Research Article

Bifurcation Analysis of a Singular Bioeconomic Model with Allee Effect and Two Time Delays

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A singular prey-predator model with time delays is formulated and analyzed. Allee effect is considered on the growth of the prey population. The singular prey-predator model is transformed into its normal form by using differential-algebraic system theory. We study its dynamics in terms of local analysis and Hopf bifurcation. The existence of periodic solutions via Hopf bifurcation with respect to two delays is established. In particular, we study the direction of Hopf bifurcation and the stability of bifurcated periodic solutions by applying the normal form theory and the center manifold argument. Finally, numerical simulations are included supporting the theoretical analysis and displaying the complex dynamical behavior of the model outside the domain of stability.

1. Introduction

The economic theory proposed by Gordon [1] in 1954 was described as follows:

net economic revenue (NER) = total revenue (TR) − total cost (TC).

This provides theoretical evidence for the establishment of singular bioeconomic model, which is described by differential-algebraic equations. Recently, there has been extensive literature dealing with such systems, describing the interactions between the different species and harvesting effort regarding activity, stabilities of equilibrium, bifurcations, and other dynamics (see, e.g., [2–5] and the references therein).

In most of ecosystems, since one species does not respond instantaneously to interactions with other species, some delays due to several reasons, such as gestation, hunting, and maturation, are required. To incorporate this idea in a modelling approach, time delays have been introduced into ecosystems. For a long time, it has been recognized that delays can have very complicated impact on the dynamics of a system (see, e.g., monographs by Hale [6], Hale and Verduyn Lunel [7], Kuang [8], Yuan and Song [9], and Wu [10]). Generally speaking, delays can cause the loss of stability and lead to periodic solutions. Some authors like in [11] described the effects of time delay in a prey-predator model incorporating parasite infection for the prey population. The literature [12] described the dynamics of a stage structured population model with time delay in fluctuating environment. The literature [13] investigated dynamics of delayed prey-predator model with harvesting. Chakraborty et al. [14] introduced a single discrete gestation delay in a differential-algebraic biological economic system and showed Hopf bifurcation in the neighborhood of coexisting equilibrium point through considering the delay as a bifurcation parameter. Zhang et al. [15] studied a ratio-dependent prey-predator singular model and analyzed the direction and stability of periodic solutions. Some other authors have discussed dynamical models with multiple delays. While analyzing, the multiply delayed models are mostly simplified by taking equal magnitude for two delays $\tau_1 = \tau_2 = \tau$ [16, 17] or choosing the sum τ of two delays as a bifurcation parameter [18, 19]. However, the delays appearing in different terms of an ecological system are not always equal. Therefore, it is necessary to discuss dynamical system with different delays.

In addition, numerous examples demonstrate that the growth of natural populations can exhibit Allee effect [20], which describes a positive relation between population density and the per capita growth rate. When populations get larger, the effect will saturate or disappear. Allee effect may result from some causes, such as mate finding, social dysfunction, inbreeding depression, fool exploitation, and predator avoidance of defense. Therefore, in this paper, we consider a differential-algebraic prey-predator model with two time delays and the Allee effect as follows:

$$\frac{dx}{dt} = x\left(\frac{x}{A+x} - d_1 - x(t-\tau_1)\right) - \frac{xy}{1+h_1x},$$

$$\frac{dy}{dt} = y\left(\frac{bx(t-\tau_2)}{1+h_1x(t-\tau_2)} - d_2\right) - Ey, \qquad (1)$$

$$0 = E(py-c) - m,$$

where A is the Allee effect constant of the prey species, respectively. d_1 and d_2 are the intrinsic mortality rate of the prey and predator population, respectively. h_1 is the handling time. b denotes food utilization efficiency. Prey dynamics is delayed by τ_1 due to crowing and the predator takes time τ_2 to convert the food into its growth; E is harvesting effort for predator, and $p > 0$, $c > 0$, and $m > 0$ are harvesting reward per unit harvesting effort for unit weight of predator, harvesting cost per unit harvesting effort for predator, and the net economic revenue per unit harvesting effort, respectively. All the parameters are positive constants.

In this paper, we investigate the effects of two delays on the dynamics of singular bioeconomic model with Allee effect. The existence of periodic solution has been explored through Hopf bifurcation. The domain of stability is defined. Then, the formula for determining the properties of Hopf bifurcation is derived by using the normal form method and center manifold theorem. Finally, numerical simulations show the bifurcation plot with respect to time delay and give the effectiveness of the result mentioned above.

2. Stability and Existence of Hopf Bifurcation

There exists a positive equilibrium $P_0 = (x_0, y_0, E_0)$ for model (1), where coordinates x_0, y_0, and E_0 satisfy the following equations:

$$\frac{x}{A+x} - d_1 - x - \frac{y}{1+h_1x} = 0,$$

$$\frac{bx(t-\tau_2)}{1+h_1x(t-\tau_2)} - d_2 - E = 0, \qquad (2)$$

$$E(py-c) - m = 0.$$

For the sake of the simplicity, denote the two differential equations by $f_1(x, y, E)$ and $f_2(x, y, E)$ and the algebraic equation by $g(x, y, E)$.

Let $u = x$, $v = y$, and $\overline{E} = (pE_0y/(py_0-c)) + E$. Therefore, the system (1) can be rewritten in the following form:

$$\frac{du}{dt} = u\left(\frac{u}{A+u} - d_1 - u(t-\tau_1)\right) - \frac{uv}{1+h_1u},$$

$$\frac{dv}{dt} = v\left(\frac{bu(t-\tau_2)}{1+h_1u(t-\tau_2)} - d_2 - \overline{E} + \frac{pE_0v}{py_0-c}\right), \qquad (3)$$

$$0 = \left(\overline{E} - \frac{pE_0v}{py_0-c}\right)(pv-c) - m.$$

Consider the following local parametric ψ of the third equation of the system (3):

$$[u, v, \overline{E}]^T = \psi(\tau_1, \tau_2, Y)$$

$$= N_0^T + U_0Y + V_0h(Y(t), g(\psi(Y(t)))) = 0, \qquad (4)$$

where $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $Y = (y_1, y_2)^T$, $N_0 = (u_0, v_0, \overline{E}_0)$, and $h(y_1, y_2) : R^2 \rightarrow R$ is a smooth mapping and has the following expression:

$$h(y_1, y_2) = \frac{pE_0(v_0 + y_2)}{pv_0 - c} + \frac{m}{p(v_0 + y_2) - c} - \frac{pE_0y_0}{pv_0 - c} - E_0. \qquad (5)$$

Then, the linearization part of the system (3) is

$$J(N_0)$$

$$= \begin{pmatrix} D_Nf_1(N_0) \\ D_Nf_2(N_0) \end{pmatrix} \begin{pmatrix} D_Ng(N_0) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{Au_0}{(A+u_0)^2} + \dfrac{h_1u_0v_0}{(1+h_1u_0)^2} - u_0e^{-\lambda\tau_1} & -\dfrac{u_0}{1+h_1u_0} \\[3mm] \dfrac{bv_0}{(1+h_1u_0)^2}e^{-\lambda\tau_2} & \dfrac{pE_0v_0}{pv_0-c} \end{pmatrix}. \qquad (6)$$

