Research Article

Multiply Warped Products with a Semisymmetric Metric Connection

Yong Wang

School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China

Correspondence should be addressed to Yong Wang; wangy581@nenu.edu.cn

We study the Einstein multiply warped products with a semisymmetric metric connection and the multiply warped products with a semisymmetric metric connection with constant scalar curvature, and we apply our results to generalized Robertson-Walker space-times with a semisymmetric metric connection and generalized Kasner space-times with a semisymmetric metric connection and find some new examples of Einstein manifolds with a semisymmetric metric connection and manifolds with constant scalar curvature with a semisymmetric metric connection.

1. Introduction

The (singly) warped product $B\times_b F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \rightarrow (0, \infty)$ is the product manifold $B \times F$ with the metric tensor $g = g_B \oplus b^2 g_F$. Here, (B, g_B) is called the base manifold and (F, g_F) is called the fiber manifold and b is called the warping function. Generalized Robertson-Walker space-times and standard static space-times are two well-known warped product spaces. The concept of warped products was first introduced by Bishop and O'Neill (see [1]) to construct examples of Riemannian manifolds with negative curvature. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new examples with interesting curvature properties since then. In [2], Dobarro and Dozo had studied from the viewpoint of partial differential equations and variational methods the problem of showing when a Riemannian metric of constant scalar curvature can be produced on product manifolds by a warped product construction. In [3], Ehrlich et al. got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson-Walker space-times. In [4], explicit solutions were also obtained for the warping function to make the space-time as Einstein when the fiber is also Einstein.

One can generalize singly warped products to multiply warped products. Briefly, a multiply warped product (M, g)

is a product manifold of form $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \ldots, m\}, b_i : B \rightarrow (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. In particular, when $B = (c, d)$ with the negative definite metric $g_B = -dt^2$ and (F_i, g_{F_i}) is a Riemannian manifold, we call M the multiply generalized Robertson-Walker space-time. In [5], Dobarro and Ünal studied Ricci-flat and Einstein-Lorentzian multiply warped products and considered the case of having constant scalar curvature for multiply warped products and applied their results to generalized Kasner space-times.

Singly warped products have a natural generalization. A twisted product (M, g) is a product manifold of form $M = B\times_b F$, with a smooth function $b : B \times F \rightarrow (0, \infty)$, and the metric tensor $g = g_B \oplus b^2 g_F$. In [6], they showed that mixed Ricci-flat twisted products could be expressed as warped products. As a consequence, any Einstein twisted products are warped products. In this paper, we define the multiply twisted products as generalizations of multiply warped products and twisted products. A multiply twisted product (M, g) is a product manifold of form $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \ldots, m\}, b_i : B \times F_i \rightarrow (0, \infty)$ is smooth.

The definition of a semisymmetric metric connection was given by Hayden in [7]. In 1970, Yano [8] considered

a semisymmetric metric connection and studied some of its properties. He proved that a Riemannian manifold admitting the semisymmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. Motivated by the Yano result, in [9], Sular and Özgür studied warped product manifolds with a semisymmetric metric connection; they computed curvature of semisymmetric metric connection and considered Einstein warped product manifolds with a semisymmetric metric connection. In this paper, we considered multiply twisted products with a semisymmetric metric connection and computed the curvature of a semisymmetric metric connection. We showed that mixed Ricci-flat multiply twisted products with a semisymmetric metric connection can be expressed as multiply warped products which generalizes the result in [6]. We also studied the Einstein multiply warped products with a semisymmetric metric connection and multiply warped products with a semisymmetric metric connection with constant scalar curvature; we applied our results to generalized Robertson-Walker space-times with a semisymmetric metric connection and generalized Kasner space-times with a semisymmetric metric connection and we found some new examples of Einstein affine manifolds and affine manifolds with constant scalar curvature. We also classified generalized Einstein Robertson-Walker space-times with a semisymmetric metric connection and generalized Einstein Kasner space-times with a semisymmetric metric connection.

Semisymmetric metric connections have some physical applications. In [10, 11], they considered orthogonal connections with arbitrary torsion on compact Riemannian manifolds. For the induced Dirac operators, twisted Dirac operators, and Dirac operators of Chamseddine-Connes type, they computed the spectral action. In addition to the Einstein-Hilbert action and the bosonic part of the standard model Lagrangian, they found the Holst term from loop quantum gravity, a coupling of the Holst term to the scalar curvature, and a prediction for the value of the Barbero-Immirzi parameter. For connections whose torsion is not zero, they showed that the Holst action can be recovered from the heat asymptotics for the natural Dirac operator acting on spinor fields. For the physical consequences of the use of torsion connections in Lorentzian geometry, we refer to the classical review [12] and the more recent overview [13] and references therein.

This paper is arranged as follows. In Section 2, we compute curvature of multiply twisted products with a semisymmetric metric connection. In Section 3, we study the special multiply warped products with a semisymmetric metric connection. In Section 4, we study the generalized Robertson-Walker space-times with a semisymmetric metric connection. In Section 5, we consider the generalized Kasner space-times with a semisymmetric metric connection.

2. Preliminaries

Let M be a Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ on a Riemannian manifold M is

called a *semisymmetric connection* if the torsion tensor T of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \tag{1}$$

satisfies

$$T(X, Y) = \pi(Y) X - \pi(X) Y, \tag{2}$$

where π is a one form associated with the vector field P on M defined by $\pi(X) = g(X \cdot P)$. $\overline{\nabla}$ is called a *semisymmetric metric connection* if it satisfies $\overline{\nabla} g = 0$. If ∇ is the Levi-Civita connection of M, the semisymmetric metric connection $\overline{\nabla}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \pi(Y) X - g(X, Y) P, \tag{3}$$

(see [8]). Let R and $\overline{R}$ be the curvature tensors of ∇ and $\overline{\nabla}$, respectively. Then R and $\overline{R}$ are related by

$$\begin{aligned}
\overline{R}(X, Y) Z = {} & R(X, Y) Z + g(Z, \nabla_X P) Y - g(Z, \nabla_Y P) X \\
& + g(X, Z) \nabla_Y P - g(Y, Z) \nabla_X P \\
& + \pi(P) \left[g(X, Z) Y - g(Y, Z) X \right] \\
& + \left[g(Y, Z) \pi(X) - g(X, Z) \pi(Y) \right] P \\
& + \pi(Z) \left[\pi(Y) X - \pi(X) Y \right],
\end{aligned} \tag{4}$$

for any vector fields X, Y, Z on M [8]. By (3) and Proposition 2, we have a *multiply twisted product* (M, g) which is a product manifold of form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \ldots, m\}$, $b_i : B \times F_i \to (0, \infty)$ is smooth.

Here, (B, g_B) is called the base manifold and (F_i, g_{F_i}) is called the fiber manifold and b_i is called the twisted function. Obviously, twisted products and multiply warped products are the special cases of multiply twisted products.

Proposition 1 (compare with [5, Proposition 2.2]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product and let $X, Y \in \Gamma(TB)$ and $U \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$ and $P \in \Gamma(TB)$. Then*

(1) $\overline{\nabla}_X Y = \overline{\nabla}_X^B Y$;

(2) $\overline{\nabla}_X U = (X(b_i)/b_i) U$;

(3) $\overline{\nabla}_U X = [X(b_i)/b_i + \pi(X)] U$;

(4) $\overline{\nabla}_U W = 0$ *if $i \neq j$*;

(5) $\overline{\nabla}_U W = U(\ln b_i) W + W(\ln b_i) U - (g_{F_i}(U, W)/b_i) \mathrm{grad}_{F_i} b_i - b_i g_{F_i}(U, W) \mathrm{grad}_B b_i + \nabla_U^{F_i} W - g(U, W) P$ *if $i = j$,*

where $\overline{\nabla}^B$ denotes the semisymmetric metric connection on B and $\mathrm{grad}_B b_i, \mathrm{grad}_{F_i} b_i$ denote the gradient vector fields on B and F_i, respectively.

Proof. Similar to Proposition 2.2 in [5] and Proposition 1 in [6], we have

(1) $\nabla_X Y = \nabla_X^B Y$;

(2) $\nabla_X U = \nabla_U X = (X(b_i)/b_i)U$;

(3) $\nabla_U W = 0$ if $i \neq j$;

(4) $\nabla_U W = U(\ln b_i)W + W(\ln b_i)U - (g_{F_i}(U,W)/b_i)\mathrm{grad}_{F_i} b_i - b_i g_{F_i}(U,W)\mathrm{grad}_B b_i + \nabla_U^{F_i} W$ if $i = j$.

So by (3), we get Proposition 1. □

Similar to the proof of Proposition 1, we get the following.

Proposition 2 (compare with [5, Proposition 2.2]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and let* $X, Y \in \Gamma(TB)$ *and* $U \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$ *and* $P \in \Gamma(TF_k)$ *for a fixed* k. *Then*

(1) $\overline{\nabla}_X Y = \nabla_X^B Y - g(X,Y)P$;

(2) $\overline{\nabla}_X U = (X(b_i)/b_i)U + g(P,U)X$;

(3) $\overline{\nabla}_U X = (X(b_i)/b_i)U$;

(4) $\overline{\nabla}_U W = g(W,P)U$ if $i \neq j$;

(5) $\overline{\nabla}_U W = U(\ln b_i)W + W(\ln b_i)U - (g_{F_i}(U,W)/b_i)\mathrm{grad}_{F_i} b_i - b_i g_{F_i}(U,W)\mathrm{grad}_B b_i + \nabla_U^{F_i} W + \pi(W)U - g(U,W)P$ if $i = j$.

