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Research Article

Non-Single-Valley Solutions for p-Order Feigenbaum's Type Functional Equation $f(\varphi(x)) = \varphi^p(f(x))$

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This work deals with Feigenbaum's functional equation $f(\varphi(x)) = \varphi^p(f(x))$, $\varphi(0) = 1$, $0 \le \varphi(x) \le 1$, $x \in [0,1]$, where $p \ge 2$ is an integer, φ^p is the p-fold iteration of φ, and $f(x)$ is a strictly increasing continuous function on $[0,1]$ that satisfies $f(0) = 0$, $f(x) < x$, $(x \in (0,1])$. Using a constructive method, we discuss the existence of non-single-valley continuous solutions of the above equation.

1. Introduction

In 1978, Feigenbaum [1, 2] and independently Couliet and Tresser [3] introduced the notion of renormalization for real dynamical systems. In 1992, Sullivan [4] proved the uniqueness of the fixed point for the period-doubling renormalization operator. This fixed point of renormalization satisfies a functional equation known as the Cvitanović-Feigenbaum equation:

$$\begin{aligned} g(x) &= -\frac{1}{\lambda} g(g(-\lambda x)), \quad 0 < \lambda < 1, \\ g(0) &= 1, \quad -1 \le g(x) \le 1, \; x \in [-1,1]. \end{aligned} \tag{1}$$

As mentioned above, this equation and its solution play an important role in the theory initiated by Feigenbaum [1, 2]. However, it is difficult to find an exact solution of the above equation in general. This problem can be studied in classes of smooth functions or of continuous functions. In classes of smooth functions, the existence of smooth solutions for (1) has been established in [5–8] and references therein. As far as we know, continuous solutions of (1) in classes of continuous functions have been relatively little researched. In this direction, we refer to [9, 10]. In particular, Yang and Zhang [9] demonstrated the existence of a single-valley continuous solution for the following equation:

$$\begin{aligned} \varphi(x) &= \frac{1}{\lambda} \varphi(\varphi(\lambda x)), \quad 0 < \lambda < 1, \\ \varphi(0) &= 1, \quad 0 \le \varphi(x) \le 1, \; x \in [0,1], \end{aligned} \tag{2}$$

which is called the second type of Feigenbaum's functional equations. In the last years, a number of authors considered the more general equation

$$\begin{aligned} \varphi(x) &= \frac{1}{\lambda} \varphi^p(\lambda x), \quad 0 < \lambda < 1, \\ \varphi(0) &= 1, \quad 0 \le \varphi(x) \le 1, \; x \in [0,1], \end{aligned} \tag{3}$$

where $p \ge 2$ is an integer and φ^p is the p-fold iteration of φ. It is easy to see that (2) is a special case of (3). For p large enough, Eckmann et al. [11] showed that there exists a solution of (3) similar to the function $\varphi(x) = |1 - 2x^2|$. For any $p \ge 2$, Zhang et al. [12] and Liao et al. [13] proved that (3) has single-valley-extended continuous solutions.

In the present paper, we will consider Feigenbaum's functional equations

$$\begin{aligned} f(\varphi(x)) &= \varphi^p(f(x)), \\ \varphi(0) &= 1, \quad 0 \le \varphi(x) \le 1, \; x \in [0,1], \end{aligned} \tag{4}$$

where $f(x)$ is a strictly increasing continuous function on $[0,1]$ that satisfies $f(0) = 0$, $f(x) < x$, $(x \in (0,1])$. We will prove the existence of single-valley-extended non-single-valley continuous solutions of (4) by the constructive method. Obviously, let $f(x) = \lambda x$; then (4) is (3).

2. Basic Definitions and Lemmas

In this section, we will give some characterizations of single-valley-extended non-single-valley continuous solutions of (4); they will be proved in the appendix.

Definition 1. One calls φ a single-valley-extended continuous solution of (4) if (i) φ is a continuous solution of (4); (ii) there exists $\alpha \in (f(1),1)$ such that φ is strictly decreasing on $[f(1),\alpha]$ and strictly increasing on $[\alpha,1]$.

