

## Research Article

# Optimal Convergence Rates for Solutions of the Monopolar Non-Newtonian Flows

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This paper is concerned with the optimal convergence rates for solutions of the monopolar non-Newtonian flows. By using the energy methods, the perturbed weak solution of perturbed system asymptotically converges to the solution of the original system with the optimal rates  $(1+t)^{-1/4}$ .

## 1. Introduction

In this paper we consider the monopolar non-Newtonian flows which is governed by the following system:

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u - \nabla \cdot \tau^v + \nabla \pi &= f, \\ \nabla \cdot u &= 0, \\ u(x, 0) &= u_0.\end{aligned}\quad (1)$$

Here  $u = u(x, t) = (u_1, u_2, u_3)$  and  $\pi$  denote the unknown velocity vector and pressure of the fluids, while  $u_0$  is the given initial velocity and  $f$  is the external force.  $\tau^v = (\tau_{ij}^v)$  is the stress tensor specified in the following form:

$$\tau_{ij}^v = 2(\mu_1 + \mu_2 |e(u)|^{p-2}) e_{ij}(u) \quad (2)$$

for the symmetric deformation velocity tensor  $e(u) = (e_{ij}(u))$  with

$$e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (3)$$

where  $\mu_1 > 0, \mu_2 > 0$  are viscous coefficients.

The system (1) was proposed for the first time by Ladyzhenskaya [1] which describes viscous non-Newtonian flows such as the molten plastics, dyes, adhesives, paints, and greases. The constitutive of those fluids is nonlinear and does not the Stokes laws. It should be mentioned that the fluid

considered here is monopolar since only first order derivative of the velocity field is involved. When  $\mu_2 = 0$ , the system becomes the classic Navier-Stokes equations [2].

Due to the importance in both mathematics and physics, the well-posedness and large time behavior of incompressible fluids have attracted more and more attention [3–6]. We refer to some classic results on the Navier-Stokes equations [3]. For the viscous non-Newtonian flows (1), there is also a large literature on the well-posedness and asymptotic behaviors [7–10]. In particular, Pokory [11] and Bae [12] recently studied the existence and uniqueness for the Cauchy problem of the system (1). When  $f = 0$  by developing the Fourier splitting method, Dong and Li [13] explored the optimal algebraic decay rate in  $\mathbb{R}^3$  as

$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/2}, \quad \forall t > 0. \quad (4)$$

It should be mentioned that the time decay properties for weak solutions of the non-Newtonian fluids (1) essentially show that the trivial solution  $u = 0$  is stable. It is desirable to consider the stability of the nontrivial solutions of the non-Newtonian fluids (1) with the nonzero and nondecay external force. To do so, we consider the perturbed non-Newtonian fluids

$$\begin{aligned}\partial_t v + (v \cdot \nabla) v - \nabla \cdot \tau^v + \nabla \pi &= f, \\ \nabla \cdot v &= 0, \\ v(x, 0) &= u_0 + w_0,\end{aligned}\quad (5)$$

where  $w_0$  is the initial perturbation. It is easy to check that if the external force satisfies  $f \in L^2(0, T; L^2(\mathbb{R}^3))$ , the non-Newtonian fluids (1) has nontrivial stationary solution and  $u = 0$  is not a solution of (1) any more. In the two-dimensional case, Dong and Chen [14] have considered the stability issue as

$$\|v(t) - u(t)\| \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \quad (6)$$

for weak solutions of the non-Newtonian fluids (1) under any perturbation  $w_0$ . It should be mentioned that Dong and Chen [14] cannot study the optimal convergence rates. One may also refer to some interesting stability results on the classic Navier-Stokes equations. When  $f = 0$ ,  $w_0 \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$  ( $r > 3$ ) with  $\|w_0\|_{L^r}$  sufficiently small; da Veiga and Secchi [15] proved that there is a unique global solution  $v(x, t)$  of perturbed Navier-Stokes equations satisfying

$$\|u(t) - v(t)\|_{L^r(\mathbb{R}^3)} \leq C(1+t)^{-3/4} \quad \forall t > 0 \quad (7)$$

for the weak solution  $u(x, t)$  of the original Navier-Stokes equations in the subcritical class

$$u(x, t) \in L^\infty(0, \infty; L^{r+2}(\mathbb{R}^3)). \quad (8)$$

When  $f = 0$ , Ponce et al. [16] further considered the weak solution  $u(x, t)$  of the original Navier-Stokes equations under the critical space

