\le (1-2s) + (2u-1) \\ &= 2(u-s). \end{aligned} \tag{22}$$

Using Lemma 6, we will prove this lemma in the other cases. In the case of $s > 1/2$, since $2\psi(s) - ((\psi(s)-1)/s)(2s-1) \le 2$, we have

$$\begin{aligned} u - s &\le (2\psi(u) + 1 - 2u)(u-s) \\ &= 2\psi(u)(u-s) - (u-s)(2u-1) \\ &\le 2\psi(u)(u-s) - (\psi(u)-\psi(s))(2u-1) \\ &= \psi(u)(1-2s) + \psi(s)(2u-1) \\ &= 2\psi(s)(u-s) - (\psi(u)-\psi(s))(2s-1) \\ &\le 2\psi(s)(u-s) - \frac{\psi(s)-1}{s}(u-s)(2s-1) \\ &= \left(2\psi(s) - \frac{\psi(s)-1}{s}(2s-1)\right)(u-s) \\ &\le 2(u-s). \end{aligned} \tag{23}$$

In the case of $u < 1/2$, since $2\psi(u) - ((1-\psi(u))/(1-u))(2u-1) \le 2$, we have

$$\begin{aligned} u - s &\le (2\psi(s) + 2s - 1)(u-s) \\ &= 2\psi(s)(u-s) + (u-s)(2s-1) \\ &\le 2\psi(s)(u-s) - (\psi(u)-\psi(s))(2s-1) \\ &= \psi(u)(1-2s) + \psi(s)(2u-1) \\ &= 2\psi(u)(u-s) - (\psi(u)-\psi(s))(2u-1) \\ &\le 2\psi(u)(u-s) - \frac{1-\psi(u)}{1-u}(u-s)(2u-1) \\ &= \left(2\psi(u) - \frac{1-\psi(u)}{1-u}(2u-1)\right)(u-s) \\ &\le 2(u-s). \end{aligned} \tag{24}$$

This completes the proof. □

We also know the following.

Lemma 8 (Bonsall and Duncan [7] page 37). *Let* $\psi \in \Psi_2$. *Then the following hold:*

(i) *the function* $t \mapsto \psi(t)/t$ *is nonincreasing;*

(ii) *the function* $t \mapsto \psi(t)/(1-t)$ *is nondecreasing.*

The following lemma follows from Lemma 8.

Lemma 9. *Let* $\psi \in \Psi_2$ *and* $s, u \in [0,1]$ *with* $s < u$. *Then*

$$\frac{s}{\psi(s)} \le \frac{u}{\psi(u)}, \qquad \frac{1-s}{\psi(s)} \ge \frac{1-u}{\psi(u)} \tag{25}$$

hold.

4. Absolute Normalized Norms on $\mathbb{C}^2$

We denote by Γ_2 the set of nondecreasing functions γ from $[0,1]$ into $[-1,1]$ satisfying $\int_0^1 \gamma(s)\,ds = 0$. The following proposition says there are many absolute normalized norms on $\mathbb{C}^2$, and we can make many such norms easily.

Proposition 10. *Define a mapping D from Ψ_2 into Γ_2 by*

$$(D\psi)(t) = \begin{cases} \psi'_+(t), & \text{if } t \in [0,1), \\ \psi'_-(t), & \text{if } t = 1 \end{cases} \tag{26}$$

for $\psi \in \Psi_2$ and $t \in [0,1]$, and define a mapping S from Γ_2 into Ψ_2 by

$$(S\gamma)(t) = 1 + \int_0^t \gamma(s)\,ds \tag{27}$$

for $\gamma \in \Gamma_2$ and $t \in [0,1]$. Then $D \circ S\gamma = \gamma$ a.e. and $S \circ D\psi = \psi$ for all $\gamma \in \Gamma_2$ and $\psi \in \Psi_2$.

Proof. Fix $\psi \in \Psi_2$ and put $\gamma = D\psi$. We will show $\gamma \in \Gamma_2$. By Lemma 3, γ is nondecreasing, $-1 \le \psi'_+(0) = \gamma(0)$ and $\gamma(1) = \psi'_-(1) \le 1$. Hence $\gamma(t) \in [-1,1]$ for all $t \in [0,1]$. By the definition of D, we have

$$1 = \psi(1) = \psi(0) + \int_0^1 \gamma(s)\,ds = 1 + \int_0^1 \gamma(s)\,ds. \tag{28}$$