For the sake of the simplicity, let

$$u_1 = u_{11} + u_{12}, \qquad u_{11} = -u_0,$$

$$u_{12} = \frac{Au_0}{(A+u_0)^2} + \frac{hu_0v_0}{(1+hu_0)^2},$$

$$u_2 = -\frac{u_0}{1+hu_0}, \qquad u_3 = \frac{bv_0}{(1+hu_0)^2}, \qquad (7)$$

$$u_4 = \frac{pv_0E_0}{pv_0-m}.$$

According to the Jacobian matrix J, the characteristic equation of (1) at P_0 can be written as follows:

$$R_1(\lambda) = \lambda^2 - (u_4 + u_{12})\lambda + u_{12}u_4$$

$$- u_{11}(\lambda - u_4)e^{-\lambda\tau_1} - u_2u_3e^{-\lambda\tau_2} = 0. \qquad (8)$$

When the time delays are not considered, the characteristic polynomial for model (1) can be expressed as the following form:

$$\lambda^2 - (u_1 + u_4)\lambda + u_1 u_4 - u_2 u_3 = 0. \tag{9}$$

It is clear that if the following conditions are satisfied:

$$u_1 + u_4 < 0, \qquad u_1 u_4 - u_2 u_3 > 0, \tag{10}$$

then P_0 is locally asymptotically stable for model (1).

Next, we consider the effects of the two time delays on the stability of model (1) in the following cases.

Case 1 ($\tau_1 = 0$ and $\tau_2 > 0$). Due to $\tau_1 = 0$, (8) becomes

$$\lambda^2 - (u_1 + u_4)\lambda + u_1 u_4 - u_2 u_3 = 0. \tag{11}$$

Assume that a purely imaginary solution of the form $\lambda = i\omega_2$ exists in (11). Substituting it into (11) and separating the real and imaginary parts, we have

$$-\omega_2^2 + (u_{11} + u_{12})u_4 = u_2 u_3 \cos \omega_2 \tau_2,$$
$$-(u_{11} + u_{12} + u_4)\omega_2 = -u_2 u_3 \sin \omega_2 \tau_2. \tag{12}$$

Taking square on both sides of (12) and summing them up, we obtain

$$\omega_2^4 + (u_1^2 + u_4^2)\omega_2^2 + u_1^2 u_4^2 - u_2^2 u_3^2 = 0. \tag{13}$$

From condition (10), (13) has unique positive real root ω_{20} if

$$u_1 u_4 + u_2 u_3 < 0. \tag{14}$$

The critical value of the delay corresponding to ω_{20} is given by

$$\tau_{2k} = \frac{1}{\omega_{20}} \arccos \frac{-\omega_{20}^2 + u_1 u_4}{u_2 u_3} + \frac{2k\pi}{\omega_{20}}, \quad k = 0, 1, 2, \ldots. \tag{15}$$

If $u_1 u_4 + u_2 u_3 > 0$, (13) has no real root and model (1) is asymptotically stable for any time delay $\tau_2 > 0$.

Now, differentiating (8) with respect to τ_2 and substituting the eigenvalue $i\omega_{20}$, it follows that

$$\text{sign}\left(\frac{d}{d\tau_2}\text{Re}(\lambda)\right)\bigg|_{\tau_{2k}} = \text{sign}\left(\omega^4 - u_1^2 u_4^2 + u_2^2 u_3^2\right) > 0. \tag{16}$$

Theorem 1. *Assume that condition (14) holds; then, there exists a $\tau_{20} > 0$ given by*

$$\tau_{20} = \frac{1}{\omega_{20}} \arccos \frac{-\omega_{20}^2 + u_1 u_4}{u_2 u_3} + \frac{2k\pi}{\omega_{20}}, \tag{17}$$

such that the positive equilibrium P_0 of model (1) is locally asymptotically stable for $\tau_2 < \tau_{20}$ and undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}$. That is, model (1) has a branch of periodic solutions bifurcating from the positive equilibrium P_0 near $\tau_2 = \tau_{20}$.

Case 2 ($\tau_1 > 0$ and $\tau_2 = 0$). According to a similar discussion as in Case 1, we can obtain the following result.

Theorem 2. *There exists a $\tau_{10} > 0$ such that the positive equilibrium P_0 of model (1) is locally asymptotically stable for $\tau_1 < \tau_{10}$ and undergoes a Hopf bifurcation when $\tau_1 = \tau_{10}$ given by*

$$\tau_{10} = \frac{1}{\omega_{10}} \arccos \frac{-u_{12}\omega_{10}^2 + u_4(-u_{12}u_4 + u_2 u_3)}{u_{11}(u_4^2 + \omega_{10}^2)}, \tag{18}$$

where $i\omega_{10}$ is the root of corresponding characteristic equation.

Case 3 ($\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$). This case states that τ_1 is regarded as a parameter and τ_2 is in its stable interval. Assume that (8) has purely imaginary solution of the form $\lambda = i\omega$. Substituting it into (8) and separating the real and imaginary parts, we have

$$-\omega^2 + u_{12}u_4 - u_2 u_3 \cos \omega\tau_2$$
$$= -u_{11}u_4 \cos \omega\tau_1 + u_{11}\omega \sin \omega\tau_1,$$
$$-(u_{12} + u_4)\omega + u_2 u_3 \sin \omega\tau_2$$
$$= u_{11}\omega \cos \omega\tau_1 + u_{11}u_4 \sin \omega\tau_1. \tag{19}$$

Eliminating τ_1 leads to

$$\omega^4 + \left[(u_4 + u_{12})^2 - 2(u_{12}u_4 - u_2 u_3 \cos \omega\tau_2) - u_{11}^2\right]\omega^2$$
$$- 2u_2 u_3 (u_4 + u_{12}) \cdot (\sin \omega\tau_2)\omega + u_{12}^2 u_4^2 + u_2^2 u_3^2 - u_{11}^2 u_4^2$$
$$- 2u_{12}u_2 u_3 u_4 \cos \omega\tau_2 = 0. \tag{20}$$

It can be seen that there exists at least one real positive root ω'_{10} for (20) if the condition $(u_{12}u_4 - u_2 u_3)^2 < u_{11}^2 u_4^2$ holds. Equation (19) can also be written as

$$-u_2 u_3 \cos \omega\tau_2 = -u_{11}u_4 \cos \omega\tau_1 + u_{11}\omega \sin \omega\tau_1$$
$$+ \omega^2 - u_{12}u_4,$$
$$u_2 u_3 \sin \omega\tau_2 = u_{11}\omega \cos \omega\tau_1 + u_{11}u_4 \sin \omega\tau_1$$
$$+ (u_{12} + u_4)\omega. \tag{21}$$

Equation (21) can be solved as

$$
\tau'_{1k} = \frac{1}{\omega'_{10}}
$$

$$
\times \left(\arccos \left(\left(\left(\omega^2 - u_{12}u_4 \right)^2 + \left(u_4 + u_{12} \right)^2 \omega^2 \right. \right. \right.
$$

$$
\left. + u_{11}^2 u_4^2 + u_{11}^2 \omega^2 - u_2^2 u_3^2 \right)
$$

$$
\times \left(-2u_{11} \sqrt{u_4^2 + \omega^2} \right.
$$

$$
\left. \left. \left. \times \sqrt{\left(\omega^2 - u_{12}u_4 \right)^2 + \omega^2 \left(u_4 + u_{12} \right)^2} \right)^{-1} \right) \right.
$$

$$
\left. + \phi_1 + \phi_2 \right) + \frac{2k\pi}{\omega'_{10}}, \quad k = 0, 1, 2, \ldots,
$$

(22)

where $\phi_1 = \arctan(u_4/\omega'_{10})$ and $\phi_2 = \arctan((\omega'^2_{10} - u_{12}u_4)/\omega'_{10}(u_4 + u_{12}))$.