$$\nabla u \in L^4(0, \infty; L^2(\mathbb{R}^3)) \quad (9)$$

and obtained that if

$$\|w_0\|_{H^1} \leq \delta \quad (10)$$

for the sufficient small constant  $\delta > 0$ , then there is a unique solution  $v(t)$  of perturbed Navier-Stokes equations satisfying

$$\sup_{t>0} \|u(t) - v(t)\|_{H^1(\mathbb{R}^3)} \leq M(\delta), \quad (11)$$

where  $M(\delta)$  is a constant satisfying  $\lim_{\delta \rightarrow 0} M(\delta) = 0$ . One may also refer to some related results on the Newtonian and viscous non-Newtonian flows [17, 18].

In this paper, we will consider the stability for weak solution of three-dimensional non-Newtonian fluids (1) and will derive the optimal convergence rates. More precisely, we will show that every perturbed solution  $v$  of the non-Newtonian fluids (2) converges asymptotically to  $u$  of the non-Newtonian fluids (1) as

$$\|v(t) - u(t)\|_{L^2} = O\left((1+t)^{-1/4}\right), \quad t \rightarrow \infty. \quad (12)$$

## 2. Preliminaries and Main Results

Throughout this paper,  $C$  stands for a generic positive constant which may vary from line to line.  $L^p(\mathbb{R}^3)$  with  $1 \leq p \leq \infty$  denotes the usual Lebesgue space [19] of all  $L^p$  integral functions associated with the norm

$$\|f\|_{L^p} = \begin{cases} \left( \int_{\mathbb{R}^3} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in \mathbb{R}^3} |f(x)|, & p = \infty. \end{cases} \quad (13)$$

We denote the Fourier transformation  $\widehat{f}$  or  $F[f]$  as

$$Ff(x) = \widehat{f}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(y) e^{-ix \cdot y} dy. \quad (14)$$

We recall the following Gronwall inequality which will be used in the following argument.

**Lemma 1** (Gronwall inequality [2]). *Let  $f(t)$ ,  $g(t)$ ,  $h(t)$  be nonnegative continuous functions and satisfy the following inequality:*

$$g(t) \leq f(t) + \int_0^t g(s) h(s) ds, \quad \forall t > 0, \quad (15)$$

where  $f'(t) \geq 0$ . Then

$$g(t) \leq f(t) \exp\left(\int_0^t h(s) ds\right), \quad \forall t > 0. \quad (16)$$

We now give the definition for weak solutions of three-dimensional non-Newtonian fluids (1) (see [11]).

**Definition 2.** Suppose  $p \geq 3$ ,  $u_0 \in L^2(\mathbb{R}^3)$ ,  $f \in L^2(0, T; L^2(\mathbb{R}^3))$ , and  $u(x, t)$  is called a weak solution of the non-Newtonian fluids (1) if the following conditions hold true.

- (i)  $u(x, t) \in L^\infty(0, T; L^2(\mathbb{R}^3))$  and  $\nabla u \in L^p(0, T; L^p(\mathbb{R}^3)) \cap L^2(0, T; L^2(\mathbb{R}^3))$  for all  $T > 0$ .
- (ii)  $u$  is weakly continuous from  $[0, \infty)$  to  $L^2(\mathbb{R}^3)$ .
- (iii)  $u$  satisfies (1) in the weak sense; that is, for all  $\varphi \in C^1([s, t]; \dot{H}^1)$

$$\begin{aligned} & \int_{\mathbb{R}^3} u(t) \varphi(t) dx + \int_s^t \int_{\mathbb{R}^3} (\tau^\nu \cdot \nabla \varphi + (u \cdot \nabla) u \varphi) dx d\tau \\ &= \int_s^t \int_{\mathbb{R}^3} u \partial_\tau \varphi dx d\tau + \int_{\mathbb{R}^3} u(s) \varphi(s) dx \\ &+ \int_s^t \int_{\mathbb{R}^3} f \varphi dx d\tau. \end{aligned} \quad (17)$$

- (iv)  $u$  satisfies energy inequality; for all  $t \geq 0$

$$\begin{aligned} & \int_{\mathbb{R}^3} |u(t)|^2 dx + 2\mu_1 \int_0^t \int_{\mathbb{R}^3} |\nabla u(\tau)|^2 dx d\tau \\ &+ 2\mu_2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(\tau)|^p dx d\tau \\ &\leq \int_{\mathbb{R}^3} |u_0|^2 dx + 2 \int_s^t \int_{\mathbb{R}^3} f u dx d\tau. \end{aligned} \quad (18)$$

The following existence and regularity of weak solutions of the incompressible non-Newtonian fluids (1) are due to the work of Pokory [11].