Remark 3. When for each $i \in \{1,\ldots,m\}, b_i : B \rightarrow (0,\infty)$ is smooth and $P = 0$, we get Proposition 2.2 in [5] by Propositions 1 and 2.

Define the curvature, Ricci curvature, and scalar curvature as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$
$$\mathrm{Ric}(X,Y) = \sum_k \varepsilon_k \langle R(X,E_k)Y, E_k \rangle, \qquad (5)$$
$$S = \sum_k \varepsilon_k \mathrm{Ric}(E_k, E_k),$$

where E_k is an orthonormal base of M with $\langle E_k, E_k \rangle = \varepsilon_k$. The Hessian of f is defined by $H^f(X,Y) = XYf - (\nabla_X Y)f$.

Proposition 4 (compare with [5, Proposition 2.4]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and let* $X, Y, Z \in \Gamma(TB)$ *and* $V \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$, $U \in \Gamma(TF_k)$, *and* $P \in \Gamma(TB)$. *Then*

(1) $\overline{R}(X,Y)Z = \overline{R}^B(X,Y)Z$;

(2) $\overline{R}(V,X)Y = -[(H_B^{b_i}(X,Y)/b_i) + (P(b_i)/b_i)g(X,Y) + \pi(P)g(X,Y) + g(Y,\nabla_X P) - \pi(X)\pi(Y)]V$;

(3) $\overline{R}(X,V)W = \overline{R}(V,W)X = \overline{R}(V,X)W = 0$ if $i \neq j$;

(4) $\overline{R}(X,Y)V = 0$;

(5) $\overline{R}(V,W)X = VX(\ln b_i)W - WX(\ln b_i)V$ if $i = j$;

(6) $\overline{R}(V,W)U = 0$ if $i = j \neq k$ or $i \neq j \neq k$;

(7) $\overline{R}(U,V)W = -g(V,W)(g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)/b_i b_k)U - g(V,W)((P(b_i)/b_i) + (P(b_k)/b_k))U - \pi(P)g(V,W)U$, if $i = j \neq k$;

(8) $\overline{R}(X,V)W = [WX(\ln b_i)]V - g(W,V) \cdot [(\nabla_X^B(\mathrm{grad}_B b_i)/b_i) + (\mathrm{grad}_{F_i}(X \ln b_i)/b_i^2) + (P(b_i)/b_i)X + \nabla_X P + \pi(P)X - \pi(X)P]$ if $i = j$;

(9) $\overline{R}(U,V)W = g(U,W)\mathrm{grad}_B(V(\ln b_i)) - g(V,W)\mathrm{grad}_B(U(\ln b_i)) + R^{F_i}(U,V)W - ((|\mathrm{grad}_B b_i|_B^2/b_i^2) + 2(P(b_i)/b_i) + \pi(P))(g(V,W)U - g(U,W)V)$ if $i = j = k$.

Proof. Similar to Proposition 2.4 in [5], we have

(1) $R(X,Y)Z = R^B(X,Y)Z$;

(2) $R(V,X)Y = -(H_B^{b_i}(X,Y)/b_i)V$;

(3) $R(X,V)W = R(V,W)X = R(V,X)W = 0$ if $i \neq j$;

(4) $R(X,Y)V = 0$;

(5) $R(V,W)X = VX(\ln b_i)W - WX(\ln b_i)V$ if $i = j$;

(6) $R(V,W)U = 0$ if $i = j \neq k$ or $i \neq j \neq k$;

(7) $R(U,V)W = -g(V,W)(g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)/b_i b_k)U$, if $i = j \neq k$;

(8) $R(X,V)W = -(g(V,W)/b_i)\nabla_X^B(\mathrm{grad}_B b_i) + [WX(\ln b_i)]V - g_{F_i}(W,V)\mathrm{grad}_{F_i}(X \ln b_i)$ if $i = j$;

(9) $R(V,W)U = g(V,U)\mathrm{grad}_B(W(\ln b_i)) - g(W,U)\mathrm{grad}_B(V(\ln b_i)) + R^{F_i}(V,W)U - (|\mathrm{grad}_B b_i|_B^2/b_i^2)(g(W,U)V - g(V,U)W)$ if $i = j = k$.

Then by (4), we get Proposition 4. □

Similarly we may get the following.

Proposition 5 (compare with [5, Proposition 2.4]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and let* $X, Y, Z \in \Gamma(TB)$ *and* $V \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$, $U \in \Gamma(TF_k)$, *and* $P \in \Gamma(TF_l)$ *for a fixed* l. *Then*

(1) $\overline{R}(X,Y)Z = R^B(X,Y)Z + [g(X,Z)(Yb_l/b_l) - g(Y,Z)(Xb_l/b_l)]P + \pi(P)[g(X,Z)Y - g(Y,Z)X]$;

(2) $\overline{R}(V,X)Y = -(H_B^{b_i}(X,Y)/b_i)V - \pi(P)g(X,Y)V$ if $i \neq l$;

(3) $\overline{R}(V,X)Y = -(H_B^{b_i}(X,Y)/b_i)V - \pi(V)(Y(b_i)/b_i)X - g(X,Y)\nabla_V P - g(X,Y)[\pi(P)V - \pi(V)P]$ if $i = l$;

(4) $\overline{R}(X,V)W = (X(b_l)/b_l)\pi(W)V$ if $i \neq j$;

(5) $\overline{R}(V,W)X = -\delta_i^l(\pi(V)/b_i)X(b_i)W + \delta_j^l(\pi(W)/b_j)X(b_j)V$ if $i \neq j$;

(6) $\overline{R}(X,Y)V = \pi(V)[(X(b_l)/b_l)Y - (Y(b_l)/b_l)X]$;

(7) $\overline{R}(V,W)X = VX(\ln b_i)W - WX(\ln b_i)V - \delta_i^l(X(b_i)/b_i)[\pi(V)W - \pi(W)V]$ if $i = j$;

(8) $\overline{R}(V,W)U = 0$ if $i = j \neq k$ or $i \neq j \neq k$;

(9) $\overline{R}(U,V)W = -g(V,W)(g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)/b_i b_k)U - g(W,\nabla_V P)U - g(V,W)\nabla_U P - \pi(P)g(V,W)U + g(V,W)\pi(U)P + \pi(W)[\pi(V)U - \pi(U)V]$, if $i = j \neq k$;

(10) $\overline{R}(X,V)W = [WX(\ln b_i)]V - g(W,V)(\nabla_X^B(\mathrm{grad}_B b_i)/b_i) - \mathrm{grad}_{F_i}(X \ln b_i)g_{F_i}(W,V) + (X(b_l)/b_l)\pi(W)V - g(W,\nabla_V P)X - g(V,W)(X(b_l)/b_l)P - g(V,W)\pi(P)X + \pi(V)\pi(W)X$ if $i = j$;

(11) $\overline{R}(U,V)W = g(U,W)\mathrm{grad}_B(V(\ln b_i)) - g(V,W)\mathrm{grad}_B(U(\ln b_i)) + R^{F_i}(U,V)W - (|\mathrm{grad}_B b_i|_B^2/b_i^2)(g(V,W)U - g(U,W)V) + \pi(P)[g(U,W)V - g(V,W)U]$ if $i = j = k \neq l$;

(12) $\overline{R}(U,V)W = g(U,W)\mathrm{grad}_B(V(\ln b_i)) - g(V,W)\mathrm{grad}_B(U(\ln b_i)) + R^{F_i}(U,V)W - (|\mathrm{grad}_B b_i|_B^2/b_i^2)(g(V,W)U - g(U,W)V) + g(W,\nabla_U P)V - g(W,\nabla_V P)U + g(U,W)\nabla_V P - g(V,W)\nabla_U P + \pi(P)[g(U,W)V - g(V,W)U] + [g(V,W)\pi(U) - g(U,W)\pi(V)]P + \pi(W)[\pi(V)U - \pi(U)V]$ if $i = j = k = l$, where δ_i^l denotes the Kronecker symbol.

Remark 6. When for each $i \in \{1,\ldots,m\}$, $b_i : B \to (0,\infty)$ is smooth and $P = 0$, we get Proposition 2.4 in [5] by Propositions 4 and 5.

By Propositions 4 and 5 and the definition of the Ricci curvature tensor, we have the following.