Now, differentiating (8) with respect to τ_1 and substituting the eigenvalue $i\omega'_{10}$ and time delay $\tau_1 = \tau'_{10}$, it follows that

$$
M \left(\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau_1} \right) \Bigg|_{\tau'_{10}} + N \left(\frac{d\omega}{d\tau_1} \right) \Bigg|_{\tau'_{10}} = P,
$$

$$
-N \left(\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau_1} \right) \Bigg|_{\tau'_{10}} + M \left(\frac{d\omega}{d\tau_1} \right) \Bigg|_{\tau'_{10}} = Q,
$$

(23)

where

$$
M = -\left(u_4 + u_{12} \right) - u_{11} \cos \omega'_{10} \tau'_{10}
$$

$$
+ \tau'_{10} u_{11} \left(-u_4 \cos \omega'_{10}\tau'_{10} + \omega'_{10} \sin \omega'_{10}\tau'_{10} \right)
$$

$$
+ \tau_2 u_2 u_3 \cos \omega'_{10}\tau_2,
$$

$$
N = -2\omega'_{10} - u_{11} \sin \omega'_{10}\tau'_{10}
$$

$$
- \tau'_{10} u_{11} \left(\omega'_{10} \cos \omega'_{10}\tau'_{10} + u_4 \sin \omega'_{10}\tau'_{10} \right)
$$

(24)

$$
+ \tau_2 u_2 u_3 \sin \omega'_{10}\tau_2,
$$

$$
P = u_{11}\omega'_{10} \left(\omega'_{10} \cos \omega'_{10}\tau'_{10} + u_4 \sin \omega'_{10}\tau'_{10} \right),
$$

$$
Q = u_{11}\omega'_{10} \left(-\omega'_{10} \sin \omega'_{10}\tau'_{10} + u_4 \cos \omega'_{10}\tau'_{10} \right).
$$

(25)

Solving (23), we obtain

$$
\left(\frac{d\,(\mathrm{Re}\,\lambda)}{d\tau_1} \right) \Bigg|_{\tau'_{10}} = \frac{MP - NQ}{M^2 + N^2} \neq 0.
$$

(26)

Theorem 3. *Assume* $(u_{12}u_4 - u_2u_3)^2 < u_{11}^2 u_4^2$ *and* $\tau_2 \in [0, \tau_{20})$. *The positive equilibrium* P_0 *of model* (1) *is asymptotically stable*

for $\tau_1 \in [0, \tau'_{10})$ *and undergoes Hopf bifurcation at* $\tau_1 = \tau'_{10}$, *where*

$$
\tau'_{10} = \frac{1}{\omega'_{10}}
$$

$$
\times \left(\arccos \left(\left(\left(\omega'^2_{10} - u_{12}u_4 \right)^2 + \left(u_4 + u_{12} \right)^2 \omega'^2_{10} \right. \right. \right.
$$

$$
\left. + u_{11}^2 u_4^2 + u_{11}^2 \omega'^2_{10} - u_2^2 u_3^2 \right)
$$

$$
\times \left(-2u_{11} \sqrt{u_4^2 + \omega'^2_{10}} \right.
$$

$$
\left. \left. \left. \times \sqrt{\left(\omega'^2_{10} - u_{12}u_4 \right)^2 + \omega'^2_{10} \left(u_4 + u_{12} \right)^2} \right)^{-1} \right) \right.
$$

$$
\left. + \phi_1 + \phi_2 \right),
$$

(27)

where $\phi_1 = \arctan(u_4/\omega'_{10})$ *and* $\phi_2 = \arctan((\omega'^2_{10} - u_{12}u_4)/\omega'_{10}(u_4 + u_{12}))$.

Case 4 ($\tau_1 \in (0, \tau_{10})$ *and* $\tau_2 > 0$). According to a similar discussion as in Case 3, we can obtain the following result.

Theorem 4. *The positive equilibrium* P_0 *of model* (1) *is asymptotically stable for* $\tau_2 \in [0, \tau'_{20})$ *and undergoes Hopf bifurcation at* $\tau_2 = \tau'_{20}$, *where*

$$
\tau'_{20} = \frac{1}{\omega'_{20}}
$$

$$
\times \left(\arccos \left(\left(\left(\omega'^2_{20} - u_{12}u_4 \right)^2 + \left(u_4 + u_{12} \right)^2 \omega'^2_{20} \right. \right. \right.
$$

$$
\left. + u_{12}^2 u_3^2 - u_{11}^2 u_4^2 - u_{11}^2 \omega'^2_{20} \right)
$$

$$
\times \left(-2u_2 u_3 \right.
$$

$$
\left. \left. \left. \times \sqrt{\left(u_4 + u_{12} \right)^2 \omega'^2_{20} + \left(\omega'^2_{20} - u_{12}u_4 \right)^2} \right)^{-1} \right) \right.
$$

$$
\left. - \psi \right),
$$

(28)

where $\psi = \arctan((u_4 + u_{12})\omega'_{20}/(\omega'^2_{20} - u_{12}u_4))$.

3. Direction and the Stability of Hopf Bifurcation

In the previous section, we obtain the conditions under which a family of periodic solutions bifurcate from the positive equilibrium at the critical values of time delays τ_1 and τ_2. As pointed out by Hassard et al., by employing the normal form

and center manifold theory, the formulae for determining the directions, stability, and period of Hopf bifurcation can be presented. Following the ideas of Hassard et al., this section discusses the directions, stability, and period of Hopf bifurcation. These properties are studied with respect to τ_1 for fixed $\tau_2 \in (0, \tau_{20})$. Throughout this section, it is considered that the system (1) undergoes Hopf bifurcation at $\tau_1 = \tau_{10}'$, $\tau_2^* \in (0, \tau_{20})$ at the equilibrium P_0. Without loss of generality, this section assumes that $\tau_2^* < \tau_{10}'$.