**Lemma 3** (existence and regularity of weak solutions). *Suppose  $p \geq 3$ ,  $u_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ , and  $f \in L^2(0, T; L^2(\mathbb{R}^3))$ .*

Then there exists a weak solution of the incompressible non-Newtonian fluids (1). Moreover, the solution is regular; that is,

$$\begin{aligned} \nabla u &\in L^\infty(0, T; L^2(\mathbb{R}^3)), \\ \Delta u &\in L^2(0, T; L^2(\mathbb{R}^3)), \\ \forall T > 0. \end{aligned} \quad (19)$$

Our results now read.

**Theorem 4.** Suppose  $p \geq 3$  and  $u(x, t)$  is a weak solution of the incompressible non-Newtonian fluids (1) with the finite energy initial data  $u_0 \in L^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$  and the nondecay external force  $f \in L^2(0, T; L^2(\mathbb{R}^3))$ . Then for any initial perturbation  $w_0 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ , the perturbed non-Newtonian fluids (2) has a weak solution  $v(x, t)$  which asymptotically converges  $u(x, t)$  with the optimal convergence rate:

$$\|v(t) - u(t)\|_{L^2} = O\left((1+t)^{-1/4}\right), \quad t \longrightarrow \infty. \quad (20)$$

**Remark 5.** Compared with some stability results of the classic Navier-Stokes equations such as da Veiga and Secchi [15] and Ponce et al. [16], we do not have any small assumptions on the initial perturbation. Moreover, our results also improve the previous results obtained by Dong and Chen [17] since we not only consider this problem in three-dimensional case instead of the two-dimensional case, but also derive the optimal convergence rates (20).

**Remark 6.** The proof of Theorem 4 is mainly based on Fourier splitting technique. The nonlinear system is essentially also a dissipative system which may allow us to divide the frequency domain into two time-dependent subdomains; the subdomain yields a first-order differential inequality for the spatial  $L^2$  norm of the Fourier transform of the weak solutions of incompressible non-Newtonian fluids (1). Compared with the derivation of Dong and Chen [17], the estimates of the nonlinear term can be dealt with in a satisfied form, for the three-dimensional case; however, the bounds of the nonlinear term become difficult. In order to overcome, we make full use of the regular properties of weak solutions and energy methods.

### 3. Proof of Theorem 4

According to the condition in Theorem 4 and the existence results of non-Newtonian fluids (1) obtained in [11], the existence of weak solutions of the perturbation of the perturbed incompressible non-Newtonian fluids (5) can be obtained by the parallel methods in [11]. Therefore, what remains is to prove the optimal convergence rates (20).

Let  $(u, \pi_1)$  and  $(v, \pi_2)$  be weak solutions of the original non-Newtonian fluids (1) and the perturbed non-Newtonian fluids (5), respectively. We denote by  $w = v - u$  and  $\pi = \pi_2 -$

$\pi_1$  the difference between the two weak solutions of (1) and (2); then  $w$  satisfies the following system in the weak sense:

$$\begin{aligned} w_t - \mu_1 \Delta w + (v \cdot \nabla) w + (w \cdot \nabla) u \\ - \mu_2 \left( \operatorname{div}(|\nabla v|^{p-2} \nabla v) - \operatorname{div}(|\nabla u|^{p-2} \nabla u) \right) \\ + \nabla \pi = 0, \\ \nabla \cdot w = 0, \\ u(x, 0) = w_0. \end{aligned} \quad (21)$$

In order to investigate the optimal convergence rates, we now need some a priori estimates of the nonlinear system (21).