Proposition 7 (compare with [5, Proposition 2.5]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and let* $X, Y, Z \in \Gamma(TB)$ *and* $V \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$, *and* $P \in \Gamma(TB)$. *Then*

(1) $\overline{\mathrm{Ric}}(X,Y) = \overline{\mathrm{Ric}}^B(X,Y) + \sum_{i=1}^m l_i[(H_B^{b_i}(X,Y)/b_i) + (P(b_i)/b_i)g(X,Y) + \pi(P)g(X,Y) + g(Y,\nabla_X P) - \pi(X)\pi(Y)]$;

(2) $\overline{\mathrm{Ric}}(X,V) = \overline{\mathrm{Ric}}(V,X) = (l_i - 1)[VX(\ln b_i)]$;

(3) $\overline{\mathrm{Ric}}(V,W) = 0$ if $i \neq j$;

(4) $\overline{\mathrm{Ric}}(V,W) = \mathrm{Ric}^{F_i}(V,W) + [(\Delta_B b_i/b_i) + (l_i - 1)(|\mathrm{grad}_B b_i|_B^2/b_i^2) + \sum_{j\neq i} l_j(g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_j)/b_i b_j) + (\overline{n} - 2)\pi(P) + \sum_{k=1}^n \varepsilon_k \langle \nabla_{E_k} P, E_k \rangle + \sum_{j\neq i} l_j(Pb_j/b_j) + (\overline{n} + l_i - 2)(Pb_i/b_i)]g(V,W)$ if $i = j$,

where $E_k, 1 \leq k \leq n$ *is an orthonormal base of* B *with* $\varepsilon_k = g(E_k, E_k)$ *and* $\dim B = n$, $\dim M = \overline{n}$.

By (2) and (3) in Proposition 7, similar to the proof of Theorem 1 in [6], we have the following.

Corollary 8. *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and* $\dim F_i > 1$ *and* $P \in \Gamma(TB)$; *then* $(M,\overline{\nabla})$ *is mixed Ricci-flat if and only if* M *can be expressed as a multiply warped product. In particular, if* $(M,\overline{\nabla})$ *is Einstein, then* M *can be expressed as a multiply warped product.*

Similar to Proposition 7, we have the following.

Proposition 9 (compare with [5, Proposition 2.5]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and let* $X, Y, Z \in \Gamma(TB)$ *and* $V \in \Gamma(TF_i)$, $W \in \Gamma(TF_j)$, *and* $P \in \Gamma(TF_r)$ *for a fixed* r. *Then*

(1) $\overline{\mathrm{Ric}}(X,Y) = \mathrm{Ric}^B(X,Y) + \sum_{i=1}^m l_i(H_B^{b_i}(X,Y)/b_i) + g(X,Y)\pi(P)(\overline{n} - 2) + g(X,Y)\sum_{q=1}^{l_r} \varepsilon_q g(\nabla_{E_q} P, E_q)$, *where* E_q *for* $1 \leq q \leq l_r$ *is an orthonormal basis of* TF_r;

(2) $\overline{\mathrm{Ric}}(X,V) = (l_i - 1)[VX(\ln b_i)] + (\overline{n} - 2)(X(b_r)/b_r)\pi(V)$;

(3) $\overline{\mathrm{Ric}}(V,X) = (l_i - 1)[VX(\ln b_i)] + (2 - \overline{n})(X(b_r)/b_r)\pi(V)$;

(4) $\overline{\mathrm{Ric}}(V,W) = 0$ if $i \neq j$;

(5) $\overline{\mathrm{Ric}}(V,W) = \mathrm{Ric}^{F_i}(V,W) + g(V,W)[(\Delta_B b_i/b_i) + (l_i - 1)(|\mathrm{grad}_B b_i|_B^2/b_i^2) + \sum_{k,k\neq i} l_k(g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)/b_i b_k) + (\overline{n} - 2)\pi(P)] + (\overline{n} - 2)g(W,\nabla_V P) + (2 - \overline{n})\pi(V)\pi(W) + g(V,W)\mathrm{div}_{F_r} P$ if $i = j$.

Remark 10. When for each $i \in \{1,\ldots,m\}$, $b_i : B \to (0,\infty)$ is smooth and $P = 0$, we get Proposition 2.5 in [5] by Propositions 7 and 9.

Corollary 11. *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and* $\dim F_i > 1$ *and* $P \in \Gamma(TF_r)$; *then* $(M,\overline{\nabla})$ *is mixed Ricci-flat if and only if* M *can be expressed as a multiply warped product and* b_r *is only dependent on* F_r. *In particular, if* $(M,\overline{\nabla})$ *is Einstein, then* M *can be expressed as a multiply warped product.*

Proof. By (2) and (3) in Proposition 9, we have that $(M,\overline{\nabla})$ is mixed Ricci-flat if and only if $[VX(\ln b_i)] = 0, (X(b_r)/b_r)\pi(V) = 0$. By $[VX(\ln b_i)] = 0$, similar to the proof of Corollary 8, we get that M can be expressed as a multiply warped product. When $i \neq r$, $\pi(V) = 0$. When $i = r$, by $(X(b_r)/b_r)\pi(V) = 0$, then b_r depends only on M_r. □

By Proposition 7 and the definition of the scalar curvature, we have the following.

Proposition 12 (compare with [5, Proposition 2.6]). *Let* $M = B\times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product and* $P \in \Gamma(TB)$; *then the scalar curvature* $\overline{S}$ *has the following expression:*

$$\overline{S} = \overline{S}^B + 2\sum_{i=1}^m \frac{l_i}{b_i}\Delta_B b_i + \sum_{i=1}^m \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^m l_i(l_i - 1)\frac{|\mathrm{grad}_B b_i|_B^2}{b_i^2}$$
$$+ \sum_{i=1}^m \sum_{j\neq i} l_i l_j \frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_j)}{b_i b_j}$$
$$+ \sum_{i=1}^m l_i(n + \overline{n} + l_i - 2)\frac{P(b_i)}{b_i}$$
$$+ \sum_{i=1}^m \sum_{j\neq i} l_i l_j \frac{P(b_j)}{b_j} + \sum_{i=1}^m l_i(n + \overline{n} - 3)\pi(P)$$
$$+ 2\sum_{i=1}^m l_i \mathrm{div}_B P.$$

$$(6)$$

By Proposition 9 and the definition of the scalar curvature, we have the following.

Proposition 13 (compare with [5, Proposition 2.6]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product and $P \in \Gamma(TF_r)$; then the scalar curvature $\overline{S}$ has the following expression:*

$$
\overline{S} = S^B + 2 \sum_{i=1}^{m} \frac{l_i}{b_i} \Delta_B b_i + \sum_{i=1}^{m} \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^{m} l_i (l_i - 1) \frac{|\mathrm{grad}_B b_i|_B^2}{b_i^2}
$$
$$
+ \sum_{i=1}^{m} \sum_{j \neq i} l_i l_j \frac{g_B \left(\mathrm{grad}_B b_i, \mathrm{grad}_B b_j \right)}{b_i b_j} + \pi(P)(\overline{n} - 1)(\overline{n} - 2)
$$
$$
+ 2(\overline{n} - 1) \mathrm{div}_{F_r} P.
$$

(7)

Remark 14. When for each $i \in \{1, \ldots, m\}$, $b_i : B \to (0, \infty)$ is smooth and $P = 0$, we get Proposition 2.6 in [5] by Propositions 12 and 13.

3. Special Multiply Warped Product with a Semisymmetric Connection

Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and I is an open interval in $\mathbb{R}$ and $b_i \in C^{\infty}(I)$.

Theorem 15. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and $P = (\partial/\partial t)$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied for any $i \in \{1, \ldots, m\}$:*

(1) *(F_i, ∇^{F_i}) is Einstein with the Einstein constant λ_i, $i \in \{1, \ldots, m\}$;*

(2) *$\sum_{i=1}^{m} l_i (b_i'/b_i - b_i''/b_i) = \lambda$;*

(3) *$\lambda_i - b_i b_i'' - (l_i - 1) b_i'^2 + (b_i^2 - b_i b_i') \sum_{j \neq i} l_j (b_j'/b_j) + (2 - \overline{n}) b_i^2 + (\overline{n} + l_i - 2) b_i b_i' = \lambda b_i^2$.*

Proof. By Proposition 7, we have

$$
\overline{\mathrm{Ric}} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = -\sum_{i=1}^{m} l_i \left(\frac{b_i'}{b_i} - \frac{b_i''}{b_i} \right);
$$
$$
\overline{\mathrm{Ric}} \left(\frac{\partial}{\partial t}, V \right) = \overline{\mathrm{Ric}} \left(V, \frac{\partial}{\partial t} \right) = 0;
$$
$$
\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}^{F_i}(V, W) + g_{F_i}(V, W)
$$
$$
\times \left[-b_i b_i'' - (l_i - 1) b_i'^2 + \left(b_i^2 - b_i b_i' \right) \sum_{j \neq i} l_j \frac{b_j'}{b_j} \right.
$$
$$
\left. + (2 - \overline{n}) b_i^2 + (\overline{n} + l_i - 2) b_i b_i' \right].
$$

(8)

By (8) and the Einstein condition, we get Theorem 15. $\square$

Theorem 16. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and $P \in \Gamma(TF_r)$ with $g_{F_r}(P, P) = 1$ and $\overline{n} > 2$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied for any $i \in \{1, \ldots, m\}$:*

(1) *(F_i, ∇^{F_i}) $(i \neq r)$ is Einstein with the Einstein constant λ_i, $i \in \{1, \ldots, m\}$;*

(2) *b_r is a constant and $\sum_{i=1}^{m} l_i(b_i''/b_i) = \mu_0$; $\mathrm{div}_{F_r} P = \mu_1$, $\mu_0 - \mu_1 + \lambda = (2 - \overline{n}) b_r^2$, where μ_0, μ_1 are constants;*