Let $x_1 = y_1 - y_1^*$, $x_2 = y_2 - y_2^*$, $t = t/\tau_1$, $\overline{x}_i(t) = x_i(\tau_1 t)$, and $\tau_1 = \tau_{10}' + \mu$ and dropping the bars for simplification of notations, the system (1) can be written as a functional differential equation in $C = C([-1, 0], R^2)$

$$\dot{x}(t) = L_\mu(x_t) + F(\mu, x_t), \qquad (29)$$

where $x(t) = (x_1(t), x_2(t))^T \in R^2$, and $L_\mu : C \to R$, $F : R \times C \to R$ are given, respectively, by

$$
\begin{aligned}
L_\mu(\phi) = \left(\tau_{10}' + \mu\right) \\
\times \Bigg(\left(\begin{matrix} \dfrac{Au_0}{(A + u_0)^2} + \dfrac{h_1 u_0 v_0}{(1 + h_1 u_0)^2} & -\dfrac{u_0}{1 + h_1 u_0} \\ 0 & \dfrac{pE_0 v_0}{pv_0 - c} \end{matrix} \right) \\
\times \phi(0) + \left(\begin{matrix} 0 & 0 \\ \dfrac{bv_0}{(1 + h_1 u_0)^2} & 0 \end{matrix} \right) \phi\left(-\dfrac{\tau_2^*}{\tau_{10}'} \right) \\
+ \left(\begin{matrix} -u_0 & 0 \\ 0 & 0 \end{matrix} \right) \phi(-1) \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
F(\mu, \phi) = \left(\tau_{10}' + \mu\right) \\
\times \Bigg(\left(\dfrac{A^2}{(A + u_0)^3} + \dfrac{h_1 v_0}{(1 + h_1 u_0)^3} \right) \phi_1^2(0) \\
- \phi_1(0)\phi_1(-1) - \dfrac{1}{(1 + h_1 u_0)^2}\phi_1(0)\phi_2(0) \\
- \left(\dfrac{A^2}{(A + u_0)^4} + \dfrac{h_1^2 v_0}{(1 + h_1 u_0)^4} \right) \cdot \phi_1^3(0) \\
+ \dfrac{h_1}{(1 + h_1 u_0)^3}\phi_1^2(0)\phi_2(0) + \cdots \\
- \dfrac{bh_1 v_0}{(1 + h_1 u_0)^3}\phi_1^2\left(-\dfrac{\tau_2^*}{\tau_{10}'} \right) \\
+ \dfrac{b}{(1 + h_1 u_0)^2}\phi_1\left(-\dfrac{\tau_2^*}{\tau_{10}'} \right)\phi_2(0)
\end{aligned}
$$

$$
\begin{aligned}
- \dfrac{pE_0 c}{(pv_0 - c)^2}\phi_2^2(0) + \dfrac{bv_0 h_1^2}{(1 + h_1 u_0)^4}\phi_1^3\left(-\dfrac{\tau_2^*}{\tau_{10}'} \right) \\
- \dfrac{bh_1}{(1 + h_1 u_0)^3}\phi_1^2\left(-\dfrac{\tau_2^*}{\tau_{10}'} \right)\phi_2(0) \\
+ \dfrac{mp^2 c}{pv_0 - c}\phi_2^3(0) + \cdots \Bigg),
\end{aligned}
$$

$$(30)$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T \in C$. By the Riesz representation theorem, there exists a 2×2 matrix $\eta(\theta, \mu)$ whose components are bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C, \qquad (31)$$

where we choose

$$
\eta(\theta, \mu)
$$

$$
= \begin{cases}
\left(\tau_{10}' + \mu\right) \\
\quad \times \left(\begin{matrix} \dfrac{Au_0}{(A + u_0)^2} + \dfrac{h_1 u_0 v_0}{(1 + h_1 u_0)^2} - u_0 & -\dfrac{u_0}{1 + h_1 u_0} \\ \dfrac{bv_0}{(1 + h_1 u_0)^2} & \dfrac{pE_0 v_0}{pv_0 - c} \end{matrix} \right), \\
\hfill \theta = 0, \\[2ex]
\left(\tau_{10}' + \mu\right)\left(\begin{matrix} -u_0 & 0 \\ \dfrac{bv_0}{(1 + h_1 u_0)^2} & 0 \end{matrix} \right), \quad \theta \in \left[-\dfrac{\tau_2^*}{\tau_{10}'}, 0 \right), \\[3ex]
\left(\tau_{10}' + \mu\right)\left(\begin{matrix} -u_0 & 0 \\ 0 & 0 \end{matrix} \right), \quad \theta \in \left(-1, -\dfrac{\tau_2^*}{\tau_{10}'} \right), \\[2ex]
0_{2 \times 2}, \quad \theta = -1.
\end{cases}
$$

$$(32)$$

For $\phi \in C^1([-1, 0], R^2)$, define

$$
A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \qquad (33)
$$

$$
R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases}
$$

Then, system (29) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \qquad (34)$$

where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0,1], (R^2)^*)$, define

$$A^* \psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \displaystyle\int_{-1}^0 d\eta^T(t,0)\,\psi(-t), & s = 0, \end{cases} \tag{35}$$

and bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) \\ - \int_{-1}^0 \int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \tag{36}$$

where $\eta(\theta) = \eta(\theta,0)$, $A = A(0)$, and A^* are adjoint operators. Since $\pm i\omega'_{10}\tau'_{10}$ are eigenvalues of $A(0)$, they are also eigenvalues of A^*. Next, we need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega'_{10}\tau'_{10}$ and $-i\omega'_{10}\tau'_{10}$, respectively.

In terms of the discussion mentioned above, we see that $\pm i\omega'_{10}\tau'_{10}$ are eigenvalues of $A(0)$ and A^*. Next, we calculate the eigenvector $q(\theta)$ of $A(0)$ corresponding to $i\omega'_{10}\tau'_{10}$ and the eigenvector $q^*(s)$ of A^* belonging to $-i\omega'_{10}\tau'_{10}$, respectively. By the definition of $A(0)$ and A^*, it is not difficult to obtain that

$$\alpha = \left(i\omega'_{10} + u_0 e^{-i\omega'_{10}\tau'_{10}} - \frac{Au_0}{(A+u_0)^2} - \frac{h_1 u_0 v_0}{(1+h_1 u_0)^2} \right) \\ \times \left(-\frac{u_0}{(1+h_1 u_0)} \right)^{-1}. \tag{37}$$

Similarly, let $q^*(s) = D(1,\alpha^*)e^{i\omega'_{10}\tau'_{10}s}$ be the eigenvector of A^* corresponding to $-i\omega'_{10}\tau'_{10}$, and we can obtain

$$\alpha^* = \left(-i\omega'_{10} + u_0 e^{i\omega'_{10}\tau'_{10}} - \frac{Au_0}{(A+u_0)^2} - \frac{h_1 u_0 v_0}{(1+h_1 u_0)^2} \right) \\ \times \left(\frac{bv_0}{(1+h_1 u_0)^2} e^{i\omega'_{10}\tau^*_2} \right)^{-1}. \tag{38}$$

It follows from (36) that

$$\langle q^*(s), q(\theta) \rangle \\ = \overline{L}(1,\overline{\alpha}^*)(1,\alpha)^T \\ - \int_{-1}^0 \int_{\xi=0}^{\theta} \overline{L}(1,\overline{\alpha}^*)e^{-i\omega'_{10}\tau'_{10}(\xi-\theta)}d\eta(\theta)(1,\alpha)^T e^{i\omega'_{10}\tau'_{10}\xi}d\xi \\ = \overline{L}\left(1 + \overline{\alpha}^*\alpha - u_0 e^{-i\omega'_{10}\tau'_{10}} + \frac{bv_0 \tau^*_2 \overline{\alpha}^*}{\tau'_{10}(1+h_1 u_0)^2}e^{-i\omega'_{10}\tau^*_2} \right). \tag{39}$$

Then, we obtain

$$\overline{L} = \left(1 + \overline{\alpha}^*\alpha - u_0 e^{-i\omega'_{10}\tau'_{10}} + \frac{bv_0 \tau^*_2 \overline{\alpha}^*}{\tau'_{10}(1+h_1 u_0)^2}e^{-i\omega'_{10}\tau^*_2} \right)^{-1}, \tag{40}$$

such that $\langle q^*(s), q(\theta) \rangle = 1$, $\langle q^*(s), \overline{q}(\theta) \rangle = 0$.