Definition 2. One calls φ a single-valley-extended non-single-valley continuous solution of (4) if (i) φ is a single-valley-extended continuous solution of (4); (ii) φ has at least an extreme point on $(0, f(1))$.

In the following, we always let $\lambda = f(1) = \varphi^{p-1}(1)$ and define the sets

$$\begin{aligned} \Delta_k &= \left[f^{k+1}(1), f^k(1)\right], \\ \Delta_k^1 &= \left[f^k(\alpha), f^k(1)\right], \\ \Delta_k^2 &= \left[f^{k+1}(1), f^k(\alpha)\right], \\ \Delta_k^\alpha &= \left[f^{k+1}(\alpha), f^k(\alpha)\right], \\ &\forall k \geq 0. \end{aligned} \tag{5}$$

Obviously,

$$\Delta_k = \Delta_k^1 \cup \Delta_k^2, \qquad \Delta_k^\alpha = \Delta_{k+1}^1 \cup \Delta_k^2, \tag{6}$$

and, from the fact that $\{f^k(1)\}$ and $\{f^k(\alpha)\}$ are, respectively, strictly decreasing and $\lim_{k\to+\infty} f^k(1) = 0$, $\lim_{k\to+\infty} f^k(\alpha) = 0$, then

$$\begin{aligned} [0,1] &= \bigcup_{k=0}^{+\infty} \Delta_k = \bigcup_{k=0}^{+\infty} \left(\Delta_k^1 \cup \Delta_k^2\right), \\ [0,\alpha] &= \bigcup_{k=0}^{+\infty} \Delta_k^\alpha = \bigcup_{k=0}^{+\infty} \left(\Delta_{k+1}^1 \cup \Delta_k^2\right). \end{aligned} \tag{7}$$

Lemma 3. *Suppose that $\varphi(x)$ is a single-valley-extended non-single-valley continuous solution of (4) and α is the extreme point of φ in $(\lambda,1)$. Then the following conclusions hold:*

(i) *$\varphi(x)$ has a unique minimum point α with $\varphi(\alpha) = 0$;*

(ii) *0 is a recurrent but not periodic point of φ;*

(iii) *λ is an extreme point of φ and $\varphi(\lambda) > \lambda$;*

(iv) *$\varphi(x)$ has a unique fixed point $\beta = \varphi(\beta)$ on $[0,1]$, and*

$$\varphi^{p-1}(1) = f(1) = \lambda < \beta < \alpha; \tag{8}$$

(v) *for $x \in [0,\lambda]$ and $0 \leq i \leq p-1$, then $\varphi^i(x) = \alpha$ if and only if $x = f(\alpha)$ and $i = p-1$.*

Lemma 4. *Suppose that $\varphi(x)$ is a single-valley-extended non-single-valley continuous solution of (4). Let $J = [0,\lambda]$, $J_0 = \varphi(J)$, and $J_i = \varphi^i(J_0)$; then the following conclusions hold:*

(i) *$J_0, J_1, \ldots, J_{p-2} \subset (\lambda, 1]$ are pairwise disjoint;*

(ii) *for all $i = 0,1,\ldots,p-2$, then $\varphi^i : J_0 \mapsto J_i$ is a homeomorphism.*

Lemma 5. *Suppose that $\varphi(x)$ is a single-valley-extended non-single-valley continuous solution of (4). Then for all $n \geq 1$, φ has infinite many extreme points $f^n(1)$ and $f^n(\alpha)$; $f^n(1)$ is the maximum point of φ on $[f^n(\alpha),\alpha]$; $f^n(\alpha)$ is the minimum point of φ on $[0, f^n(1)]$.*

Lemma 6. *Suppose that $\varphi(x)$ is a single-valley-extended non-single-valley continuous solution of (4). Then the equation $\varphi^{p-1}(x) = f(x)$ has only one solution $x = 1$ on $(\varphi(f(\alpha)), 1]$.*

Lemma 7. *Let φ_1, φ_2 be two single-valley-extended non-single-valley continuous solutions of (4). If*

$$\varphi_1(x) = \varphi_2(x), \quad x \in [\lambda, 1], \tag{9}$$

then $\varphi_1(x) = \varphi_2(x)$ on $[0,1]$.