(3) *$\mathrm{Ric}^{F_r}(V, W) + \overline{\lambda} g_{F_r}(V, W) = (\overline{n} - 2)[\pi(V)\pi(W) - g(W, \nabla_V P)]$, for $V, W \in \Gamma(TF_r)$;*

(4) *$\lambda_i - b_i b_i'' + (\overline{n} - 2) b_i^2 b_r^2 - b_i b_i' \sum_{j \neq i} l_j (b_j'/b_j) - (l_i - 1) b_i'^2 = (\lambda - \mu_1) b_i^2$.*

Proof. By Proposition 9(2) and $g_{F_r}(P, P) = 1$, we have that b_r is a constant. By Proposition 9, then

$$
\overline{\mathrm{Ric}} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = \sum_{i=1}^{m} l_i \frac{b_i''}{b_i} + (2 - \overline{n}) b_r^2 - \mathrm{div}_{F_r} P = -\lambda. \quad (9)
$$

By variables separation, we have

$$
\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = \mu_0,
$$
$$
\mathrm{div}_{F_r} P = \mu_1, \qquad \mu_0 - \mu_1 + \lambda = (2 - \overline{n}) b_r^2,
$$
$$
\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}^{F_i}(V, W) + b_i^2 g_{F_i}(V, W)
$$
$$
\times \left[-\frac{b_i''}{b_i} + (l_i - 1) \frac{-b_i'^2}{b_i^2} + \sum_{j \neq i} l_j \frac{-b_i' b_j'}{b_i b_j} \right.
$$
$$
\left. + (\overline{n} - 2) \pi(P) \right] + (\overline{n} - 2) g(W, \nabla_V P)
$$
$$
+ (2 - \overline{n}) \pi(V) \pi(W) + g(V, W) \mathrm{div}_{F_r} P.
$$

(10)

When $i \neq r$, then $\nabla_V P = \pi(V) = 0$, so

$$
\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}^{F_i}(V, W) + b_i^2 g_{F_i}(V, W)
$$
$$
\times \left[-\frac{b_i''}{b_i} + (l_i - 1) \frac{-b_i'^2}{b_i^2} + \sum_{j \neq i} l_j \frac{-b_i' b_j'}{b_i b_j} \right.
$$
$$
\left. + (\overline{n} - 2) b_r^2 \right] + \mu_1 b_i^2 g_{F_i}(V, W)
$$
$$
= \lambda b_i^2 g_{F_i}(V, W).
$$

(11)

By variables separation, we have that (F_i, ∇^{F_i}) $(i \neq r)$ is Einstein with the Einstein constant λ_i and

$$\lambda_i - b_i b_i'' + (\bar{n} - 2) b_i^2 b_r^2 - b_i b_i' \sum_{j \neq i} l_j \frac{b_j'}{b_j} - (l_i - 1) b_i'^2 \tag{12}$$
$$= (\lambda - \mu_1) b_i^2.$$

When $i = r$ and b_r is a constant, then

$$\mathrm{Ric}^{F_i}(V, W) + b_r^2 \left[(\bar{n} - 2) b_r^2 + \mu_1 - \lambda \right] g_{F_i}(V, W)$$
$$= (\bar{n} - 2) \left[\pi(V) \pi(W) - g(W, \nabla_V P) \right]. \tag{13}$$

So we prove the above theorem. □

Remark 17. Comparing with Theorem 3.3 in [5], in Theorems 15 and 16, the unit vector field P emerges. For Theorem 3.3 in [5], P equals zero. So Theorem 3.3 in [5] is not the special case of Theorems 15 and 16. Then equations in our theorems are different from the equations in theorems in [5].

When $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ is a multiply warped product and $P = \partial/\partial t$, by Proposition 12, we have

$$\bar{S} = -2 \sum_{i=1}^m l_i \frac{b_i''}{b_i} + \sum_{i=1}^m \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^m l_i (l_i - 1) \frac{-b_i'^2}{b_i^2}$$
$$+ \sum_{i=1}^m \sum_{j \neq i} l_i l_j \frac{-b_i b_j}{b_i b_j} + \sum_{i=1}^m l_i (\bar{n} + l_i - 1) \frac{b_i'}{b_i} \tag{14}$$
$$+ \sum_{i=1}^m \sum_{j \neq i} l_i l_j \frac{b_j'}{b_j} - \sum_{i=1}^m l_i (\bar{n} - 2).$$

The following result just follows from the method of separation of variables and the fact that each S^{F_i} is function defined on F_i.

Proposition 18. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product and $P = \partial/\partial t$. If $(M, \bar{\nabla})$ has constant scalar curvature $\bar{S}$, then each (F_i, ∇^{F_i}) has constant scalar curvature S^{F_i}.*

When $P \in \Gamma(TF_r)$, by Proposition 13, we have

$$\bar{S} = -2 \sum_{i=1}^m l_i \frac{b_i''}{b_i} + \sum_{i=1}^m \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^m l_i (l_i - 1) \frac{-b_i'^2}{b_i^2}$$
$$+ \sum_{i=1}^m \sum_{j \neq i} l_i l_j \frac{-b_i' b_j'}{b_i b_j} + \pi(P)(\bar{n} - 1)(\bar{n} - 2) \tag{15}$$
$$+ 2(\bar{n} - 1) \mathrm{div}_{F_r} P.$$

Similarly we have the following.

Proposition 19 (compare with [5, Proposition 3.5]). *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product and $P \in \Gamma(TF_r)$. If $(M, \bar{\nabla})$ has constant scalar curvature $\bar{S}$, then*

(1) *each (F_i, ∇^{F_i}) $(i \neq r)$ has constant scalar curvature S^{F_i};*

(2) *moreover, if $g_{F_r}(P, P)$, $\mathrm{div}_{F_r} P$ are also constants, then S^{F_r} is a constant.*

Remark 20. When $P = 0$ in Proposition 19, we get Proposition 3.5 in [5].

4. Generalized Robertson-Walker Space-Times with a Semisymmetric Metric Connection

In this section, we study $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$. As a corollary of Theorem 15, we obtain the following.

Corollary 21. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$. Then $(M, \bar{\nabla})$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied:*

(1) *(F, ∇^F) is Einstein with the Einstein constant λ_F;*

(2) *$l((f'/f) - (f''/f)) = \lambda$;*

(3) *$\lambda_F - ff'' + (1 - l) f'^2 + (1 - l - \lambda) f^2 + (2l - 1) f' f = 0$.*

Remark 22. In Theorem 5.1 in [9], they got the Einstein condition of $M = I \times F$ with a semisymmetric metric connection, but they did not consider conditions (2) and (3).

Corollary 23. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$ and $\dim F = 1$. Then $(M, \bar{\nabla})$ is Einstein with the Einstein constant λ if and only if $f'' = f' - \lambda f$.*

By Corollary 21 (2) and (3), we get the following.

Corollary 24. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$ and $\dim F > 1$. Then $(M, \bar{\nabla})$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied:*

(1) *(F, ∇^F) is Einstein with the Einstein constant λ_F;*

(2) *$f'' = f' - (\lambda/l) f$;*

(3) *$(\lambda_F/(1 - l)) + f'^2 + (1 + (\lambda/l)) f^2 - 2ff' = 0$.*

By Corollary 23 and elementary methods for ordinary differential equations, we get the following.

Theorem 25. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$ and $\dim F = 1$. Then $(M, \bar{\nabla})$ is Einstein with the Einstein constant λ if and only if*

(1) *$\lambda < 1/4$, $f(t) = c_1 e^{((1+\sqrt{1-4\lambda})/2)t} + c_2 e^{((1-\sqrt{1-4\lambda})/2)t}$,*

(2) *$\lambda = 1/4$, $f(t) = c_1 e^{(1/2)t} + c_2 t e^{(1/2)t}$,*

(3) *$\lambda > 1/4$, $f(t) = c_1 e^{(1/2)t} \cos((\sqrt{4\lambda - 1}/2)t) + c_2 e^{(1/2)t} \sin((\sqrt{4\lambda - 1}/2)t)$.*

Let $\lambda/l = d_0$, $\lambda_F/(1-l) = \overline{d_0}$, $(1+\sqrt{1-4d_0})/2 = a_0$, $(1-\sqrt{1-4d_0})/2 = b_0$; then, $a_0 + b_0 = 1$, $d_0 = a_0 b_0$. When $\dim F > 1$, by Corollary 24(2), we have the following three cases.

Case (i) ($d_0 < 1/4$). We have $f = c_1 e^{a_0 t} + c_2 e^{b_0 t}$. By Corollary 24(3), then

$$\overline{d_0} + c_1^2 \left(a_0^2 + 1 + a_0 b_0 - 2a_0\right) e^{2a_0 t}$$
$$+ c_2^2 \left(b_0^2 + 1 + a_0 b_0 - 2b_0\right) e^{2b_0 t} \tag{16}$$
$$+ 2c_1 c_2 \left(2a_0 b_0 + 1 - a_0 - b_0\right) e^{(a_0 + b_0)t} = 0.$$

When $b_0 = 0$, we get $d_0 = 0$, $a_0 = 1$, $\lambda = 0$. By (16), $\overline{d_0} + c_2^2 = 0$, so $\lambda_F = (l-1)c_2^2$. In this case $f = c_1 e^t + c_2$. When $b \neq 0$, then $e^{2a_0 t}$, $e^{2b_0 t}$, and $e^{(a_0 + b_0)t}$ are linearly independent, so $c_2^2(b_0^2 + 1 + a_0 b_0 - 2b_0) = c_2^2(1 - b_0) = 0$ and $c_2 = 0$. Then $c_1 \neq 0$, by $c_1^2(a_0^2 + 1 + a_0 b_0 - 2a_0) = c_1^2(1 - a_0) = 0$, so $a_0 = 1$; then $d_0 = \lambda = 0$. Thus $f = c_1 e^t$.