Based on the ideas in Hassard et al. [21], we can obtain $g(z,\overline{z}) = \overline{q}^*(0)F_0(z,\overline{z})$ and the following expressions:

$$x_{1t}(0) = W^{(1)}(t,0) + z + \overline{z},$$

$$x_{2t}(0) = W^{(2)}(t,0) + \alpha z + \overline{\alpha}\overline{z},$$

$$x_{1t}\left(-\frac{\tau^*_2}{\tau'_{10}} \right) = W^{(1)}\left(t, -\frac{\tau^*_2}{\tau'_{10}} \right) + ze^{-i\omega'_{10}\tau^*_2} + \overline{z}e^{i\omega'_{10}\tau^*_2},$$

$$x_{2t}\left(-\frac{\tau^*_2}{\tau'_{10}} \right) = W^{(2)}\left(t, -\frac{\tau^*_2}{\tau'_{10}} \right) + \alpha ze^{-i\omega'_{10}\tau^*_2} + \overline{\alpha}\overline{z}e^{i\omega'_{10}\tau^*_2}, \tag{41}$$

$$x_{1t}(-1) = W^{(1)}(t,-1) + ze^{-i\omega'_{10}\tau'_{10}} + \overline{z}e^{i\omega'_{10}\tau'_{10}},$$

$$x_{2t}(-1) = W^{(2)}(t,-1) + \alpha ze^{-i\omega'_{10}\tau'_{10}} + \overline{\alpha}\overline{z}e^{i\omega'_{10}\tau'_{10}},$$

where those important parameters can be computed as follows:

g_{20}

$$= 2\overline{D}\tau'_{10}\left[\left(\frac{A^2}{(A+u_0)^3} + \frac{h_1 v_0}{(1+h_1 u_0)^3} \right) \\ - e^{-i\omega'_{10}\tau'_{10}} - \frac{\alpha}{(1+h_1 u_0)^2} \\ + \overline{\alpha}^*\left(\frac{b\alpha}{(1+h_1 u_0)^2}e^{-i\omega'_{10}\tau^*_2} \\ - \frac{bh_1 v_0}{(1+h_1 u_0)^3}e^{-2i\omega'_{10}\tau^*_2} \\ - \frac{pE_0 c\alpha^2}{(pv_0 - c)^2} \right) \right],$$

g_{11}

$$= 2\overline{D}\tau'_{10}\left[\left(\frac{A^2}{(A+u_0)^3} + \frac{h_1 v_0}{(1+h_1 u_0)^3} \right) \\ - \text{Re}\left\{ e^{i\omega'_{10}\tau'_{10}} \right\} - \frac{\text{Re}\{\alpha\}}{(1+h_1 u_0)^2} \\ + \overline{\alpha}^*\left(\frac{b}{(1+h_1 u_0)^2}\text{Re}\left\{ \alpha e^{i\omega'_{10}\tau^*_2} \right\} \\ - \frac{bh_1 v_0}{(1+h_1 u_0)^3} - \frac{pE_0 c\alpha\overline{\alpha}}{(pv_0 - c)^2} \right) \right],$$

g_{02}

$$= 2\overline{D}\tau'_{10}\left[\left(\frac{A^2}{(A+u_0)^3} + \frac{h_1 v_0}{(1+h_1 u_0)^3}\right)\right.$$

$$- e^{i\omega'_{10}\tau'_{10}} - \frac{\overline{\alpha}}{(1+h_1 u_0)^2}$$

$$+ \overline{\alpha}^*\left(\frac{b\overline{\alpha}}{(1+h_1 u_0)^2}e^{i\omega'_{10}\tau_2^*}\right.$$

$$- \frac{bh_1 v_0}{(1+h_1 u_0)^3}e^{2i\omega'_{10}\tau_2^*}$$

$$\left.\left. - \frac{pE_0 c\overline{\alpha}^2}{(pv_0 - c)^2}\right)\right],$$

g_{21}

$$= 2\overline{D}\tau'_{10}\left\{\left(\frac{A^2}{(A+u_0)^3} + \frac{h_1 v_0}{(1+h_1 u_0)^3}\right)\right.$$

$$\times \left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right)$$

$$- \left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}(-1)}{2} + \frac{W_{20}^1(0)}{2}e^{i\omega'_{10}\tau'_{10}}\right.$$

$$\left. + W_{11}^{(1)}(0)e^{-i\omega'_{10}\tau'_{10}}\right) - \frac{1}{(1+h_1 u_0)^2}$$

$$\times \left(W_{11}^2(0) + \frac{W_{20}^{(2)}(0)}{2}\right.$$

$$\left. + \frac{W_{20}^{(1)}(0)}{2}\overline{\alpha} + W_{11}^{(1)}(0)\alpha\right)$$

$$- 3\left(\frac{A^2}{(A+u_0)^4} + \frac{v_0 h_1^2}{(1+h_1 u_0)^4}\right)$$

$$+ \frac{h_1}{(1+h_1 u_0)^3}(\overline{\alpha} + 2\alpha)$$

$$+ \overline{\alpha}^*\left(\frac{b}{(1+h_1 u_0)^2}\right.$$

$$\times \left(W_{11}^{(2)}(0)e^{-i\omega'_{10}\tau_2^*}\right.$$

$$+ \frac{W_{20}^{(2)}(0)}{2}e^{i\omega'_{10}\tau_2^*}$$

$$+ \frac{W_{20}^{(1)}(-\tau_2^*/\tau'_{10})}{2}\overline{\alpha}$$

$$\left. + W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right)\alpha\right) - \frac{bh_1 v_0}{(1+h_1 u_0)^3}$$

$$\times \left(2W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right)e^{-i\omega'_{10}\tau_2^*}\right.$$

$$\left. + W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right)e^{i\omega'_{10}\tau_2^*}\right)$$

$$- \frac{pE_0 c}{(pv_0 - c)^2}\left(2W_{11}^{(2)}(0)\alpha + W_{20}^{(2)}(0)\overline{\alpha}\right)$$

$$+ \frac{3bv_0 h_1^2}{(1+h_1 u_0)^4}e^{-i\omega'_{10}\tau_2^*} - \frac{bh_1}{(1+h_1 u_0)^3}$$

$$\left.\times \left(\overline{\alpha}e^{-2i\omega'_{10}\tau_2^*} + 2\alpha\right) + \frac{3mp^2 c\alpha^2\overline{\alpha}}{(pv_0 - c)^4}\right\}.$$

$$\tag{42}$$

According to the computation process similar to Hassard's, parameters of g_{21} can be written as follows:

S_1

$$= \begin{pmatrix} \dfrac{A^2}{(A+u_0)^3} + \dfrac{h_1 v_0}{(1+h_1 u_0)^3} - e^{-i\omega'_{10}\tau'_{10}} - \dfrac{\alpha}{(1+h_1 u_0)^2} \\ \dfrac{b\alpha}{(1+h_1 u_0)^2}e^{-i\omega'_{10}\tau_2^*} - \dfrac{bh_1 v_0}{(1+h_1 u_0)^3}e^{-2i\omega'_{10}\tau_2^*} - \dfrac{pE_0 c\alpha^2}{(pv_0 - c)^2} \end{pmatrix},$$