3. Constructive Method of Solutions

In this section, we will prove constructively the existence of single-valley-extended non-single-valley continuous solutions of (4).

Theorem 8. *Fix a strictly increasing continuous function $f(x)$ on $[0,1]$ with $f(0) = 0$, $f(x) < x$ $(x \in (0,1])$. Denote $f(1) = \lambda$. If $\varphi_0(x)$ is a continuous function on $[\lambda,1]$ and satisfies the following conditions,*

(i) *there exists an $\alpha \in (\lambda,1)$ such that $\varphi_0(\alpha) = 0$ and φ_0 is strictly decreasing on $[\lambda,\alpha]$ and strictly increasing on $[\alpha,1]$;*

(ii) *$\varphi_0^{p-1}(1) = f(1) = \lambda$, $\varphi_0^p(\lambda) = f(\varphi_0(1))$;*

(iii) *there exists an $\alpha_0 \in (\lambda,1]$ with $\varphi_0^{p-1}(\alpha_0) = 0$, $\varphi_0(\lambda) > \alpha_0$; denote $J_0 = [\alpha_0, 1]$, $J_i = \varphi_0^i(J_0)$; then*

  (a) *$J_0, J_1, \ldots, J_{p-2} \subset (\lambda,1]$ are pairwise disjoint,*

  (b) *for all $i = 0,1,\ldots,p-2$, then $\varphi_0^i : J_0 \mapsto J_i$ is a homeomorphism,*

  (c) *α is an endpoint of J_{p-2} and $\alpha = \varphi_0^{p-2}(\alpha_0)$;*

(iv) *the equation $\varphi_0^{p-1}(x) = f(x)$ has only one solution $x = 1$ on $(\alpha_0, 1]$,*

then there exists a uniquely single-valley-extended non-single-valley continuous function $\varphi(x)$ satisfying the equation

$$\begin{aligned} f(\varphi(x)) &= \varphi^p(f(x)), \quad x \in [0,1], \\ \varphi(x) &= \varphi_0(x), \quad x \in [\lambda,1]. \end{aligned} \tag{10}$$

And φ has infinitely many extreme points. Conversely, if φ_0 is the restriction on $[\lambda, 1]$ of a single-valley-extended non-single-valley continuous solution to (4), then conditions (i)–(iv) above must hold.

Proof. Suppose that φ_0 satisfies conditions (i)–(iv). Define $\psi_+ = \varphi_0^{p-1}|_{[\alpha_0,1]}$. By condition (iii), we have that ψ_+ is a homeomorphism. And by $\psi_+(\alpha_0) = \varphi_0^{p-1}(\alpha_0) = 0 \le \psi_+(1)$ we know that ψ_+ is strictly increasing.

Firstly, we define φ on Δ_k by induction. Obviously, $\varphi = \varphi_0$ is well defined on Δ_0. Suppose that $\varphi(x)$ is well defined as $\varphi_k(x)$ on Δ_k and strictly increasing and decreasing, respectively, on Δ_k^1 and Δ_k^2 for all $k \le m$, where $m \ge 0$ is a certain integer. Let

$$\varphi_{m+1}(x) = \psi_+^{-1}\left(f\left(\varphi_m\left(f^{-1}(x)\right)\right)\right), \quad (x \in \Delta_{m+1}); \tag{11}$$

then $\varphi(x)$ is well defined as $\varphi_{m+1}(x)$ on Δ_{m+1} and strictly increasing and decreasing, respectively, on Δ_{m+1}^1 and Δ_{m+1}^2. Thereby $\varphi(x)$ is well defined as a continuous function $\varphi_k(x)$ on Δ_k and strictly increasing and decreasing, respectively, on Δ_k^1 and Δ_k^2 for all $k \ge 0$ by induction. And

$$\psi_+\left(\varphi_{k+1}(f(x))\right) = f\left(\varphi_k(x)\right), \quad x \in \Delta_k. \tag{12}$$