Case (ii) ($d_0 = 1/4$). One has $f = c_1 e^{(1/2)t} + c_2 t e^{(1/2)t}$. By Corollary 24(3), then

$$\overline{d_0} + e^t \left[\left(\frac{1}{2}c_1 + c_2 + \frac{1}{2}c_2 t\right)^2 + \frac{5}{4}(c_1 + c_2 t)^2 \right.$$
$$\left. - 2\left(\frac{1}{2}c_1 + c_2 + \frac{1}{2}c_2 t\right)(c_1 + c_2 t) \right] = 0. \tag{17}$$

The coefficient of $t^2 e^t$ is $(1/2)c_2^2$, so $c_2 = 0$. The coefficient of e^t is $((1/2)c_1 + c_2)^2 + (5/4)(c_1)^2 - 2c_1((1/2)c_1 + c_2)$, so $c_1 = 0$; in this case we have no solutions.

Case (iii) ($d_0 > 1/4$). One has $f(t) = c_1 e^{(1/2)t} \cos(h_0 t) + c_2 e^{(1/2)t} \sin(h_0 t)$, where $h_0 = \sqrt{4d_0 - 1}/2$. By Corollary 24(3), then

$$\overline{d_0} + e^t \left\{ \left[\left(\frac{c_1}{2} + c_2 h_0\right)\cos(h_0 t) + \left(\frac{c_2}{2} - c_1 h_0\right)\sin(h_0 t)\right]^2 \right.$$
$$+ (1 + d_0)(c_1 \cos(h_0 t) + c_2 \sin(h_0 t))^2$$
$$- 2(c_1 \cos(h_0 t) + c_2 \sin(h_0 t))$$
$$\left. \times \left[\left(\frac{c_1}{2} + c_2 h_0\right)\cos(h_0 t) + \left(\frac{c_2}{2} - c_1 h_0\right)\sin(h_0 t)\right] \right\}$$
$$= 0. \tag{18}$$

Considering the coefficients of $\cos^2(h_0 t)e^t$ and $\sin^2(h_0 t)e^t$, we get

$$\left(\frac{1}{4} + d_0\right)c_1^2 + c_2^2 h_0^2 - c_1 c_2 h_0 = 0;$$
$$\left(\frac{1}{4} + d_0\right)c_2^2 + c_1^2 h_0^2 + c_1 c_2 h_0 = 0. \tag{19}$$

Adding the above two equalities, then $1/4 + d_0 + h_0^2 = 0$ and $d_0 = 0$. There is a contradiction with $d_0 > 1/4$ and in this case we have no solutions. So we obtain the following theorem.

Theorem 26. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$ and $\dim F > 1$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if $\lambda = 0$ and $f = c_1 e^t + c_2$ and (F, ∇^F) is Einstein with the Einstein constant $(l-1)c_2^2$.*

By (14) and (15), we have the following.

Corollary 27. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if (F, ∇^F) has constant scalar curvature S^F and*

$$\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{f'^2}{f^2} + 2l^2\frac{f'}{f} + (1-l)l. \tag{20}$$

Corollary 28. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P \in \Gamma(TF)$ and $g_F(P, P) = c_0$, $\mathrm{div}_{F_r} P = c_0'$, where c_0 and c_0' are constants. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if (F, ∇^F) has constant scalar curvature S^F and*

$$\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{f'^2}{f^2} + c_0(l-1)lf^2 + 2c_0'l. \tag{21}$$

In (20), we make the change of variable $f(t) = \sqrt{v(t)}$ and have the following equation:

$$v''(t) + \frac{l-3}{4}\frac{v'(t)^2}{v(t)} - lv'(t) + \left(l-1+\frac{\overline{S}}{l}\right)v(t) - \frac{S^F}{l} = 0. \tag{22}$$

Theorem 29. *Let $M = I \times F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \partial/\partial t$ and $\dim F = l = 3$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if (F, ∇^F) has constant scalar curvature S^F*

(1) *$\overline{S} < 3/4$ and $\overline{S} \neq -6$, $v(t) = c_1 e^{((3+\sqrt{1-(4/3)\overline{S}})/2)t} + c_2 e^{((3-\sqrt{1-(4/3)\overline{S}})/2)t} + (S^F/(6+\overline{S}))$;*

(2) *$\overline{S} = 3/4$, $v(t) = c_1 e^{(3/2)t} + c_2 t e^{(3/2)t} + (S^F/(6+\overline{S}))$;*

(3) *$\overline{S} > 3/4$, $v(t) = c_1 e^{(3/2)t} \cos((\sqrt{(4/3)\overline{S}-1}/2)t) + c_2 e^{(3/2)t} \sin((\sqrt{(4/3)\overline{S}-1}/2)t) + (S^F/(6+\overline{S}))$;*

(4) *$\overline{S} = -6$, $v(t) = c_1 - (S^F/9)t + c_2 e^{3t}$.*

Proof. If $l = 3$, then we have a simple differential equation as follows:

$$v''(t) - 3v'(t) + \left(2 + \frac{\overline{S}}{3}\right)v(t) - \frac{S^F}{3} = 0. \tag{23}$$

If $\overline{S} \neq -6$, putting $h(t) = (2 + \overline{S}/3)v(t) - S^F/3$, it follows that $h''(t) - 3h'(t) + (2 + \overline{S}/3)h(t) = 0$. The above solutions (1)–(3) follow directly from elementary methods for ordinary differential equations. When $\overline{S} = -6$, then $v''(t) - 3v'(t) - S^F/3 = 0$, and we get solution (4). □

Theorem 30. *Let* $M = I \times F$ *with the metric tensor* $-dt^2 + f(t)^2 g_F$ *and* $P = \partial/\partial t$ *and* $\dim F = l \neq 3$ *and* $S^F = 0$. *If* $(M, \overline{\nabla})$ *has constant scalar curvature* $\overline{S}$ *if and only if*

(1) $\overline{S} < l/(l + 1)$, $v(t) = (c_1 e^{((l + \sqrt{1 - ((l+1)/l)\overline{S}})/2)t} + c_2 e^{((l - \sqrt{1 - ((l+1)/l)\overline{S}})/2)t})^{4/(l+1)}$;

(2) $\overline{S} = l/(l + 1)$, $v(t) = (c_1 e^{(l/2)t} + c_2 t e^{(l/2)t})^{4/(l+1)}$;

(3) $\overline{S} > l/(l + 1)$, $v(t) = (c_1 e^{(l/2)t} \cos((\sqrt{((l+1)/l)\overline{S} - 1}/2)t) + c_2 e^{(l/2)t} \sin((\sqrt{((l+1)/l)\overline{S} - 1}/2)t))^{4/(l+1)}$.

Proof. In this case, (22) is changed into the simpler form

$$\frac{v''(t)}{v(t)} + \frac{l - 3}{4} \frac{v'(t)^2}{v(t)^2} - l \frac{v'(t)}{v(t)} + \left(l - 1 + \frac{\overline{S}}{l}\right) = 0. \quad (24)$$

Putting $v(t) = w(t)^{4/(l+1)}$, then $w(t)$ satisfies the equation $w'' - lw' + ((l+1)/4)(l-1+\overline{S}/l)w = 0$; by the elementary methods for ordinary differential equations we prove the above theorem. $\qquad \square$

When $\dim F = l \neq 3$ and $S^F \neq 0$, putting $v(t) = w(t)^{4/(l+1)}$, then $w(t)$ satisfies the following equation:

$$w'' - lw' + \frac{(l+1)}{4}\left(l - 1 + \frac{\overline{S}}{l}\right)w - \frac{(l+1)}{4}\frac{S^F}{l}w^{1-4/(l+1)} = 0. \quad (25)$$

5. Generalized Kasner Space-Times with a Semisymmetric Metric Connection

In this section, we consider the scalar and Ricci curvature of generalized Kasner space-times with a semisymmetric metric connection. We recall the definition of generalized Kasner space-times [5].

Definition 31. A generalized Kasner space-time (M, g) is a Lorentzian multiply warped product of the form $M = I \times_{\phi^{p_1}} F_1 \times \cdots \times_{\phi^{p_m}} F_m$ with the metric $g = -dt^2 \oplus \phi^{2p_1} g_{F_1} \oplus \cdots \oplus \phi^{2p_m} g_{F_m}$, where $\phi : I \to (0, \infty)$ is smooth and $p_i \in \mathbb{R}$, for any $i \in \{1, \ldots, m\}$ and also $I = (t_1, t_2)$.

We introduce the following parameters $\zeta = \sum_{i=1}^m l_i p_i$ and $\eta = \sum_{i=1}^m l_i p_i^2$ for generalized Kasner space-times. By Theorem 15 and direct computations, we get the following.