S_2

$$= \begin{pmatrix} \dfrac{A^2}{(A+u_0)^3} + \dfrac{h_1 v_0}{(1+h_1 u_0)^3} - \mathrm{Re}\left(e^{i\omega'_{10}\tau'_{10}}\right) - \dfrac{\mathrm{Re}(\alpha)}{(1+h_1 u_0)^2} \\ \dfrac{b}{(1+h_1 u_0)^2}\mathrm{Re}\left(\alpha e^{i\omega'_{10}\tau_2^*}\right) - \dfrac{bh_1 v_0}{(1+h_1 u_0)^3} - \dfrac{pE_0 c\alpha\overline{\alpha}}{(pv_0 - c)^2} \end{pmatrix},$$

R_1

$$= 2\begin{pmatrix} 2i\omega'_{10} - \dfrac{Au_0}{(A+u_0)^2} - \dfrac{h_1 u_0 v_0}{(1+h_1 u_0)^2} + u_0 e^{-2i\omega'_{10}\tau'_{10}} & \dfrac{u_0}{1+h_1 u_0} \\ -\dfrac{bv_0}{1+h_1 u_0}e^{-2i\omega'_{10}\tau_2^*} & 2i\omega'_{10} - \dfrac{pE_0 v_0}{pv_0 - c} \end{pmatrix}^{-1} S_1,$$

$$R_2 = 2\begin{pmatrix} u_0 - \dfrac{Au_0}{(A+u_0)^2} - \dfrac{h_1 u_0 v_0}{(1+h_1 u_0)^2} & \dfrac{u_0}{1+h_1 u_0} \\ -\dfrac{bv_0}{(1+h_1 u_0)^2} & -\dfrac{pE_0 v_0}{pv_0 - c} \end{pmatrix}^{-1} S_2,$$

$$W_{20}(\theta) = \frac{ig_{20}}{\omega'_{10}\tau'_{10}}q(0)e^{i\omega'_{10}\tau'_{10}\theta}$$

$$+ \frac{i\overline{g}_{02}}{3\omega'_{10}\tau'_{10}}\overline{q}(0)e^{-i\omega'_{10}\tau'_{10}\theta} + R_1 e^{2i\omega'_{10}\tau'_{10}\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega'_{10}\tau'_{10}}q(0)e^{i\omega'_{10}\tau'_{10}\theta} + \frac{i\overline{g}_{11}}{\omega'_{10}\tau'_{10}}\overline{q}(0)e^{-i\omega'_{10}\tau'_{10}\theta} + R_2.$$
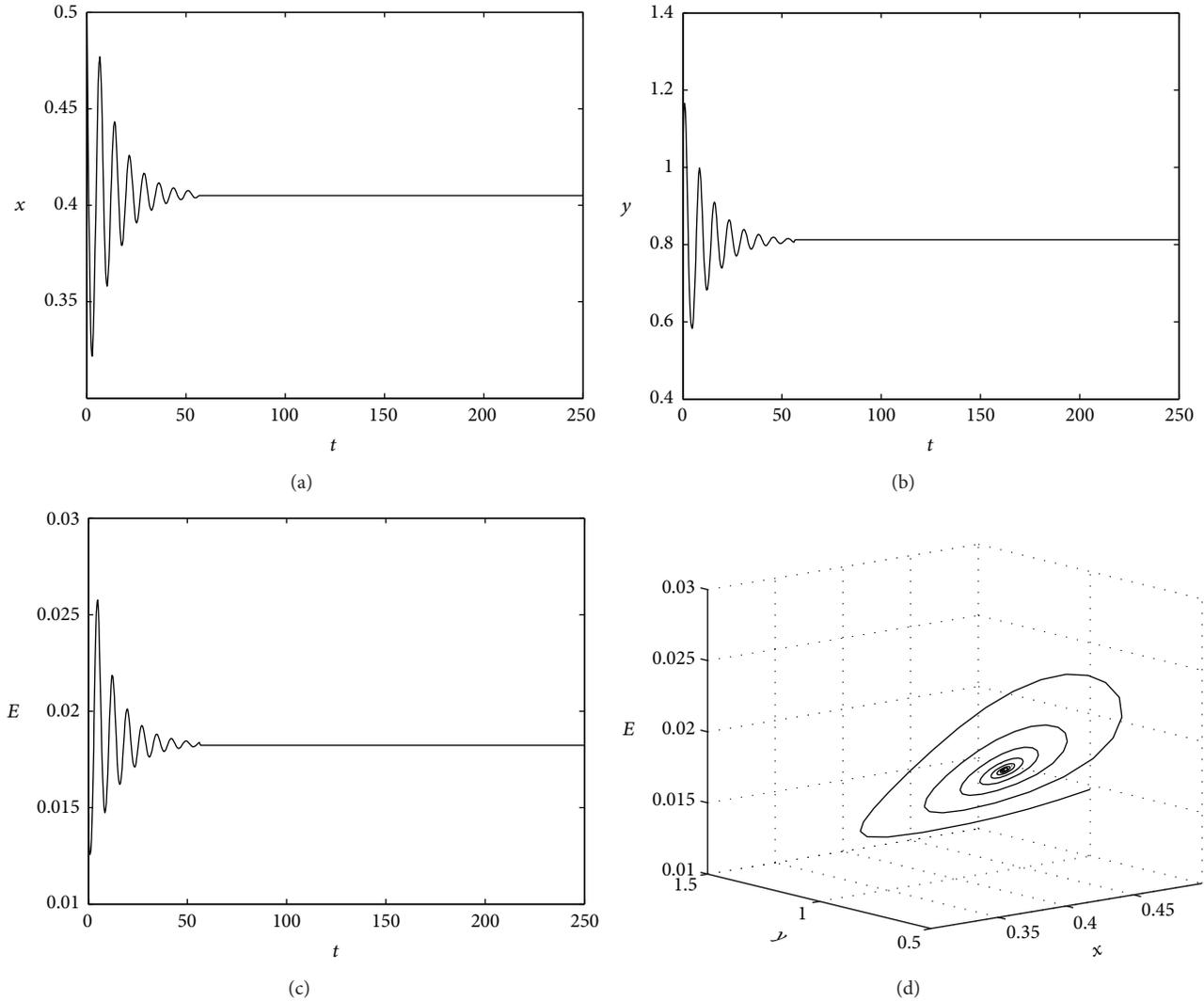
$$\tag{43}$$

FIGURE 1: Without any time delay, the positive equilibrium point P_0 is asymptotically stable.

Therefore, we can compute g_{21} by the parameters and delays. Then, the following results can be expressed as

$$c_1(0) = \frac{i}{2\omega'_{10}\tau'_{10}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau'_{10})\}},$$

$$\beta_2 = 2\,\mathrm{Re}\{c_1(0)\},$$

$$T_2 = -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\,\mathrm{Im}\{\lambda'(\tau'_{10})\}}{\omega'_{10}\tau'_{10}}.$$

(44)

These parameters give a description of the bifurcating periodic solutions in the center manifold of system (1) at critical values $\tau_1 = \tau'_{10}$, which can be expressed as follows.