**Lemma 7.** Under the same condition in Theorem 4, then the difference  $w = v - u$  satisfies the following inequality:

$$\widehat{w}(\xi, t) \leq \left| \widehat{w}_0(\xi) e^{-\mu_1 |\xi|^2 t} \right| + c |\xi| + c |\xi| t. \quad (22)$$

*Proof of Lemma 7.* We first formally derive inequality (22). The rigorous derivation is obtained by considering the approximated solutions of the following approximate system:

$$\begin{aligned} \partial_t w_k - \mu_1 \Delta w + (v_k \cdot \nabla) \widehat{w}_k + (\widehat{w}_k \cdot \nabla) u_k \\ - \mu_2 \left( \operatorname{div}(|\nabla v_k|^{p-2} \nabla v_k) - \operatorname{div}(|\nabla u_k|^{p-2} \nabla u_k) \right) \\ + \nabla \pi = 0, \end{aligned} \quad (23)$$

$$\nabla \cdot w_k = 0,$$

$$w_k(x, 0) = w_0.$$

Here the retarded modification  $\tilde{f}_k$  of  $f_k$  is defined by

$$\begin{aligned} \tilde{f}_k(x, t) = \delta^{-4} \iint \psi\left(\frac{y}{\delta}, \frac{s}{\delta}\right) \bar{f}_k(x - y, t - s) dy ds, \\ k = \frac{T}{\delta}, \end{aligned} \quad (24)$$

where  $\psi \in C_0^\infty(\mathbb{R}^3 \times (0, \infty))$  is a positive modifier.  $\bar{f}_k$  is the zero extension of the function  $u_k$  which is originally defined for  $t \geq 0$ . For the similar estimates (22) of the approximate solutions  $w_k$ , we only need to take the limit as  $k \rightarrow \infty$ . Therefore we now deal with this settle in the following argument only for weak solution  $w$  directly.

Taking Fourier transformation of (23), it follows that by solving an ordinary equation

$$\begin{aligned} \widehat{w} = \widehat{w}_0(\xi) e^{-\mu_1 |\xi|^2 t} + \int_0^t e^{-\mu_1 |\xi|^2 (t-s)} \mathcal{N}(\xi, s) ds \\ \leq e^{-\mu_1 \Delta t} \widehat{w}_0 + \int_0^t |\mathcal{N}(\xi, s)| ds, \end{aligned} \quad (25)$$

where

$$\begin{aligned}
 N(\xi, t) &= -F[(v \cdot \nabla)w + (w \cdot \nabla)u] \\
 &\quad + F[\mu_2(\operatorname{div}(|\nabla v|^{p-2}\nabla v) - \operatorname{div}(|\nabla u|^{p-2}\nabla u))] \\
 &\quad - F[\nabla p] \\
 &=: I + J + K.
 \end{aligned} \tag{26}$$

We now compute  $I, J, K$  one by one. For  $I$ , applying the divergence free properties of the velocity fields and Hölder inequality, it follows that

$$\begin{aligned}
 I &\leq \left| \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot y} \{ (v \cdot \nabla)w + (w \cdot \nabla)u \} dy \right| \\
 &\leq \sum_{i,j} \left| \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot y} \{ \partial_i(v_i w_j) + \partial_i(w_i u_j) \} dy \right| \\
 &\leq C \sum_{i,j} |\xi| \int_{\mathbb{R}^3} |v_i w_j + w_i u_j| dy \\
 &\leq C |\xi| (\|v\|_{L^2} + \|u\|_{L^2}) \|w\|_{L^2} \\
 &\leq C |\xi|,
 \end{aligned} \tag{27}$$

where we have used the bounds of  $u, v, w$ ; that is,

$$\operatorname{ess\,sup}_t (\|v\|_{L^2} + \|u\|_{L^2}) \leq C, \quad \operatorname{ess\,sup}_t \|w\|_{L^2} \leq C. \tag{28}$$

For  $J$ , one shows that

$$\begin{aligned}
 J &\leq \left| \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix \cdot y} \{ \mu_2(\operatorname{div}(|\nabla v|^{p-2}\nabla v) \right. \\
 &\quad \left. - \operatorname{div}(|\nabla u|^{p-2}\nabla u)) \} dy \right| \\
 &\leq C |\xi| \int_{\mathbb{R}^3} |\nabla u|^{p-2} \nabla u + |\nabla v|^{p-2} \nabla v \, dy \\
 &\leq C |\xi| (\|\nabla v\|_{L^{p-1}}^{p-1} + \|\nabla u\|_{L^{p-1}}^{p-1}).
 \end{aligned} \tag{29}$$