Proposition 32. *Let* $M = I \times_{\phi^{p_1}} F_1 \times \cdots \times_{\phi^{p_m}} F_m$ *be a generalized Kasner space-time and* $P = \partial/\partial t$. *Then* $(M, \overline{\nabla})$ *is Einstein with the Einstein constant* λ *if and only if the following conditions are satisfied for any* $i \in \{1, \ldots, m\}$:

(1) (F_i, ∇^{F_i}) *is Einstein with the Einstein constant* λ_i, $i \in \{1, \ldots, m\}$;

(2) $\zeta((\phi' - \phi'')/\phi) - (\eta - \zeta)(\phi'^2/\phi^2) = \lambda$;

(3) $\lambda_i/\phi^{2p_i} - p_i(\phi''/\phi) - (\zeta - 1)p_i(\phi'^2/\phi^2) + [\zeta + (\overline{n} - 2)p_i](\phi'/\phi) = \overline{n} + \lambda - 2$.

By (14), we obtain the following.

Proposition 33. *Let* $M = I \times_{\phi^{p_1}} F_1 \times \cdots \times_{\phi^{p_m}} F_m$ *be a generalized Kasner space-time and* $P = \partial/\partial t$. *Then* $(M, \overline{\nabla})$ *has constant scalar curvature* $\overline{S}$ *if and only if each* (F_i, ∇^{F_i}) *has constant scalar curvature* S^{F_i} *and*

$$\overline{S} = \sum_{i=1}^m \frac{S^{F_i}}{\phi^{2p_i}} - 2\zeta \frac{\phi''}{\phi} - \left(\eta + \zeta^2 - 2\zeta\right)\frac{\phi'^2}{\phi^2} + 2(\overline{n} - 1)\zeta \frac{\phi'}{\phi}$$
$$+ (2 - \overline{n})(\overline{n} - 1). \quad (26)$$

Next, we first give a classification of four-dimensional generalized Kasner space-times with a semisymmetric metric connection and then consider Ricci tensors and scalar curvatures of them.

Definition 34. Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ with the metric $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Consider the following:

(i) (M, g) is said to be of type (I) if $m = 1$ and $\dim(F) = 3$;

(ii) (M, g) is said to be of type (II) if $m = 2$ and $\dim(F_1) = 1$ and $\dim(F_2) = 2$;

(iii) (M, g) is said to be of type (III) if $m = 3$ and $\dim(F_1) = 1$, $\dim(F_2) = 1$ and $\dim(F_3) = 1$.

By Theorems 26 and 29, we have given a classification of type (I) Einstein spaces and type (I) spaces with the constant scalar curvature.

5.1. Classification of Einstein Type (II) Generalized Kasner Space-Times with a Semisymmetric Metric Connection. Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2$ be an Einstein type (II) generalized Kasner space-time and $P = \partial/\partial t$. Then $\zeta = p_1 + 2p_2$, $\eta = p_1^2 + 2p_2^2$. By Theorem 15, we have

$$\zeta\left(\frac{\phi' - \phi''}{\phi}\right) - (\eta - \zeta)\frac{\phi'^2}{\phi^2} = \lambda, \quad (27a)$$

$$-p_1\frac{\phi''}{\phi} - (\zeta - 1)p_1\frac{\phi'^2}{\phi^2} + [\zeta + 2p_1]\frac{\phi'}{\phi} = \lambda + 2, \quad (27b)$$

$$\frac{\lambda_2}{\phi^{2p_2}} - p_2\frac{\phi''}{\phi} - (\zeta - 1)p_2\frac{\phi'^2}{\phi^2} + [\zeta + 2p_2]\frac{\phi'}{\phi} = \lambda + 2, \quad (27c)$$

where λ_2 is a constant. Consider the following two cases.
Case (i) ($\zeta = 0$). In this case, $p_2 = -(1/2)p_1$, $\eta = (3/2)p_1^2$. Then by (27a)–(5.2iii), we have

$$-\eta\frac{\phi'^2}{\phi^2} = \lambda, \quad (28a)$$

$$p_1\left(-\frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} + 2\frac{\phi'}{\phi}\right) = \lambda + 2, \quad (28b)$$

$$\frac{\lambda_2}{\phi^{-p_1}} - \frac{1}{2}p_1\left(-\frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} + 2\frac{\phi'}{\phi}\right) = \lambda + 2. \quad (28c)$$

Case (i)(a) ($\eta = 0$). One has $p_i = 0$; by (28a), $\lambda = 0$. By (28b), $\lambda + 2 = 0$; this is a contradiction.

Case (i)(b) ($\eta \neq 0$). One has $p_i \neq 0$.

Case (i)(b)(1) ($\lambda_2 = 0$). By (28b) and (28c), $\lambda = -2$ and

$$-\frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} + 2\frac{\phi'}{\phi} = 0, \qquad \frac{\phi'^2}{\phi^2} = \frac{2}{\eta}; \qquad (29)$$

then $\phi = c_0 e^{\pm(\sqrt{2/\eta})t}$ which does not satisfy the first equation in (29); this a contradiction.

Case (i)(b)(2) ($\lambda_2 \neq 0$). By (28b) and (28c), we have $\lambda_2/\phi^{-p_1} = (3/2)(\lambda + 2)$, so ϕ is a constant. By (28b), $\lambda + 2 = 0$, so $\lambda_2 = 0$; this is a contradiction. In a word, we have no solutions when $\zeta = 0$.

Case (ii) ($\zeta \neq 0$). One has $\eta \neq 0$. Putting $\phi = \psi^{\zeta/\eta}$, then $\psi'' - \psi' + (\lambda\eta/\zeta^2)\psi = 0$. Hence,

(1) $\lambda < \zeta^2/4\eta$, $\psi = c_1 e^{((1+\sqrt{1-4\lambda\eta/\zeta^2})/2)t} + c_2 e^{((1-\sqrt{1-4\lambda\eta/\zeta^2})/2)t}$,

(2) $\lambda = \zeta^2/4\eta$, $\psi = c_1 e^{(1/2)t} + c_2 t e^{(1/2)t}$,

(3) $\lambda > \zeta^2/4\eta$, $\psi = c_1 e^{(1/2)t} \cos((\sqrt{4\lambda\eta/\zeta^2 - 1}/2)t) + c_2 e^{(1/2)t} \sin((\sqrt{4\lambda\eta/\zeta^2 - 1}/2)t)$.

We make (27a), (5.2ii), (5.2iii) into

$$\frac{\zeta^2}{\eta}\frac{\psi' - \psi''}{\psi} = \lambda, \qquad \psi = \phi^{\eta/\zeta}, \qquad (30a)$$

$$-\frac{p_1}{\zeta}\frac{(\phi^\zeta)''}{\phi^\zeta} + \frac{\zeta + 2p_1}{\zeta}\frac{(\phi^\zeta)'}{\phi^\zeta} = \lambda + 2, \qquad (30b)$$

$$\frac{\lambda_2}{\phi^{2p_2}} - \frac{p_2}{\zeta}\frac{(\phi^\zeta)''}{\phi^\zeta} + \frac{\zeta + 2p_2}{\zeta}\frac{(\phi^\zeta)'}{\phi^\zeta} = \lambda + 2. \qquad (30c)$$

When $p_1 = p_2$, type (II) spaces turn into type (I) spaces, so we assume $p_1 \neq p_2$. By (30b) and (30c), then

$$\psi' = \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}\psi^{1-2p_2\zeta/\eta} + \frac{(\lambda + 2)\eta}{\zeta^2}\psi. \qquad (31)$$

Case (ii)(1). Consider $\lambda < \zeta^2/4\eta$, $\psi = c_1 e^{at} + c_2 e^{bt}$, where $a = (1 + \sqrt{1 - 4\lambda\eta/\zeta^2})/2$, $b = ((1 - \sqrt{1 - 4\lambda\eta/\zeta^2})/2)t$.
By (31),

$$ac_1 e^{at} + bc_2 e^{bt}$$
$$= \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}\left(c_1 e^{at} + c_2 e^{bt}\right)^{1-2p_2\zeta/\eta} \qquad (32)$$
$$+ \frac{(\lambda + 2)\eta}{\zeta^2}\left(c_1 e^{at} + c_2 e^{bt}\right).$$

Case (ii)(1)(a) ($c_1 = 0$). One has

$$\left[b - \frac{(\lambda + 2)\eta}{\zeta^2}\right]c_2 e^{bt} = \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}\left(c_2 e^{bt}\right)^{1-2p_2\zeta/\eta}. \qquad (33)$$

Case (ii)(1)(a)(1) ($b \neq 0$, $p_1\lambda_2 \neq 0$). One has $p_2 = 0$ and $\zeta = p_1$, $\eta = p_1^2$ and $b = (1 - \sqrt{1 - 4\lambda})/2$ and $\psi = c_2 e^{bt}$. By (30b) and $b^2 - b + \lambda = 0$, we get $-b^2 + 3b = \lambda + 2$ and $b = 1$. But $b < 1/2$, and this is a contradiction.

Case (ii)(1)(a)(2) ($b \neq 0$, $p_1\lambda_2 = 0$). If $p_1 = 0$, then $\zeta = 2p_2$, $\eta = 2p_2^2$ and $b = (\lambda + 2)\eta/\zeta^2$, so $\lambda = -4$ and $b = -1$. By (30c), we get $\lambda_2 = 0$ and $-2b^2 + 4b = \lambda + 2$ which is a contradiction.