This implies $\int_0^1 \gamma(s)ds = 0$. Therefore, we have shown $\gamma \in \Gamma_2$. Next, we fix $\gamma \in \Gamma_2$ and put $\psi = S\gamma$. We will will $S\gamma \in \Psi_2$. Since γ is nondecreasing, we have that ψ is convex. It is obvious that $\psi(0) = \psi(1) = 1$. From the convexity of ψ, $\psi(t) \le 1$ for all $t \in [0,1]$. Since $-1 \le \gamma(t)$ for $t \in [0,1]$, we have

$$\psi(t) = 1 + \int_0^t \gamma(s)\,ds \ge 1 + \int_0^t (-1)\,ds = 1 - t \tag{29}$$

for $t \in [0,1]$. Since $\gamma(t) \le 1$ for $t \in [0,1]$, we also have

$$\begin{aligned} \psi(t) &= 1 + \int_0^t \gamma(s)\,ds \\ &= 1 + \int_0^1 \gamma(s)\,ds - \int_t^1 \gamma(s)\,ds \\ &= 1 - \int_t^1 \gamma(s)\,ds \ge 1 - \int_t^1 1\,ds = t \end{aligned} \tag{30}$$

for $t \in [0,1]$. Therefore $\psi \in \Psi_2$. The remains are obvious. □

We next discuss the convexity and smoothness. In [11], Takahashi et al. proved that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex. See also [8]. Using this fact, we can obtain the following.

Proposition 11. *Let $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex if and only if $D\psi$ is injective.*

Proof. We assume that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex. Then ψ is strictly convex. That is, for $s,t,u \in [0,1]$ with $0 \le s < t < u \le 1$, we have

$$\psi'_+(s) < \psi'_-(t) \le \psi'_+(t) < \psi'_-(u). \tag{31}$$

Hence $D\psi$ is injective. We can easily prove the converse implication. □

In [10], Mitani et al. proved that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if ψ is differentiable at any $t \in (0,1)$ and $\psi'_+(0) = -1$ and $\psi'_-(1) = 1$. Using this fact, we can prove the following.

Proposition 12. *Let $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if $D\psi$ is surjective.*

Proof. We assume that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth. Then ψ is differentiable at any $t \in (0,1)$ and $\psi'_+(0) = -1$ and $\psi'_-(1) = 1$. So $(D\psi)(0) = -1$ and $(D\psi)(1) = 1$ are obvious. We note that $\partial\psi(0) = (-\infty, -1]$ and $\partial\psi(1) = [1, +\infty)$. For $a \in (-1,1)$, there exists $t \in [0,1]$ with $a \in \partial\psi(t)$. From the above note, we have $t \in (0,1)$. From the differentiability, we obtain

$$\begin{aligned} a \in \partial\psi(t) &= \left[\psi'_-(t), \psi'_+(t)\right] \\ &= \left\{\psi'(t)\right\} = \left\{\psi'_+(t)\right\} = \left\{(D\psi)(t)\right\}. \end{aligned} \tag{32}$$

That is, $(D\psi)(t) = a$. Therefore we have shown $D\psi$ is surjective. Conversely, we next assume that $D\psi$ is surjective. We suppose that ψ is not differentiable at some $t \in (0,1)$. Then we have $\psi'_-(t) < \psi'_+(t)$. By Lemma 3, we have

$$(D\psi)([0,1]) \subset [-1,1] \setminus \left(\psi'_-(t), \psi'_+(t)\right) \subsetneqq [-1,1]. \tag{33}$$

This contradicts the surjectivity of $D\psi$. Hence, ψ is differentiable at any $t \in (0,1)$. We next suppose that $-1 < \psi'_+(0)$. Then by Lemma 3 again, we have

$$(D\psi)([0,1]) \subset [-1,1] \setminus \left[-1, \psi'_+(0)\right) \subsetneqq [-1,1]. \tag{34}$$

This is a contradiction. Hence, $\psi'_+(0) = -1$. We can similarly prove $\psi'_-(1) = 1$. Therefore, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth. □

5. Examples

In this section, we present examples of absolute normalized norms on $\mathbb{C}^2$ satisfying that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly convex and is not p-uniformly convex. We also present examples of such norms satisfying that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly smooth and is not q-uniformly smooth. We note that, in finite dimensional Banach spaces, strict convexity and uniform convexity are equivalent. Smoothness and uniform smoothness are also equivalent.

Theorem 13. *Let $\gamma \in \Gamma_2$ and $p \in [2, \infty)$. Assume that there exist sequences $\{s_n\}$ and $\{u_n\}$ in $[0,1]$ such that $s_n < u_n$ for $n \in \mathbb{N}$,*

$$\lim_{n\to\infty}(u_n - s_n) = 0, \qquad \lim_{n\to\infty}\frac{\gamma(u_n) - \gamma(s_n)}{(u_n - s_n)^{p-1}} = 0. \tag{35}$$

Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is not p-uniformly convex.