Secondly, we prove that φ_k and φ_{k+1} have the same value on the common endpoint $f^{k+1}(1)$ of Δ_k and Δ_{k+1} for all $k \ge 0$. From condition (ii) we have

$$\psi_+\left(\varphi_0(\lambda)\right) = \varphi_0^{p-1}\left(\varphi_0(\lambda)\right) = \varphi_0^p(\lambda) = f\left(\varphi_0(1)\right). \tag{13}$$

And letting $m = 0$, $x = \lambda$ in (11) we get

$$\varphi_1\left(f(1)\right) = \psi_+^{-1}\left(f\left(\varphi_0(1)\right)\right) = \varphi_0(\lambda) = \varphi_0\left(f(1)\right). \tag{14}$$

That is, φ_0 and φ_1 have the same value on the common endpoint $f(1) = \lambda$ of Δ_0 and Δ_1. Suppose that

$$\varphi_m\left(f^m(1)\right) = \varphi_{m-1}\left(f^m(1)\right), \tag{15}$$

where $m \ge 1$ is a certain integer. Let $x = f^{m+1}(1)$ in (11); then we have

$$\begin{aligned}\varphi_{m+1}\left(f^{m+1}(1)\right) &= \psi_+^{-1}\left(f\left(\varphi_m\left(f^m(1)\right)\right)\right)\\ &= \psi_+^{-1}\left(f\left(\varphi_{m-1}\left(f^m(1)\right)\right)\right)\\ &= \varphi_m\left(f^{m+1}(1)\right).\end{aligned} \tag{16}$$

That is, φ_k and φ_{k+1} have the same value on the common endpoint $f^{k+1}(1)$ of Δ_k and Δ_{k+1} for all $k \ge 0$ by induction. Therefore, we can let

$$\varphi(x) = \begin{cases} 1, & (x = 0), \\ \varphi_k(x), & (x \in \Delta_k). \end{cases} \tag{17}$$

Since φ_k is continuous on Δ_k and increasing and decreasing, respectively, on Δ_k^1 and Δ_k^2 for $k \ge 0$ and (14), (15), and (16), we have that φ is a non-single-valley continuous function and has infinitely many extreme points on $(0, 1]$.

Thirdly, we prove that φ is continuous at $x = 0$ as follows. It is trivial that $\{f^k(\alpha)\}$ is strictly decreasing and $\lim_{k\to\infty} f^k(\alpha) = 0$. We prove $\{\varphi_k(f^k(\alpha))\}|_{k=1}^{\infty}$ is strictly increasing on $[\alpha_0, 1]$ by induction as follows. Since $f(\varphi_1(f(\alpha))) > 0$ and from (11), we get

$$\begin{aligned}\varphi_2\left(f^2(\alpha)\right) &= \psi_+^{-1}\left(f\left(\varphi_1\left(f(\alpha)\right)\right)\right)\\ &> \psi_+^{-1}(0) = \psi_+^{-1}\left(f\left(\varphi_0(\alpha)\right)\right) = \varphi_1\left(f(\alpha)\right).\end{aligned} \tag{18}$$

Suppose that $\varphi_m(f^m(\alpha)) > \varphi_{m-1}(f^{m-1}(\alpha))$, where $m \ge 2$ is a certain integer. Therefore, by (11) and the fact that $\psi_+^{-1} \circ f$ is strictly increasing, we have that

$$\begin{aligned}&\varphi_{m+1}\left(f^{m+1}(\alpha)\right)\\ &\quad= \psi_+^{-1}\left(f\left(\varphi_m\left(f^m(\alpha)\right)\right)\right)\\ &\quad> \psi_+^{-1}\left(f\left(\varphi_{m-1}\left(f^{m-1}(\alpha)\right)\right)\right) = \varphi_m\left(f^m(\alpha)\right).\end{aligned} \tag{19}$$