(i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical).

(ii) β_2 gives the stability of periodic solution: if $\beta_2 < 0$ ($\beta_2 > 0$), the periodic solution is stable (unstable).

(iii) T_2 expresses the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solution increases (decreases).

4. Numerical Simulation

With the help of MATLAB, we present some numerical simulations to verify and extend our theoretical analysis proved in Sections 2 and 3. The parameters of model (1) are chosen as $A = 0.02$, $d_1 = 0.01$, $h_1 = 1.25$, $b = 7.5$, $d_2 = 2$, $p = 35$, $c = 1$, and $m = 0.5$.

According to the parameters given above, we can obtain the positive equilibrium point $P_0(0.4055, 0.8099, 0.0183)$ of model (1). It is easy to obtain that P_0 is asymptotically stable in the absence of delay (see Figure 1). Moreover, since $u_1u_4 + u_2u_3 = -0.7233 < 0$, there is a positive root $\omega_{20} = 0.8389$ for (13). Hence, when $\tau_1 = 0$, the critical value of the delay
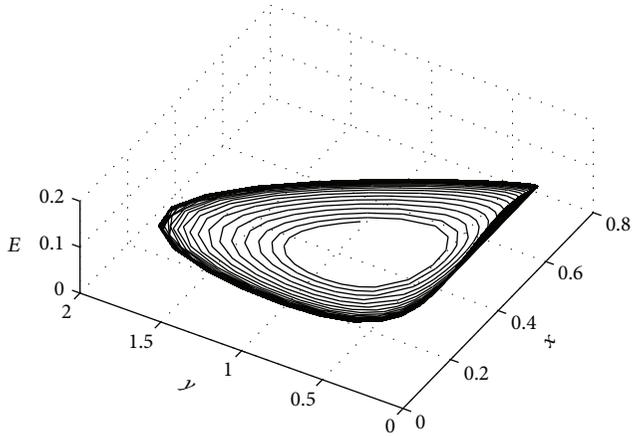
FIGURE 2: When $\tau_2 = 0.288 > \tau_{20}$, bifurcating periodic solutions from the equilibrium point P_0 occur.

FIGURE 3: The critical curve of stable domain with respect to τ_1 and τ_2.

corresponding to ω_{20} is $\tau_{20} = 0.2247$, which is shown in Figure 2. When $\tau_2 = 0$, by a similar computation, we can obtain that the critical delay for τ_1 is $\tau_{10} = 1.272$.

Based on the results of Cases 3 and 4 in Section 2, we draw a critical curve $\tau_2 = f(\tau_1)$ with respect to two parameters τ_1 and τ_2 (see Figure 3). The main algorithm for drawing the critical curve is searching the values $\tau'_{10}(\tau'_{20})$ for fixed positive values of $\tau_2 \in (0, \tau_{20})$ ($\tau_1 \in (0, \tau_{10})$). From Figure 3, it is clear that the domain surrounded by the critical curve $f(\tau_1)$ and the two axes, that is, the gray region, is stable for model (1).

In addition, according to the algorithm derived in Section 3, we can obtain the following values: $c_1(0) = -1.9243 - 3.0887i$, $\omega'_{10} = 0.7538$, $\tau'_{10} = 0.6546$, $\lambda'(\tau'_{10}) = 0.0879 + 0.1826i$, $\mu_2 = 21.8851$, $\beta_2 = -3.8487$, and $T_2 = -1.8402$. By the discussion in Section 3, we know that model (1) can undergo a supercritical Hopf bifurcation at the positive equilibrium P_0 and the bifurcating periodic solution occurs when τ_1 crosses τ'_{10} to the right with fixed $\tau^*_2 \in (0, \tau_{20})$ and the bifurcating periodic solution is stable. Figure 4 plots the bifurcating periodic solution of the model (1) at $\tau_1 = \tau'_{10} = 0.6546$ and $\tau^*_2 = 0.15$.

To explore the possibility of occurrence of chaos, bifurcation diagrams are plotted for the key parameter τ_1. Bifurcation diagram in Figure 5 of model (1) in ($\tau_1 - x$) plane is given for the fixed $\tau_2 = 0.15$. We can see that model (1) experiences the processes of periodic, periodic doubling cascade and chaos in the region $0.98 < \tau_1 < 1.018$. When $0.98 < \tau_1 < 0.996$, model (1) exhibits a stable limit cycle. When τ_1 is increasing, the limit cycle is unstable and there is a cascade of periodic doubling bifurcation (see Figure 6) leading to chaos.

5. Conclusions

In this paper, dynamical complexity of a multiple-delayed predator-prey model with Allee effect is analyzed by using differential-algebraic system theory. The prey-predator mo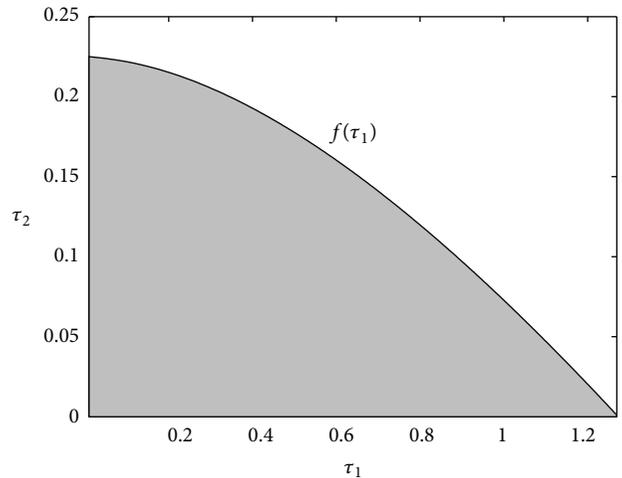del with single delay has been investigated by many researchers (see, e.g., [22–25]). However, to our best knowledge, there are few papers on the bifurcations of differential-algebraic population dynamics with two or multiple delays. On the other hand, it should be noted that almost the existing bioeconomic models (see [2–5, 14, 15]) only investigate the simplest case of a logistic prey growth function. Compared with these works, the introduction of Allee effect makes the work studied in this paper novel.

This paper provides a new and efficient method for the qualitative analysis of the Hopf bifurcation of differential-algebraic biological economic system with multiple delays. From the analysis of the proposed model, we have obtained some interesting and useful results. The main contribution of this paper is that the stability domain with respect to two delays is defined and plotted. By choosing the time delay τ_1 as a bifurcation parameter, the direction of Hopf bifurcation and stability of the bifurcating periodic orbit are discussed by using normal form and center manifold. Moreover, numerical simulations are carried out which substantiate the theoretical results. At the same time, bifurcation diagrams and attracting sets are plotted, which validate the existence of chaos due to periodic doubling bifurcation. However, those theoretical analyses, such as the existence of chaos and conditions of periodic doubling bifurcation, will be the topics of the future research.