Proof. Put $\psi = S\gamma$. Without loss of generality, we may assume

$$\frac{\gamma(u_n) - \gamma(s_n)}{(u_n - s_n)^{p-1}} \le \frac{1}{n} \tag{36}$$

for $n \in \mathbb{N}$, and $\{s_n\}$ and $\{u_n\}$ converge to some number $t_0 \in [0,1]$. We put

$$t_n = \frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} \tag{37}$$

for $n \in \mathbb{N}$. It is clear that $s_n < t_n < u_n$ for $n \in \mathbb{N}$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}^2$ by

$$x_n = \frac{1}{\psi(s_n)}\ (1-s_n, s_n), \qquad y_n = \frac{1}{\psi(u_n)}(1-u_n, u_n) \tag{38}$$

for $n \in \mathbb{N}$. It is obvious $\|x_n\| = \|y_n\| = 1$. Then we have

$$\begin{aligned} x_n + y_n &= \left(\frac{1-s_n}{\psi(s_n)} + \frac{1-u_n}{\psi(u_n)}, \frac{s_n}{\psi(s_n)} + \frac{u_n}{\psi(u_n)}\right) \\ &= \left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)}\right)(1-t_n, t_n). \end{aligned} \tag{39}$$

Thus,

$$\|x_n + y_n\| = \left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)}\right)\psi(t_n). \tag{40}$$

We put

$$v_n = \frac{(u_n/\psi(u_n)) - (s_n/\psi(s_n))}{((1-2s_n)/\psi(s_n)) + ((2u_n-1)/\psi(u_n))}. \tag{41}$$

By Lemma 9,

$$0 \le \frac{u_n}{\psi(u_n)} - \frac{s_n}{\psi(s_n)} \le \frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)}. \tag{42}$$

From this inequality and (46), $v_n \in [0,1]$ holds. Using v_n, we also have

$$\begin{aligned} \|x_n - y_n\| &= \left\|\left(\frac{1-s_n}{\psi(s_n)} - \frac{1-u_n}{\psi(u_n)}, \frac{s_n}{\psi(s_n)} - \frac{u_n}{\psi(u_n)}\right)\right\| \\ &= \left\|\left(\frac{1-s_n}{\psi(s_n)} - \frac{1-u_n}{\psi(u_n)}, \frac{u_n}{\psi(u_n)} - \frac{s_n}{\psi(s_n)}\right)\right\| \\ &= \left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)}\right)\|(1-v_n, v_n)\| \\ &= \left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)}\right)\psi(v_n). \end{aligned} \tag{43}$$

Therefore, we obtain

$$\begin{aligned} &\delta\left(\left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)}\right)\psi(v_n)\right) \\ &\qquad \le 1 - \frac{1}{2}\left(\frac{1}{\psi(s_n)} + \frac{1}{\psi(u_n)}\right)\psi(t_n). \end{aligned} \tag{44}$$

We will show $\liminf_{\varepsilon \to +0} \delta(\varepsilon)/\varepsilon^p = 0$. Before showing it, we need some inequalities:

$$\begin{aligned} &2\psi(s_n)\psi(u_n) - (\psi(u_n) + \psi(s_n))\psi(t_n) \\ &\quad = \psi(s_n)(\psi(u_n) - \psi(t_n)) - \psi(u_n)(\psi(t_n) - \psi(s_n)) \\ &\quad = \psi(s_n)\int_{t_n}^{u_n}\gamma(s)\,ds - \psi(u_n)\int_{s_n}^{t_n}\gamma(s)\,ds \\ &\quad \le \psi(s_n)\gamma(u_n)(u_n - t_n) - \psi(u_n)\gamma(s_n)(t_n - s_n) \\ &\quad = \psi(s_n)\gamma(u_n)\left(u_n - \frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))}\right) \\ &\qquad - \psi(u_n)\gamma(s_n)\left(\frac{(s_n/\psi(s_n)) + (u_n/\psi(u_n))}{(1/\psi(s_n)) + (1/\psi(u_n))} - s_n\right) \\ &\quad = \frac{1}{(1/\psi(s_n)) + (1/\psi(u_n))}(\gamma(u_n) - \gamma(s_n))(u_n - s_n) \\ &\quad \le (\gamma(u_n) - \gamma(s_n))(u_n - s_n) \\ &\quad \le \frac{1}{n}(u_n - s_n)^p, \end{aligned} \tag{45}$$

$$\begin{aligned} &\left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)}\right)\psi(v_n) \\ &\qquad = (\psi(u_n)(1-2s_n) + \psi(s_n)(2u_n-1)) \\ &\qquad\quad \times \frac{\psi(v_n)}{\psi(s_n)\psi(u_n)} \\ &\qquad \ge (u_n - s_n)\frac{\psi(v_n)}{\psi(s_n)\psi(u_n)} \\ &\qquad \ge \frac{1}{2}(u_n - s_n) > 0 \end{aligned} \tag{46}$$