Thereby, $\{\varphi_k(f^k(\alpha))\}|_{k=1}^{\infty}$ is strictly increasing in $[\alpha_0, 1]$ by induction. Let $\lim_{k\to\infty}\varphi_k(f^k(\alpha)) = \gamma$; then $\gamma \in [\alpha_0, 1]$. From (12), we have that

$$\begin{aligned}\varphi_0^{p-1}\left(\varphi_{k+1}\left(f^{k+1}(\alpha)\right)\right) &= \psi_+\left(\varphi_{k+1}\left(f^{k+1}(\alpha)\right)\right)\\ &= f\left(\varphi_k\left(f^k(\alpha)\right)\right).\end{aligned} \tag{20}$$

Let $k \to \infty$; we get $\varphi_0^{p-1}(\gamma) = f(\gamma)$. By condition (iv), we know $\gamma = 1 = \varphi(0)$. This proves that φ is continuous at $x = 0$. Thereby, φ is a continuous function on $[0, 1]$. We have that $\varphi(x)$ satisfies (10) by (12) and $\varphi(x)$ is unique from Lemma 7.

Obviously, if φ_0 is the restriction on $[\lambda, 1]$ of a single-valley-extended non-single-valley continuous solution to (4), then conditions (i)–(iv) must hold by the lemmas in Section 2. □

Example 9. Let $\varphi_0(x) : [1/4, 1] \mapsto [0, 1]$ be defined by

$$\varphi_0(x) = \begin{cases} -\dfrac{13}{8}x + \dfrac{39}{32}, & \left(\dfrac{1}{4} \le x \le \dfrac{3}{4}\right), \\ x - \dfrac{3}{4}, & \left(\dfrac{3}{4} \le x \le 1\right). \end{cases} \tag{21}$$

Obviously, φ_0 satisfies the conditions of Theorem 8 with $p = 2$, $f(x) = x/4$, and $\lambda = f(1) = 1/4$, $\alpha = 3/4$. Hence it is the restriction to $[1/4, 1]$ of a single-valley-extended continuous solution φ to (4). Since φ_0 has the minimum point $\alpha = 3/4$ and $\varphi_0(1/4) = 13/16 > 3/4$, φ is a single-valley-extended non-single-valley continuous solution. Its graph is depicted in Figure 1.

Example 10. Let $\varphi_0(x) : [1/5, 1] \mapsto [0, 1]$ be defined by

$$\varphi_0(x) = \begin{cases} -\dfrac{109}{25}x + \dfrac{218}{125}, & \left(\dfrac{1}{5} \le x \le \dfrac{2}{5}\right), \\ x - \dfrac{2}{5}, & \left(\dfrac{2}{5} \le x \le 1\right). \end{cases} \tag{22}$$

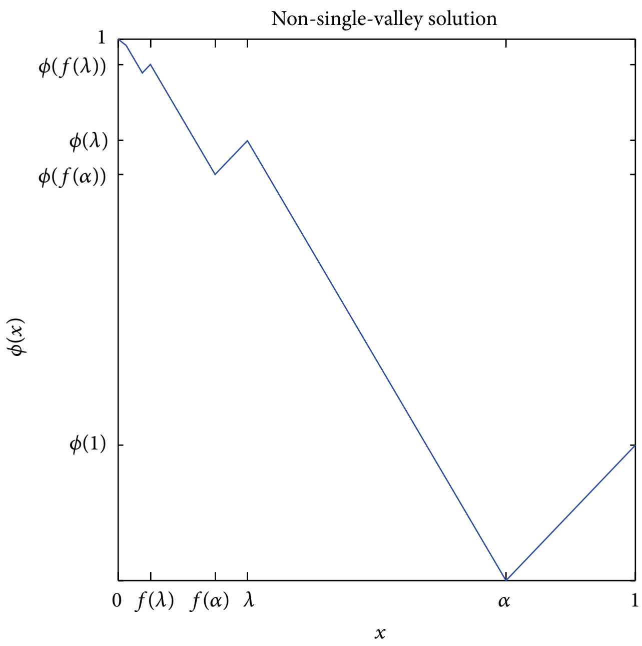

Figure 1: The graph of non-single-valley solution.