Conflict of Interests

The authors do not have conflict of interests with any commercial identities.
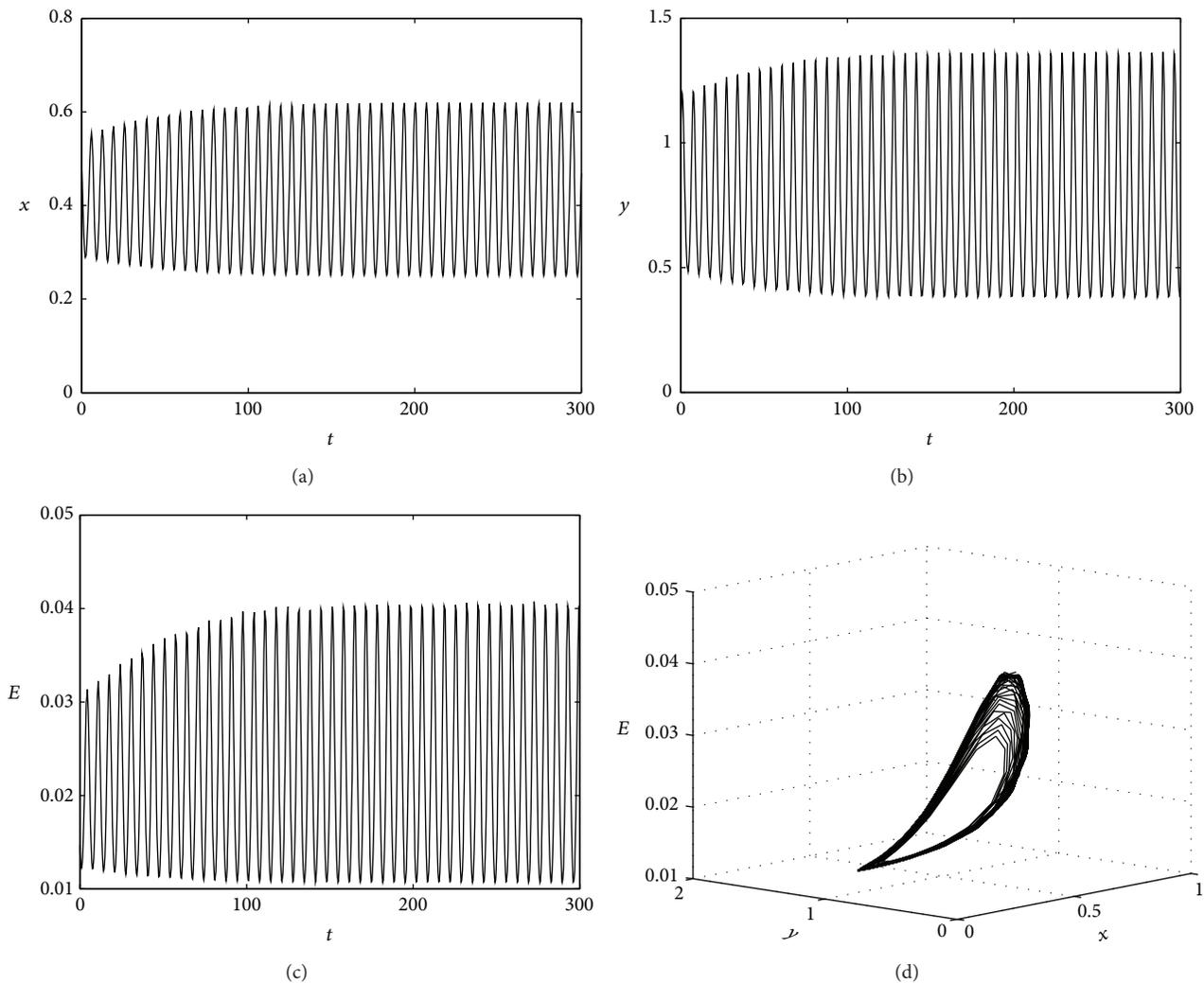
Acknowledgments

(a)

(b)

(c)

(d)

FIGURE 4: When at $\tau_1 = \tau_{10}' = 0.6546$ and $\tau_2^* = 0.15$, bifurcating periodic solutions of model (1) from P_0 occur.
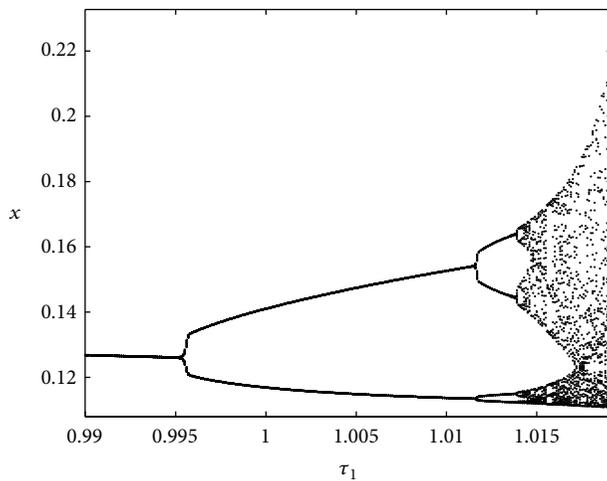

FIGURE 5: Bifurcation diagram of model (1) showing the effect of τ_1 on the dynamical behavior.
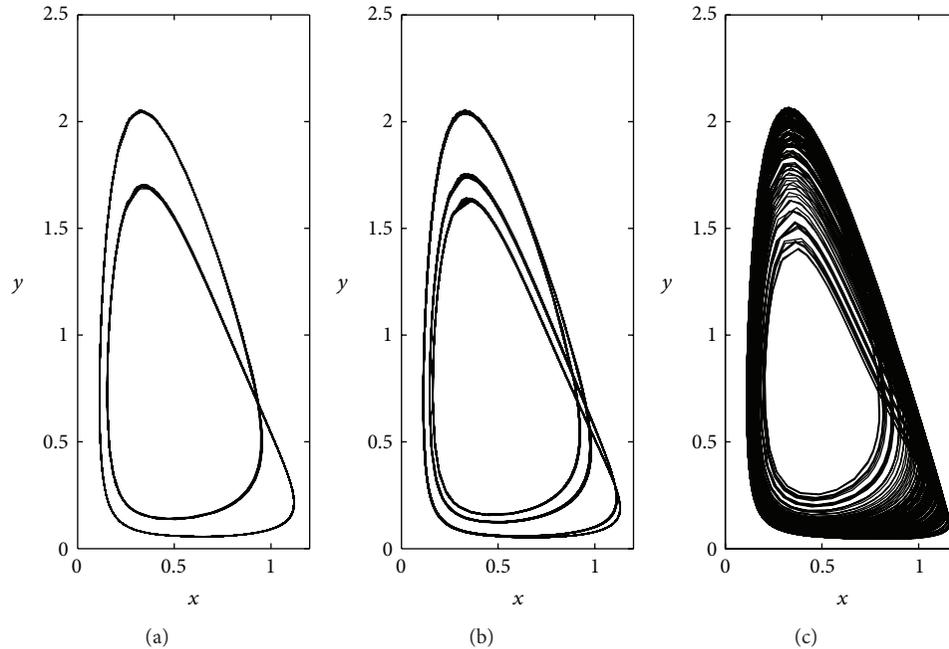
FIGURE 6: The projection of the attracting set of model (1) in the x-y plane for (a) $\tau_1 = 1.01$, (b) $\tau_1 = 1.0125$, and (c) $\tau_1 = 1.017$, respectively.

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