by Lemma 7. From (45) and (46), we have

$$\begin{aligned} &\frac{\delta((((1-2s_n)/\psi(s_n)) + ((2u_n-1)/\psi(u_n)))\psi(v_n))}{((((1-2s_n)/\psi(s_n)) + ((2u_n-1)/\psi(u_n)))\psi(v_n))^p} \\ &\quad \le \frac{1 - (1/2)((1/\psi(s_n)) + (1/\psi(u_n)))\psi(t_n)}{((((1-2s_n)/\psi(s_n)) + ((2u_n-1)/\psi(u_n)))\psi(v_n))^p} \\ &\quad = \frac{1}{2\psi(s_n)\psi(u_n)} \\ &\qquad \times \frac{2\ \psi(s_n)\psi(u_n) - (\psi(u_n) + \psi(s_n))\psi(t_n)}{((((1-2s_n)/\psi(s_n)) + ((2u_n-1)/\psi(u_n)))\psi(v_n))^p} \\ &\quad \le \frac{1}{2\psi(s_n)\psi(u_n)} \end{aligned}$$

$$\times \frac{(u_n - s_n)^p}{n((((1-2s_n)/\psi(s_n))+((2u_n-1)/\psi(u_n)))\,\psi(v_n))^p}$$
$$\le \frac{1}{2n\psi(s_n)\psi(u_n)} 2^p$$
$$\le \frac{2}{n} 2^p,$$
$$\limsup_{n\to\infty} \left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)} \right) \psi(v_n)$$
$$\le \lim_{n\to\infty} \left(\frac{1-2s_n}{\psi(s_n)} + \frac{2u_n-1}{\psi(u_n)} \right)$$
$$= \frac{1-2t_0}{\psi(t_0)} + \frac{2t_0-1}{\psi(t_0)}$$
$$= 0. \tag{47}$$

These imply $\liminf_{\varepsilon\to+0}\delta(\varepsilon)/\varepsilon^p = 0$. So by Proposition 1, we obtain the desired result. □

Corollary 14. *Let $\gamma \in \Gamma_2$. Assume that γ is injective, γ is infinitely differentiable on the neighborhood of some $t_0 \in (0,1)$, and*

$$\gamma'(t_0) = \gamma''(t_0) = \gamma'''(t_0) = \cdots = 0. \tag{48}$$

Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is uniformly convex and is not p-uniformly convex for all $p \in [2,\infty)$.

Proof. Put $\psi = S\gamma$. By Proposition 11, since γ is injective, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is strictly convex and hence it is uniformly convex. By the L'Hospital theorem, for $n \in \mathbb{N}$ with $n \ge 2$, we have

$$\begin{aligned} 0 &= \lim_{u\to t_0+0} \frac{\gamma^{(n-1)}(u)}{(n-1)!} \\ &= \lim_{u\to t_0+0} \frac{\gamma^{(n-2)}(u)}{(n-1)!/1!\,(u-t_0)} \\ &= \lim_{u\to t_0+0} \frac{\gamma^{(n-3)}(u)}{(n-1)!/2!(u-t_0)^2} \\ &\;\;\vdots \\ &= \lim_{u\to t_0+0} \frac{\gamma'(u)}{(n-1)(u-t_0)^{n-2}} \\ &= \lim_{u\to t_0+0} \frac{\gamma(u)-\gamma(t_0)}{(u-t_0)^{n-1}}. \end{aligned} \tag{49}$$

So, by Theorem 13, we have that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not n-uniformly convex for every $n \in \mathbb{N}$ with $n \ge 2$. Therefore, we obtain the desired result. □