Obviously, φ_0 satisfies the conditions of Theorem 8 with $p = 3$, $f(x) = x^2/5$, and $\lambda = f(1) = 1/5, \alpha = 2/5$. Hence it is the restriction to $[1/5, 1]$ of a single-valley-extended continuous solution φ to (4). Since φ_0 has the minimum point $\alpha = 2/5$ and $\varphi_0(1/5) = 109/125 > 2/5$, $\varphi_0^2(1/5) = 59/125 > 2/5$, φ is a single-valley-extended non-single-valley continuous solution. Its graph is similar to Figure 1.

Appendix

Proof of Lemma 3. (i) Suppose that γ is a minimum point of φ. By (4) we have

$$f(\varphi(\gamma)) = \varphi^p(f(\gamma)) \geq \varphi(\gamma). \tag{A.1}$$

And from $f(0) = 0$, $f(x) < x\,(x \in (0,1])$ we know $\varphi(\gamma) = 0$. If $\gamma < \lambda$, then for all $i = 1, 2, \ldots$ we have $\varphi^i([0,\lambda]) = [0,1]$. Thus there exists $x_0 \in [0,\lambda]$ such that $\varphi^{p-1}(x_0) = 0$. And by (4) we have

$$f\left(\varphi\left(f^{-1}(x_0)\right)\right) = \varphi^p(x_0) = \varphi(0) = 1. \tag{A.2}$$

This contradicts $f(0) = 0$, $f(x) < x\,(x \in (0,1])$. Thus $\gamma > \lambda$. And from the definition of α we know $\gamma = \alpha$ and $\varphi(\alpha) = \varphi(\gamma) = 0$.

(ii) We now prove that, for all $n \geq 0$ and each $x \in [0,1]$, we have

$$f^n(\varphi(x)) = \varphi^{p^n}(f^n(x)). \tag{A.3}$$

Obviously, (A.3) holds for $n = 1$ by (4). Suppose that (A.3) holds for $n \leq k$, where k is a certain integer. Therefore, by induction and (4), we have that

$$\begin{aligned}
&\varphi^{p^{k+1}}\left(f^{k+1}(x)\right)\\
&\quad = \left(\varphi^{p^k}\right)^p\left(f^{k+1}(x)\right) = \left(\varphi^{p^k}\right)^{p-1} \circ \varphi^{p^k}\left(f^{k+1}(x)\right)\\
&\quad = \left(\varphi^{p^k}\right)^{p-1}\left(f^k(\varphi(f(x)))\right)\\
&\quad = \left(\varphi^{p^k}\right)^{p-2} \circ \varphi^{p^k}\left(f^k(\varphi(f(x)))\right)\\
&\quad = \left(\varphi^{p^k}\right)^{p-2}\left(f^k\left(\varphi^2(f(x))\right)\right) = \cdots\\
&\quad = \left(\varphi^{p^k}\right)^{p-i}\left(f^k\left(\varphi^i(f(x))\right)\right) = \cdots\\
&\quad = f^k\left(\varphi^p(f(x))\right) = f^k(f(\varphi(x))) = f^{k+1}(\varphi(x));
\end{aligned} \tag{A.4}$$

that is, (A.3) holds for $n = k + 1$. Thereby, (A.3) is proved by induction. Let $x = 0$ in (A.3); we have that

$$f^n(1) = f^n(\varphi(0)) = \varphi^{p^n}(f^n(0)) = \varphi^{p^n}(0). \tag{A.5}$$

And it is trivial that $\{f^n(1)\}$ is strictly decreasing and $\lim_{n\to+\infty} f^n(1) = 0$. Thereby, we have that

$$\lim_{n\to+\infty}\varphi^{p^n}(0) = \lim_{n\to+\infty} f^n(1) = 0; \tag{A.6}$$

that is, we proved that 0 is a recurrent but not periodic point of φ.