If $\lambda_2 = 0$, by (30b) and $b = (\lambda + 2)\eta/\zeta^2$, we get $\lambda = 0$ or -2. When $\lambda = 0$, then $b = 2\eta/\zeta^2 = 0$; this is a contradiction. There is a similar contradiction for $\lambda = -2$.

Case (ii)(1)(a)(3) ($b = 0$). One has $\psi = c_2$; by (30a), $\lambda = 0$. By (30b), $\lambda = -2$; this is a contradiction.

Case (ii)(1)(b) ($c_2 = 0$). One has

$$\left[a - \frac{(\lambda + 2)\eta}{\zeta^2}\right]c_1 e^{at} = \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}\left(c_1 e^{at}\right)^{1-2p_2\zeta/\eta}. \qquad (34)$$

Case (ii)(1)(b)(1) ($p_1\lambda_2 \neq 0$). One has $p_2 = 0$ and $\zeta = 2p_2$, $\eta = 2p_2^2$ and $a = (1 + \sqrt{1 - 4\lambda})/2$ and $\lambda_2 = \lambda + 2 - (1 + \sqrt{1 - 4\lambda})/2$. By (30b), then $a = 1$ and $\lambda = 0$, so $\lambda_2 = 1$ and $\psi = c_1 e^t$ and ϕ satisfies (30c). In this case, we get $p_2 = 0$, $p_1 \neq 0$, $\phi = c_0 e^{t/p_1}$, $\lambda = 0$, $\lambda_2 = 1$.

Case (ii)(1)(b)(2) ($p_1\lambda_2 = 0$). If $p_1 = 0$, then $\zeta = 2p_2$, $\eta = 2p_2^2$ and $\psi = c_1 e^{at}$ and $a = (\lambda + 2)\eta/\zeta^2$, so $\lambda = 0$ and $a = 1$. By (30c), we get $\lambda_2 = 0$ and ϕ satisfies (30b) and (30c). In this case, $p_1 = 0$, $p_2 \neq 0$ $\lambda = 0$, $\lambda_2 = 0$, $\phi = c_0 e^{t/p_2}$.

If $\lambda_2 = 0$, by (30b) and $a = (\lambda + 2)\eta/\zeta^2$, then $\lambda = 0$ and $a = 2\eta/\zeta^2 = 1$. By (30c), then $\lambda_2 = 0$ and ϕ satisfies (30b) and (30c). In this case, $p_1 \neq 0$, $p_2 \neq 0$, $\lambda = \lambda_2 = 0$, $p_1 = 4p_2$, $\phi = c_0 e^{t/3p_2}$.

Case (ii)(1)(c) ($c_1 \neq 0, c_2 \neq 0, b \neq 0$). If $p_2 \neq 0$, then e^{at}, e^{bt}, $(c_1 e^{at} + c_2 e^{bt})^{1-2p_2\zeta/\eta}$ are linearly independent; by (32), then

$$\left[a - \frac{(\lambda + 2)\eta}{\zeta^2}\right]c_1 = 0, \qquad \left[b - \frac{(\lambda + 2)\eta}{\zeta^2}\right]c_2 = 0,$$
$$\frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}\left(c_1 e^{at}\right)^{1-2p_2\zeta/\eta}. \qquad (35)$$

So $a = b = (\lambda + 2)\eta/\zeta^2$, and this is a contradiction.
If $p_2 = 0$, then by (32),

$$a - \frac{(\lambda + 2)\eta}{\zeta^2} - \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2} = 0,$$
$$b - \frac{(\lambda + 2)\eta}{\zeta^2} - \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2} = 0, \qquad (36)$$

so $a = b$ and we get a contradiction.

Case (ii)(1)(d) ($c_1 \neq 0, c_2 \neq 0, b = 0$). When $1 - 2p_2\zeta/\eta \neq 0$, we have similar discussions. When $1 - 2p_2\zeta/\eta = 0$, we have ($\lambda +$

$2)\eta/\zeta^2 = 1$. By $b = 0$, then $\lambda = 0$ and $2\eta = \zeta^2 = 4p_2\zeta$, so $4p_2 = \zeta$ and $p_1 = 2p_2$. But $\eta = 2p_2\zeta$; then $p_1 = p_2 = 0$. This is a contradiction.

Case (ii)(2) ($\lambda = \zeta^2/4\eta$, $\psi = c_1 e^{(1/2)t} + c_2 t e^{(1/2)t}$). By (31), we have

$$\left[\frac{1}{2}c_1 + c_2 - a_0 c_1 + \left(\frac{c_2}{2} - a_0 c_2\right)t\right] e^{(1/2)t}$$
$$= \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}(c_1 + c_2 t)^{1-2p_2\zeta/\eta}\left(e^{(1/2)t}\right)^{1-2p_2\zeta/\eta}, \quad (37)$$

where $a_0 = 1/4 + 2\eta/\zeta^2$.

Case (ii)(2)(a) ($c_2 \neq 0$). One has $(1/2)c_1 + c_2 - a_0 c_1 + (c_2/2 - a_0 c_2)t \neq 0$ and $p_2 = 0$. By (30c), $\lambda_2 + (\phi^\zeta)'/\phi^\zeta = \lambda + 2$, then $\phi^\zeta = c_0 e^{(-\lambda_2 + \lambda + 2)t}$ and $(c_1 e^{(1/2)t} + c_2 t e^{(1/2)t})^{\zeta^2/\eta} = c_0 e^{(-\lambda_2 + \lambda + 2)t}$; this is a contradiction with $c_2 \neq 0$.

Case (ii)(2)(b) ($c_2 = 0$). By (37), we have

$$\left(\frac{1}{2}c_1 - a_0 c_1\right) e^{(1/2)t}$$
$$= \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}(c_1)^{1-2p_2\zeta/\eta}\left(e^{(1/2)t}\right)^{1-2p_2\zeta/\eta}. \quad (38)$$

If $a_0 = 1/2$, then $p_1\lambda_2 = 0$ and $\lambda = 2$. If $p_1 = 0$, then $\zeta = 2p_2$ and by (30b), $(\phi^\zeta)'/\phi^\zeta = 4$ and $\phi^\zeta = c_0 e^{4t}$. By (30c),

$$\frac{\lambda_2}{(c_0' e^{(1/2)t})^{2p_2\zeta/\eta}} - \frac{16p_2}{\zeta} + 4\frac{\zeta + 2p_2}{\zeta} = 4, \quad (39)$$

so $\lambda_2 = 0$ and we have a contradiction by (39).

If $p_1 \neq 0$, then $\lambda_2 = 0$, so $\psi = c_1 e^{(1/2)t}$ and $\phi^\zeta = c_0 e^{4t}$. Then by (30b), we have $p_1 = 0$ which contradicts with $p_1 \neq 0$.

If $a_0 \neq 1/2$, then $p_2 = 0$ and $\zeta = p_1, \eta = p_1^2$ and $\lambda = 1/4$ and $a_0 = 9/4$. By $\phi^\xi = c_1 e^{(1/2)t}$ and (30b), we have a contradiction. In a word, we have no solutions in case (ii)(2).

Case (ii)(3). One has $\lambda > \zeta^2/4\eta$, $\psi = c_1 e^{(1/2)t}\cos(at) + c_2 e^{(1/2)t}\sin(at)$, where $a = \sqrt{4\lambda\eta/\zeta^2 - 1/2}$. By (31), we have

$$\left(\frac{c_1}{2} + ac_2\right)\cos(at) + \left(-ac_1 + \frac{c_2}{2}\right)\sin(at)$$
$$= \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}(c_1\cos(at) + c_2\sin(at))^{1-2p_2\zeta/\eta}e^{-2p_2\zeta/\eta}$$
$$+ \frac{(\lambda + 2)\eta}{\zeta^2}(c_1\cos(at) + c_2\sin(at)). \quad (40)$$

If $p_2 \neq 0$, then $p_1\lambda_2 = 0$ and

$$\frac{c_1}{2} + ac_2 = \frac{(\lambda + 2)\eta}{\zeta^2}c_1, \qquad -ac_1 + \frac{c_2}{2} = \frac{(\lambda + 2)\eta}{\zeta^2}c_2, \quad (41)$$

so $c_1^2 + c_2^2 = 0$. This is a contradiction.

If $p_2 = 0$, then

$$\frac{c_1}{2} + ac_2 = \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}c_1 + \frac{(\lambda + 2)\eta}{\zeta^2}c_1,$$
$$-ac_1 + \frac{c_2}{2} = \frac{p_1\lambda_2\eta}{(p_2 - p_1)\zeta^2}c_2 + \frac{(\lambda + 2)\eta}{\zeta^2}c_2. \quad (42)$$

Then $c_1^2 + c_2^2 = 0$. This is a contradiction. By the above discussions, we get the following theorem.

Theorem 35. *Let $M = I\times_{\phi^{p_1}}F_1\times_{\phi^{p_2}}F_2$ be a generalized Kasner space-time and $\dim F_1 = 1, \dim F_2 = 2$ and $P = \partial/\partial t$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if (F_2, ∇^{F_2}) is Einstein with the Einstein constant λ_2, and one of the following conditions is satisfied:*

(1) $p_2 = 0$, $p_1 \neq 0$, $\phi = c_0 e^{t/p_1}$, $\lambda = 0$, $\lambda_2 = 1$;

(2) $p_1 = 0$, $p_2 \neq 0$, $\lambda = 0$, $\lambda_2 = 0$, $\phi = c_0 e^{t/p_2}$;

(3) $p_1 \neq 0$, $p_2 \neq 0, \lambda = \lambda_2 = 0$, $p_1 = 4p_2$, $\phi = c_0 e^{t/3p_2}$.