In order to estimate  $K$ , we act the divergence operator to both sides of (21); it follows that

$$\begin{aligned}
 \Delta \pi &= \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (-v_i w_j - u_i w_j + |\nabla u|^{p-2} \partial_i u_j - |\nabla v|^{p-2} \partial_i v_j), \\
 |\xi|^2 F[\pi] &= \sum_{i,j} \xi_i \xi_j F[-v_i w_j - u_i w_j + |\nabla u|^{p-2} \partial_i u_j - |\nabla v|^{p-2} \partial_i v_j].
 \end{aligned} \tag{30}$$

Now together with the estimates of  $I$  and  $J$ ,  $K$  can be estimated as

$$K \leq C |\xi| + C |\xi| (\|\nabla v\|_{L^{p-1}}^{p-1} + \|\nabla u\|_{L^{p-1}}^{p-1}). \tag{31}$$

Hence plugging the estimates (27)–(31) of  $I, J, K$  into (26) and then (25), we may obtain that

$$|\widehat{w}| \leq e^{-\mu_1 \Delta t} \widehat{w}_0 + C |\xi| t + C |\xi| \int_0^t (\|\nabla v\|_{L^{p-1}}^{p-1} + \|\nabla u\|_{L^{p-1}}^{p-1}) ds. \tag{32}$$

Thanks to

$$\begin{aligned}
 &\left( \int_0^\infty \|\nabla u\|_{L^{p-1}}^{p-1} ds \right)^{1/(p-1)} \\
 &\leq \left( \int_0^\infty \|\nabla u\|_2^{\theta(p-1)} \|\nabla u\|_p^{(1-\theta)(p-1)} ds \right)^{1/(p-1)} \\
 &\leq \left( \int_0^\infty \|\nabla u\|_2^2 ds \right)^{\theta/2} \left( \int_0^\infty \|\nabla u\|_p^p ds \right)^{(1-\theta)/p} \\
 &\leq C
 \end{aligned} \tag{33}$$

for

$$\frac{1}{p-1} = \frac{\theta}{2} + \frac{1-\theta}{p} \tag{34}$$

and the following facts due to the energy inequality

$$\nabla u, \nabla v \in L^2(0, \infty; L^2) \cap L^p(0, \infty; L^p) \quad \text{for } p \geq 3, \tag{35}$$

thus (32) is written as

$$|\widehat{w}| \leq \left| e^{-\mu_1 \Delta t} \widehat{w}_0 \right| + C |\xi| t + C |\xi| \tag{36}$$

which completes the proof of this lemma.  $\square$

Now we are in positive to prove (20). Taking the  $L^2$  inner product of (21) with  $w$ , it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |w|^2 dx + \mu_1 \int_{\mathbb{R}^3} |\nabla w|^2 dx \leq \int_{\mathbb{R}^3} (w \cdot \nabla) w \cdot u \, dx, \tag{37}$$

where we have used the following fact:

$$\begin{aligned}
 &\int_{\mathbb{R}^3} \mu_2 (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u) \cdot \nabla w \, dx \\
 &\geq \mu_2 \{ \|\nabla v\|_{L^p}^p - \|\nabla u\|_{L^p}^p + \|\nabla v\|_{L^p}^{p-1} \|\nabla u\|_{L^p} \\
 &\quad - \|\nabla u\|_{L^p}^{p-1} \|\nabla v\|_{L^p} \} \\
 &\geq (\|\nabla v\|_{L^p}^{p-1} - \|\nabla u\|_{L^p}^{p-1}) (\|\nabla v\|_{L^p} - \|\nabla u\|_{L^p}) \geq 0.
 \end{aligned} \tag{38}$$

Applying Hölder inequality, Gagliardi-Nirenberg inequality, and Young inequality gives that

$$\begin{aligned}
 \left| \int_{\mathbb{R}^3} (w \cdot \nabla) w \cdot u \, dx \right| &\leq C \|w\|_{L^3} \|\nabla w\|_{L^2} \|u\|_{L^6} \\
 &\leq C \|\nabla w\|_{L^2}^{3/2} \|w\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \\
 &\leq \frac{\mu_1}{2} \|\nabla w\|_{L^2}^2 + C (\mu_1) \|w\|_{L^2}^2 \|\nabla u\|_{L^2}^4,
 \end{aligned} \tag{39}$$

from which we rewrite (37) as

$$\frac{d}{dt} \|w(t)\|_{L^2}^2 + \mu_1 \|\nabla w(t)\|_{L^2}^2 \leq C(\mu_1) \|w\|_{L^2}^2 \|\nabla u\|_{L^2}^4. \quad (40)$$