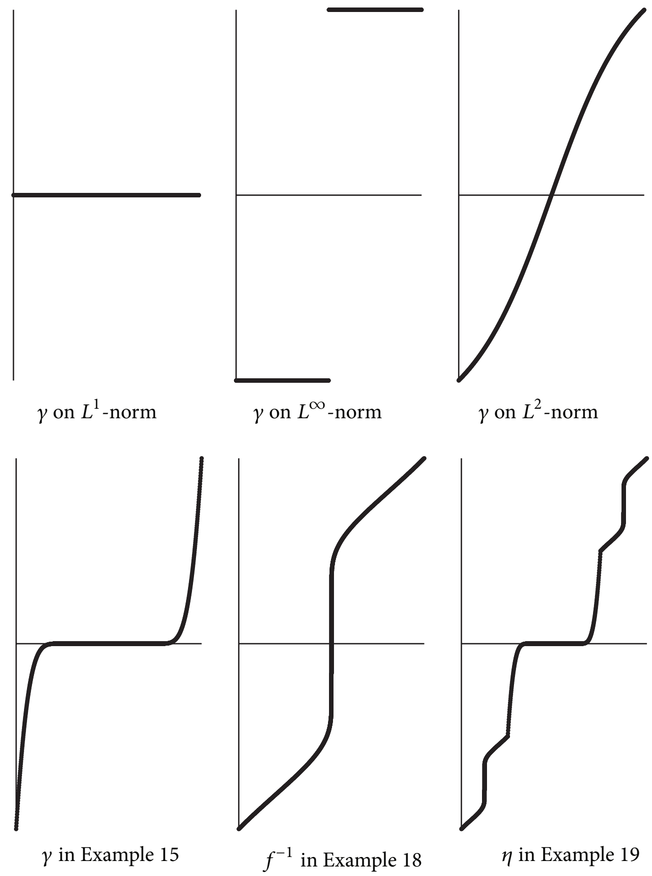

FIGURE 1

It is well known that a function f from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(t) = \begin{cases} 0, & \text{if } t \le 0, \\ \exp(-t^{-2}), & \text{if } t > 0 \end{cases} \tag{50}$$

for $t \in \mathbb{R}$ is strictly increasing on $[0,\infty)$, infinitely differentiable and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Example 15. Define $\gamma \in \Gamma_2$ by

$$\gamma(t) = \begin{cases} -\exp\left(4 - \left(t - \frac{1}{2}\right)^{-2}\right), & \text{if } t < \frac{1}{2}, \\ 0, & \text{if } t = \frac{1}{2}, \\ +\exp\left(4 - \left(t - \frac{1}{2}\right)^{-2}\right), & \text{if } t > \frac{1}{2} \end{cases} \tag{51}$$

for $t \in [0,1]$. Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is uniformly convex and not p-uniformly convex for all $p \in [2,\infty)$. See Figure 1.

Theorem 16. *Let $\gamma \in \Gamma_2$ and $q \in (1,2]$. Assume that there exist a constant $\lambda \in (0,1/2)$ and sequences $\{s_n\}$ and $\{u_n\}$ in $[0,1]$ such that $s_n < u_n$ for $n \in \mathbb{N}$,*

$$\lim_{n\to\infty}(u_n - s_n) = 0,$$
$$\lim_{n\to\infty}\frac{\gamma(\lambda s_n + (1-\lambda)u_n) - \gamma((1-\lambda)s_n + \lambda u_n)}{(u_n - s_n)^{q-1}} = \infty. \tag{52}$$

Then $(\mathbb{C}^2, \|\cdot\|_{S\gamma})$ is not q-uniformly smooth.

Proof. Put $\psi = S\gamma$. Without loss of generality, we may assume

$$\frac{\gamma(\lambda s_n + (1-\lambda)u_n) - \gamma((1-\lambda)s_n + \lambda u_n)}{(u_n - s_n)^{q-1}} \geq n \tag{53}$$

for $n \in \mathbb{N}$, and $\{s_n\}$ and $\{u_n\}$ converge to some number $t_0 \in [0,1]$. We define a sequence $\{t_n\}$ by (37). Since

$$\lim_{n\to\infty}\frac{\psi(s_n)}{\psi(s_n)+\psi(u_n)} = \frac{1}{2}, \qquad \lim_{n\to\infty}\frac{\psi(u_n)}{\psi(s_n)+\psi(u_n)} = \frac{1}{2}, \tag{54}$$

we may also assume that

$$\frac{\psi(s_n)}{\psi(s_n)+\psi(u_n)} \in [\lambda, 1-\lambda],$$
$$\frac{\psi(u_n)}{\psi(s_n)+\psi(u_n)} \in [\lambda, 1-\lambda] \tag{55}$$

for $n \in \mathbb{N}$. We note that

$$(1-\lambda)s_n + \lambda u_n \leq t_n \leq \lambda s_n + (1-\lambda)u_n \tag{56}$$

because

$$t_n = \frac{\psi(u_n)}{\psi(s_n)+\psi(u_n)}s_n + \frac{\psi(s_n)}{\psi(s_n)+\psi(u_n)}u_n \tag{57}$$

for $n \in \mathbb{N}$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{C}^2$ by

$$x_n = \frac{1}{\psi(t_n)}(1-t_n, t_n),$$
$$y_n = \frac{(\psi(s_n)(1-u_n) - \psi(u_n)(1-s_n), \psi(s_n)u_n - \psi(u_n)s_n)}{(\psi(s_n)+\psi(u_n))\psi(t_n)} \tag{58}$$

for $n \in \mathbb{N}$. It is obvious that $\|x_n\| = 1$. We put $v_n \in [0,1]$ by (41). We have