(iii) Firstly, we prove that λ is an extreme point of φ. By the fact that φ is strictly increasing on $[\alpha, 1]$ and (4), we have that φ^p is strictly increasing in $[f(\alpha), f(1)]$. Thereby φ is strictly monotone in $[f(\alpha), f(1)]$. Suppose that λ is not an extreme point; then φ is strictly decreasing on $[f(\alpha), \alpha]$. We prove φ is strictly decreasing on Δ_k^α $(k \geq 0)$, respectively, as follows.

Obviously, φ is strictly decreasing on Δ_0^α. Suppose that φ is strictly decreasing on Δ_k^α for all $k \leq m$, where $m \geq 0$ is a certain integer. By (4), we have that φ^p is strictly decreasing on Δ_{m+1}^α. Thereby φ is strictly monotone on Δ_{m+1}^α. Suppose that φ is strictly increasing on Δ_{m+1}^α; then we claim that

$$\varphi(f^n(\alpha)) < \varphi\left(f^{m+1}(\alpha)\right), \quad \forall n \geq m+2. \tag{A.7}$$

We first prove that

$$\varphi(f^s(\alpha)) \neq \varphi\left(f^l(\alpha)\right), \quad \forall s \neq l. \tag{A.8}$$

Suppose that there exists $s \neq l$ such that $\varphi(f^s(\alpha)) = \varphi(f^l(\alpha)) = \delta$. And from (A.3) we get

$$\varphi^{p^s-1}(\delta) = \varphi^{p^s}(f^s(\alpha)) = f^s(\varphi(\alpha)) = f^s(0) = 0. \tag{A.9}$$

Similarly, we have $\varphi^{p^l-1}(\delta) = 0$. If $s > l$, then

$$0 = \varphi^{p^s-1}(\delta) = \varphi^{(p^s-p^l)+(p^l-1)}(\delta) = \varphi^{(p^s-p^l)}(0). \tag{A.10}$$

This contradicts that 0 is not a periodic point. That is, we proved (A.8). Thereby, we have $\varphi(f^n(\alpha)) \neq \varphi(f^{m+1}(\alpha))$, $\forall n \geq m+2$. And from the fact that φ is strictly increasing on Δ_{m+1}^{α}, we get $\varphi(f^{m+2}(\alpha)) < \varphi(f^{m+1}(\alpha))$. Suppose that $\varphi(f^n(\alpha)) < \varphi(f^{m+1}(\alpha))$, where $n \geq m+2$ is a certain integer. If $\varphi(f^{n+1}(\alpha)) > \varphi(f^{m+1}(\alpha)) > \varphi(f^n(\alpha))$, and by the fact that $\varphi^{p^{m+1}-1}$ is strictly monotone on $\varphi([f^{n+1}(\alpha), f^n(\alpha)])$ and (A.3), we get

$$\begin{aligned}
&\varphi^{p^{m+1}-1}\left(\varphi\left(f^{n+1}(\alpha)\right)\right)\\
&\quad< \varphi^{p^{m+1}-1}\left(\varphi\left(f^{m+1}(\alpha)\right)\right)\\
&\quad= \varphi^{p^{m+1}}\left(f^{m+1}(\alpha)\right) = f^{m+1}(0) = 0,
\end{aligned} \tag{A.11}$$

or

$$\begin{aligned}
&\varphi^{p^{m+1}-1}\left(\varphi\left(f^{n}(\alpha)\right)\right)\\
&\quad< \varphi^{p^{m+1}-1}\left(\varphi\left(f^{m+1}(\alpha)\right)\right)\\
&\quad= \varphi^{p^{m+1}}\left(f^{m+1}(\alpha)\right) = f^{m+1}(0) = 0.
\end{aligned} \tag{A.12}$$