Taking Parseval inequality into consideration

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} |\widehat{w}(t)|^2 d\xi + \mu_1 \int_{\mathbb{R}^3} |\xi|^2 |\widehat{w}(t)|^2 d\xi \\ & \leq C(\mu_1) \|w\|_{L^2}^2 \|\nabla u\|_{L^2}^4. \end{aligned} \quad (41)$$

And then multiplying both sides by  $(1+t)^4$

$$\begin{aligned} & \frac{d}{dt} \left( (1+t)^3 \int_{\mathbb{R}^3} |\widehat{w}(t)|^2 d\xi \right) + \mu_1 (1+t)^4 \int_{\mathbb{R}^3} |\xi|^2 |\widehat{w}(t)|^2 d\xi \\ & \leq 4(1+t)^3 \int_{\mathbb{R}^3} |\widehat{w}(t)|^2 d\xi + C(1+t)^4 \|\widehat{w}\|_{L^2}^2 \|\nabla u\|_{L^2}^4. \end{aligned} \quad (42)$$

Let

$$S(t) = \left\{ \xi \in \mathbb{R}^3 : |\xi|^2 \leq \frac{4}{\mu_1 (1+t)} \right\}, \quad (43)$$

and then we divide the domain  $\mathbb{R}^3$  of the second integral in (37) into  $S(t)$  and  $S(t)^c$ ; that is to say,

$$\begin{aligned} & \frac{d}{dt} \left( (1+t)^4 \int_{\mathbb{R}^3} |\widehat{w}(t)|^2 d\xi \right) \\ & \leq C(1+t)^3 \int_{S(t)^c} |\widehat{w}(t)|^2 d\xi + C(1+t)^4 \|\widehat{w}\|_{L^2}^2 \|\nabla u\|_{L^2}^4. \end{aligned} \quad (44)$$

Integrating in time from 0 to  $t$  gives

$$\begin{aligned} & (1+t)^4 \int_{\mathbb{R}^3} |\widehat{w}(t)|^2 d\xi \\ & \leq C \|\widehat{w}_0\|_{L^2}^2 + C \int_0^t (1+s)^3 \int_{S(s)} |\widehat{w}(s)|^2 d\xi ds \\ & \quad + \int_0^t (1+t)^4 \|\widehat{w}\|_{L^2}^2 \|\nabla u\|_{L^2}^4 ds. \end{aligned} \quad (45)$$

By employing Lemma 7, we have after the direct computation

$$\begin{aligned} & \int_0^t (1+s)^3 \int_{S(s)} |\widehat{w}(s)|^2 d\xi ds \\ & \leq C \int_0^t (1+s)^3 \int_{S(s)} \left| e^{-\widehat{\mu_1 \Delta t} w_0} + |\xi| s + |\xi| \right|^2 d\xi ds \\ & \leq C \int_0^t (1+s)^3 \left\| e^{-\widehat{\mu_1 \Delta t} w_0} \right\|_{L^2}^2 ds + C(1+t)^{7/2} \\ & \quad + C(1+t)^{3/2} \\ & \leq C(1+t)^{7/2}, \end{aligned} \quad (46)$$

where we have used the following  $L^p - L^q$  estimates of heat equations:

$$\left\| e^{-\mu_1 \Delta t} w_0 \right\|_{L^q(\mathbb{R}^3)} \leq C t^{-(3/2)(1/r-1/q)} \|w_0\|_{L^r(\mathbb{R}^3)}, \quad t > 0 \quad (47)$$

with  $1 \leq r \leq q \leq \infty$ .

Hence we insert (46) into (45) and take Gronwall inequality into consideration:

$$(1+t)^4 \|w(t)\|^2 \leq \{C(1+t)^{7/2}\} \exp \left( C \int_0^\infty \|\nabla u(s)\|_{L^2}^4 ds \right). \quad (48)$$

Since

$$\nabla u \in L^2(0, \infty; L^2(\mathbb{R}^3)) \cap L^\infty(0, \infty; L^2(\mathbb{R}^3)), \quad (49)$$

then by the interpolation inequality we have

$$\nabla u \in L^4(0, \infty; L^2(\mathbb{R}^3)). \quad (50)$$

Hence (48) implies

$$\|v(t) - u(t)\|_{L^2} = O((1+t)^{-1/4}), \quad t \longrightarrow \infty \quad (51)$$

which completes the proof of Theorem 4.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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