$$\begin{aligned}
&\|y_n\| \\
&= \frac{\|(\psi(s_n)(1-u_n) - \psi(u_n)(1-s_n), \psi(s_n)u_n - \psi(u_n)s_n)\|}{(\psi(s_n)+\psi(u_n))\psi(t_n)} \\
&= \frac{\|(\psi(s_n)(u_n-1) + \psi(u_n)(1-s_n), \psi(s_n)u_n - \psi(u_n)s_n)\|}{(\psi(s_n)+\psi(u_n))\psi(t_n)} \\
&= \frac{\psi(s_n)(2u_n-1) + \psi(u_n)(1-2s_n)}{(\psi(s_n)+\psi(u_n))\psi(t_n)}\|(1-v_n, v_n)\| \\
&= \frac{\psi(s_n)(2u_n-1) + \psi(u_n)(1-2s_n)}{(\psi(s_n)+\psi(u_n))\psi(t_n)}\psi(v_n) \\
&\leq 2(\psi(s_n)(2u_n-1) + \psi(u_n)(1-2s_n)) \\
&\leq 4(u_n - s_n)
\end{aligned} \tag{59}$$

by Lemma 7. We note that $\lim_n\|y_n\| = 0$. We will calculate $\|x_n + y_n\|$ and $\|x_n - y_n\|$. We have

$$\begin{aligned}
x_n + y_n &= \frac{1}{\psi(t_n)}\left(1 - t_n + \frac{\psi(s_n)(1-u_n) - \psi(u_n)(1-s_n)}{\psi(s_n)+\psi(u_n)},\right. \\
&\qquad \left. t_n + \frac{\psi(s_n)u_n - \psi(u_n)s_n}{\psi(s_n)+\psi(u_n)}\right) \\
&= \frac{1}{\psi(t_n)}\left(1 + \frac{\psi(s_n)-\psi(u_n)}{\psi(s_n)+\psi(u_n)}\right)(1-u_n, u_n) \\
&= \frac{1}{\psi(t_n)}\frac{2\psi(s_n)}{\psi(s_n)+\psi(u_n)}(1-u_n, u_n)
\end{aligned} \tag{60}$$

because

$$\frac{t_n + ((\psi(s_n)u_n - \psi(u_n)s_n)/(\psi(s_n)+\psi(u_n)))}{1 + ((\psi(s_n)-\psi(u_n))/(\psi(s_n)+\psi(u_n)))} = u_n. \tag{61}$$

Hence,

$$\|x_n + y_n\| = \frac{1}{\psi(t_n)}\frac{2\psi(s_n)\psi(u_n)}{\psi(s_n)+\psi(u_n)} \tag{62}$$

for $n \in \mathbb{N}$. Similarly, we obtain

$$x_n - y_n = \frac{1}{\psi(t_n)}\frac{2\psi(u_n)}{\psi(s_n)+\psi(u_n)}(1-s_n, s_n) \tag{63}$$

and hence

$$\|x_n - y_n\| = \frac{1}{\psi(t_n)}\frac{2\psi(s_n)\psi(u_n)}{\psi(s_n)+\psi(u_n)} \tag{64}$$

for $n \in \mathbb{N}$. Therefore, we obtain

$$\rho(\|y_n\|) \geq \frac{\|x_n+y_n\|+\|x_n-y_n\|}{2}-1 = \frac{1}{\psi(t_n)}\frac{2\psi(s_n)\psi(u_n)}{\psi(s_n)+\psi(u_n)}-1. \tag{65}$$