5.2. Type (II) Generalized Kasner Space-Times with a Semisymmetric Metric Connection with Constant Scalar Curvature. By Proposition 33, then (F_2, ∇^{F_2}) has constant scalar curvature S^{F_2} and

$$\overline{S} = \frac{S^{F_2}}{\phi^{2p_2}} - 2\zeta\frac{\phi''}{\phi} - \left(\eta + \zeta^2 - 2\zeta\right)\frac{\phi'^2}{\phi^2} + 6\zeta\frac{\phi'}{\phi} - 6. \quad (43)$$

If $\zeta = 0$, when $\eta = 0$, then $p_1 = p_2 = 0$ and $\overline{S} = S^{F_2} - 6$. If $\eta \neq 0$, then

$$\eta\frac{\phi'^2}{\phi^2} = \frac{S^{F_2}}{\phi^{2p_2}} - \left(\overline{S} + 6\right). \quad (44)$$

If $\zeta \neq 0$, putting $\phi = \psi^{2\zeta/(\eta + \zeta^2)}$, we get

$$-\frac{4\zeta^2}{\eta + \zeta^2}\psi'' + \frac{12\zeta^2}{\eta + \zeta^2}\psi' - \left(\overline{S} + 6\right)\psi + S^{F_2}\psi^{1-4p_2\zeta/(\eta + \zeta^2)} = 0. \quad (45)$$

5.3. Type (III) Generalized Kasner Space-Times with a Semisymmetric Metric Connection with Constant Scalar Curvature. By Proposition 33, then

$$\overline{S} = -2\zeta\frac{\phi''}{\phi} - \left(\eta + \zeta^2 - 2\zeta\right)\frac{\phi'^2}{\phi^2} + 6\zeta\frac{\phi'}{\phi} - 6. \quad (46)$$

If $\zeta = \eta = 0$, then $p_1 = p_2 = p_3 = 0$, and we get $\overline{S} = -6$.

If $\zeta = 0, \eta \neq 0$, then $[(\ln\phi)']^2 = -(\overline{S}+6)/\eta$, so when $\overline{S}+6 > 0$, there are no solutions, when $\overline{S} + 6 = 0$, ϕ is a constant, and when $\overline{S} + 6 < 0$, $\phi = c_0 e^{\pm(\sqrt{-(\overline{S}+6)/\eta})t}$.

If $\zeta \neq 0$, then $\eta \neq 0$; putting $\phi = \psi^{2\zeta/(\eta+\zeta^2)}$, then

$$\psi'' - 3\psi' + \frac{(\overline{S}+6)(\eta+\zeta^2)}{4\zeta^2}\psi = 0. \tag{47}$$

So, we get

(1) $\overline{S} + 6 < 9\zeta^2/(\eta+\zeta^2)$, $\psi = c_1 e^{((3+\sqrt{9-(\overline{S}+6)(\eta+\zeta^2)/\zeta^2})/2)t} + c_2 e^{((3-\sqrt{9-(\overline{S}+6)(\eta+\zeta^2)/\zeta^2})/2)t}$,

(2) $\overline{S} + 6 = 9\zeta^2/(\eta+\zeta^2)$, $\psi = c_1 e^{(3/2)t} + c_2 t e^{(3/2)t}$,

(3) $\overline{S} + 6 > 9\zeta^2/(\eta+\zeta^2)$, $\psi = c_1 e^{(3/2)t}\cos((\sqrt{-9+(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t) + c_2 e^{(3/2)t}\sin((\sqrt{-9+(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t)$. So we get the following theorem.

Theorem 36. *Let $M = I\times_{\phi^{p_1}} F_1\times_{\phi^{p_2}} F_2\times_{\phi^{p_3}} F_3$ be a generalized Kasner space-time and $\dim F_1 = \dim F_2 = \dim F_3 = 1$, and $P = \partial/\partial t$. Then $\overline{S}$ is a constant if and only if one of the following cases holds.*

(1) *One has $\zeta = \eta = 0$, $\overline{S} = -6$.*

(2) *One has $\zeta = 0, \eta \neq 0$; when $\overline{S} + 6 > 0$, there are no solutions, when $\overline{S} + 6 = 0$, ϕ is a constant, and when $\overline{S} + 6 < 0$, $\phi = c_0 e^{\pm\sqrt{-((\overline{S}+6)/\eta)}t}$.*

(3) *If $\zeta \neq 0$*

   (a) *$\overline{S} + 6 < 9\zeta^2/(\eta+\zeta^2)$, $\phi = (c_1 e^{(3+\sqrt{9-(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t} + c_2 e^{(3-\sqrt{9-(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t})^{2\zeta/(\eta+\zeta^2)}$,*

   (b) *$\overline{S} + 6 = 9\zeta^2/(\eta+\zeta^2)$, $\phi = (c_1 e^{(3/2)t} + c_2 t e^{(3/2)t})^{2\zeta/(\eta+\zeta^2)}$,*

   (c) *$\overline{S} + 6 > 9\zeta^2/(\eta+\zeta^2)$, $\phi = (c_1 e^{(3/2)t}\cos((\sqrt{-9+(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t) + c_2 e^{(3/2)t}\sin((\sqrt{-9+(\overline{S}+6)(\eta+\zeta^2)/\zeta^2}/2)t))^{2\zeta/(\eta+\zeta^2)}$.*

5.4. Einstein Type (III) Generalized Kasner Space-Times with a Semisymmetric Metric Connection. By Proposition 32, we have

$$\zeta\left(\frac{\phi'-\phi''}{\phi}\right) - (\eta-\zeta)\frac{\phi'^2}{\phi^2} = \lambda, \tag{48a}$$

$$-p_1\left[\frac{\phi''}{\phi} + (\zeta-1)\frac{\phi'^2}{\phi^2} - 2\frac{\phi'}{\phi}\right] + \zeta\frac{\phi'}{\phi} = \lambda + 2, \tag{48b}$$

$$-p_2\left[\frac{\phi''}{\phi} + (\zeta-1)\frac{\phi'^2}{\phi^2} - 2\frac{\phi'}{\phi}\right] + \zeta\frac{\phi'}{\phi} = \lambda + 2, \tag{48c}$$

$$-p_3\left[\frac{\phi''}{\phi} + (\zeta-1)\frac{\phi'^2}{\phi^2} - 2\frac{\phi'}{\phi}\right] + \zeta\frac{\phi'}{\phi} = \lambda + 2. \tag{48d}$$

If $\zeta = \eta = 0$, by (48a), $\lambda = 0$ and by (48b), $\lambda = -2$; this is a contradiction.

If $\zeta = 0, \eta \neq 0$, adding (48b), (48c), and (48d), we get $\lambda = -2$. By (48a), $\phi'^2/\phi^2 = 2/\eta$ and $\phi = c_0 e^{\pm\sqrt{2/\eta}t}$. But by (48b), then $\phi''/\phi + (\zeta-1)(\phi'^2/\phi^2) - 2\phi'/\phi = 0$, and this is a contradiction.

Consider $\zeta \neq 0$. If $p_1 = p_2 = p_3$, we get type (I), so we may let $p_1 \neq p_2$. By (48b) and (48c), we have $(\phi^\zeta)'/\phi^\zeta = \lambda + 2$ and $(\phi^\zeta)''/\phi^\zeta - 2(\phi^\zeta)'/\phi^\zeta = 0$, so $\phi^\zeta = c_0 e^{(\lambda+2)t}$ and $\lambda = -2$ or 0. When $\lambda = -2$, ψ is a constant; by (48a), $\lambda = 0$, this is a contradiction. When $\lambda = 0$, $\psi = c_0 e^{2\eta t/\zeta^2}$, and $\psi'' - \psi' = 0$, so $2\eta/\zeta^2 = 1$. In this case, we get when $p_i \neq p_j$ for some $i, j \in \{1, 2, 3\}$, $\lambda = 0$, $2\eta/\zeta^2 = 1$, $\phi = c_0 e^{2t/\zeta}$. We get the following theorem.

Theorem 37. *Let $M = I\times_{\phi^{p_1}} F_1\times_{\phi^{p_2}} F_2\times_{\phi^{p_3}} F_3$ be a generalized Kasner space-time for $p_i \neq p_j$ for some $i, j \in \{1, 2, 3\}$ and $\dim F_1 = \dim F_2 = \dim F_3 = 1$, and $P = \partial/\partial t$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if $\lambda = 0, 2\eta/\zeta^2 = 1, \phi = c_0 e^{2t/\zeta}$.*

Remark 38. Comparing with Proposition 4.3, Proposition 4.11, Section 5 in [5], in Proposition 32, Proposition 33, and Theorems 35–37, the unit vector field $P = \partial/\partial t$ emerges. For Proposition 4.3, Proposition 4.11, Section 5 in [5], P equals zero. So equations in our theorems are different from the equations in theorems in [5].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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