From

$$\begin{aligned}
&2\psi(s_n)\psi(u_n)-(\psi(u_n)+\psi(s_n))\psi(t_n) \\
&= \psi(s_n)(\psi(u_n)-\psi(t_n))-\psi(u_n)(\psi(t_n)-\psi(s_n)) \\
&= \psi(s_n)\left(\int_{t_n}^{\lambda s_n+(1-\lambda)u_n}\gamma(s)\,ds+\int_{\lambda s_n+(1-\lambda)u_n}^{u_n}\gamma(s)\,ds\right) \\
&\quad -\psi(u_n)\left(\int_{s_n}^{(1-\lambda)s_n+\lambda u_n}\gamma(s)\,ds + \int_{(1-\lambda)s_n+\lambda u_n}^{t_n}\gamma(s)\,ds\right) \\
&\geq \psi(s_n)\gamma(t_n)(\lambda s_n+(1-\lambda)u_n-t_n) \\
&\quad +\psi(s_n)\gamma(\lambda s_n+(1-\lambda)u_n)\lambda(u_n-s_n) \\
&\quad -\psi(u_n)\gamma((1-\lambda)s_n+\lambda u_n)\lambda(u_n-s_n) \\
&\quad -\psi(u_n)\gamma(t_n)(t_n-(1-\lambda)s_n-\lambda u_n) \\
&= \psi(s_n)\gamma(t_n) \\
&\quad \times\left(\lambda s_n+(1-\lambda)u_n-\frac{(s_n/\psi(s_n))+(u_n/\psi(u_n))}{(1/\psi(s_n))+(1/\psi(u_n))}\right) \\
&\quad +\psi(s_n)\gamma(\lambda s_n+(1-\lambda)u_n)\lambda(u_n-s_n) \\
&\quad -\psi(u_n)\gamma((1-\lambda)s_n+\lambda u_n)\lambda(u_n-s_n) \\
&\quad -\psi(u_n)\gamma(t_n) \\
&\qquad \times\left(\frac{(s_n/\psi(s_n))+(u_n/\psi(u_n))}{(1/\psi(s_n))+(1/\psi(u_n))}-(1-\lambda)s_n-\lambda u_n\right) \\
&= -\psi(s_n)\gamma(t_n)\lambda(u_n-s_n) \\
&\quad +\psi(s_n)\gamma(\lambda s_n+(1-\lambda)u_n)\lambda(u_n-s_n) \\
&\quad -\psi(u_n)\gamma((1-\lambda)s_n+\lambda u_n)\lambda(u_n-s_n) \\
&\quad +\psi(u_n)\gamma(t_n)\lambda(u_n-s_n) \\
&= \psi(s_n)\lambda(u_n-s_n)(\gamma(\lambda s_n+(1-\lambda)u_n)-\gamma(t_n)) \\
&\quad +\psi(u_n)\lambda(u_n-s_n)(\gamma(t_n)-\gamma((1-\lambda)s_n+\lambda u_n)) \\
&\geq \frac{1}{2}\lambda(u_n-s_n)(\gamma(\lambda s_n+(1-\lambda)u_n)-\gamma(t_n)) \\
&\quad +\frac{1}{2}\lambda(u_n-s_n)(\gamma(t_n)-\gamma((1-\lambda)s_n+\lambda u_n)) \\
&= \frac{1}{2}\lambda(u_n-s_n) \\
&\quad \times(\gamma(\lambda s_n+(1-\lambda)u_n)-\gamma((1-\lambda)s_n+\lambda u_n)) \\
&\geq \frac{1}{2}\lambda n(u_n-s_n)^q
\end{aligned} \tag{66}$$

and (59), we have

$$\begin{aligned}
\frac{\rho(\|y_n\|)}{\|y_n\|^q} &\geq \frac{1}{(4(u_n-s_n))^q}\,\frac{(1/2)\lambda n(u_n-s_n)^q}{\psi(t_n)(\psi(s_n)+\psi(u_n))} \\
&\geq \frac{1}{4^q}\frac{(1/2)\lambda n}{\psi(t_n)(\psi(s_n)+\psi(u_n))} \geq \frac{1}{4^{q+1}}\lambda n.
\end{aligned} \tag{67}$$

Hence we obtain $\limsup_{\tau\to+0}\rho(\tau)/\tau^q = \infty$. So by Proposition 2, we obtain the desired result. □

Corollary 17. *Let f be a bijective and strictly increasing function from $[-1,1]$ into $[0,1]$ with $\int_{-1}^{1} f(a)da = 1$. Assume that f is infinitely differentiable on the neighborhood of some $a_0 \in (-1,1)$, and*

$$f'(a_0)=f''(a_0)=f'''(a_0)=\cdots=0. \tag{68}$$

Then $f^{-1} \in \Gamma_2$ and $(\mathbb{C}^2, \|\cdot\|_{Sf^{-1}})$ is uniformly smooth and is not q-uniformly smooth for all $q \in (1,2]$.

Proof. It is not difficult to check $f^{-1} \in \Gamma_2$. Put $\gamma = f^{-1}$ and $\psi = S\gamma$. By Proposition 12, since γ is surjective, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth and hence it is uniformly smooth. Fix $\nu \in \mathbb{N}$. As in the proof of Corollary 14, we can prove $\lim_{b\to a_0+0}(f(b)-f(a_0))/(b-a_0)^\nu = 0$. Since f is strictly increasing, we have

$$\lim_{b\to a_0+0}\frac{(b-a_0)^\nu}{f(b)-f(a_0)}=\infty. \tag{69}$$

Putting $u = f(b)$ and $t_0 = f(a_0)$, we have

$$\begin{aligned}
\lim_{u\to t_0+0}\frac{\gamma(u)-\gamma(t_0)}{(u-t_0)^{1/\nu}} &= \lim_{b\to a_0+0}\frac{b-a_0}{(f(b)-f(a_0))^{1/\nu}} \\
&= \lim_{b\to a_0+0}\left(\frac{(b-a_0)^\nu}{f(b)-f(a_0)}\right)^{1/\nu} \\
&= \infty.
\end{aligned} \tag{70}$$

We choose a strictly increasing sequence $\{s_n\}$ and a strictly decreasing sequence $\{u_n\}$ in $[0,1]$ satisfying $t_0 = (2/3)s_n + (1/3)u_n$ for $n \in \mathbb{N}$ and $\lim_n s_n = \lim_n u_n = t_0$. Then it is obvious that $t_0 < (1/3)s_n + (2/3)u_n$ for $n \in \mathbb{N}$ and $\lim_n((1/3)s_n + (2/3)u_n) = t_0$. We have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\gamma((1/3)s_n+(2/3)u_n)-\gamma((2/3)s_n+(1/3)u_n)}{(u_n-s_n)^{1/\nu}} \\
&= \frac{1}{3^{1/\nu}}\lim_{n\to\infty}\frac{\gamma((1/3)s_n+(2/3)u_n)-\gamma(t_0)}{((1/3)s_n+(2/3)u_n-t_0)^{1/\nu}}=\infty.
\end{aligned} \tag{71}$$

Thus, by Theorem 16, we have that $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not $(1+1/\nu)$-uniformly smooth. Since ν is arbitrary, $(\mathbb{C}^2, \|\cdot\|_\psi)$ is not q-uniformly smooth for every $q \in (1,2]$. □

Example 18. Define a function f from $[-1,1]$ onto $[0,1]$ by

$$f(a) = \begin{cases} \dfrac{-\exp\left(1-a^{-2}\right)}{2} + \dfrac{1}{2}, & \text{if } a < 0, \\ \dfrac{1}{2}, & \text{if } a = 0, \\ \dfrac{+\exp\left(1-a^{-2}\right)}{2} + \dfrac{1}{2} & \text{if } a > 0 \end{cases} \quad (72)$$

for $a \in [-1,1]$. Then $(\mathbb{C}^2, \|\cdot\|_{Sf^{-1}})$ is uniformly smooth and not q-uniformly smooth for all $q \in (1,2]$. See Figure 1.

Example 19. Let γ be as in Example 15 and let f be as in Example 18. Define a function η from $[0,1]$ into $[-1,1]$ by

$$\eta(t) = \begin{cases} \dfrac{f^{-1}(4t)}{4} - \dfrac{3}{4}, & \text{if } t \le \dfrac{1}{4}, \\ \dfrac{\gamma(2t-1/2)}{2}, & \text{if } \dfrac{1}{4} \le t \le \dfrac{3}{4}, \\ \dfrac{f^{-1}(4t-3)}{4} + \dfrac{3}{4}, & \text{if } t \ge \dfrac{3}{4} \end{cases} \quad (73)$$

for $t \in [0,1]$. Then $(\mathbb{C}^2, \|\cdot\|_{S\eta})$ is uniformly convex, uniformly smooth, not p-uniformly convex for all $p \in [2,\infty)$, and not q-uniformly smooth for all $q \in (1,2]$. See Figure 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The author is supported in part by Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science.

References

[1] K. Ball, E. A. Carlen, and E. H. Lieb, "Sharp uniform convexity and smoothness inequalities for trace norms," *Inventiones Mathematicae*, vol. 115, no. 3, pp. 463–482, 1994.

[2] J. A. Clarkson, "Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40, no. 3, pp. 396–414, 1936.

[3] O. Hanner, "On the uniform convexity of L^p and ℓ^p," *Arkiv för Matematik*, vol. 3, pp. 239–244, 1956.

[4] N. Shioji, "On uniformly convex functions and uniformly smooth functions," *Mathematica Japonica*, vol. 41, no. 3, pp. 641–655, 1995.

[5] Y. Takahashi, K. Hashimoto, and M. Kato, "On sharp uniform convexity, smoothness, and strong type, cotype inequalities," *Journal of Nonlinear and Convex Analysis*, vol. 3, no. 2, pp. 267–281, 2002.

[6] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[7] F. F. Bonsall and J. Duncan, *Numerical Ranges. II*, vol. 10 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, New York, NY, USA, 1973.

[8] K.-S. Saito, M. Kato, and Y. Takahashi, "Absolute norms on $\mathbb{C}^n$," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 879–905, 2000.

[9] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, USA, 1970.

[10] K.-I. Mitani, K.-S. Saito, and T. Suzuki, "Smoothness of absolute norms on $\mathbb{C}^n$," *Journal of Convex Analysis*, vol. 10, no. 1, pp. 89–107, 2003.

[11] Y. Takahashi, M. Kato, and K.-S. Saito, "Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces," *Journal of Inequalities and Applications*, vol. 7, no. 2, pp. 179–186, 2002.