This contradicts that $\varphi^{p^{m+1}-1}(x) \geq 0$. That is, $\varphi(f^{n+1}(\alpha)) < \varphi(f^{m+1}(\alpha))$. Thereby, we proved (A.7) by induction. If $\varphi(f^{m+1}(\alpha)) = 1$, then by (A.3) we have

$$\begin{aligned}
\varphi^{p^{m+1}}(1) &= \varphi^{p^{m+1}}\left(\varphi\left(f^{m+1}(\alpha)\right)\right) = \varphi\left(\varphi^{p^{m+1}}\left(f^{m+1}(\alpha)\right)\right)\\
&= \varphi\left(f^{m+1}(\varphi(\alpha))\right) = \varphi(0) = 1.
\end{aligned} \tag{A.13}$$

This contradicts conclusion (ii). Thereby, we get $\varphi(f^n(\alpha)) < \varphi(f^{m+1}(\alpha)) < 1$, $\forall n \geq m+2$. This contradicts that $\varphi(0) = 1$. Thereby, φ is strictly decreasing on Δ_{m+1}^{α}. Furthermore, we proved that φ is strictly decreasing on Δ_k^{α} $(k \geq 0)$, respectively, by induction; that is, φ is single-valley solution of (4). This contradicts the condition that φ is a non-single-valley solution of (4). That is, we proved that λ is an extreme point of φ.

Secondly, we prove that $\varphi(\lambda) > \lambda$. We claim that

$$\varphi\left(f^n(1)\right) \neq \lambda, \quad \varphi\left(f^n(\alpha)\right) \neq \lambda, \quad \forall n \geq 1. \tag{A.14}$$

Suppose that $\varphi(f^n(1)) = \lambda$; by (A.5) we get $\varphi(f^n(1)) = \varphi^{p^n+1}(0)$. And from $\varphi^p(0) = \lambda$, we have that $\varphi^p(0) = \varphi^{p^n+1}(0)$. Let

$$L = \left\{0, \varphi(0), \ldots, \varphi^p(0), \varphi^{p+1}(0), \ldots, \varphi^{p^n}(0)\right\}; \tag{A.15}$$

obviously, L is a limited set and $\varphi^i(0) \in L$, $\forall i \in Z^+$. By (A.5), we have that $f^i(1) = \varphi^{p^i}(0) \in L$, $\forall i \in Z^+$. This contradicts that L is limited. Thereby, we proved that $\varphi(f^n(1)) \neq \lambda$. Suppose that $\varphi(f^n(\alpha)) = \lambda$; from (A.3), we have

$$\begin{aligned}
0 &= f^n(\varphi(\alpha)) = \varphi^{p^n}\left(f^n(\alpha)\right)\\
&= \varphi^{p^n-1}(\lambda) = \varphi^{p^n+p-1}(0).
\end{aligned} \tag{A.16}$$

This contradicts that 0 is not a periodic point of φ. That is, (A.14) holds. Suppose that $\varphi(\lambda) < \lambda$. Since λ is an extreme point of φ, we have that φ is strictly increasing on $[f(\alpha), \lambda]$. Furthermore, we have $\varphi(f(\alpha)) < \varphi(\lambda) < \lambda$. We claim that

$$\varphi\left(f^n(1)\right) < \lambda, \quad \varphi\left(f^n(\alpha)\right) < \lambda, \quad \forall n \geq 1. \tag{A.17}$$

It is trivial that (A.17) holds for $n = 1$. Suppose that (A.17) holds for $n = m$, where m is a certain integer. By (A.3), we know φ^{p^m} is strictly monotone on $[f^{m+1}(1), f^m(\alpha)]$. If $\varphi(f^{m+1}(1)) > \lambda$, then there exists $x_0 \in (f^{m+1}(1), f^m(\alpha))$ such that $\varphi(x_0) = \lambda$. Thus, x_0 is an extreme point of φ^2 on $(f^{m+1}(1), f^m(\alpha))$. That is, φ^2 is not monotone on $[f^{m+1}(1), f^m(\alpha)]$. Furthermore, φ^{p^m} is not monotone on $[f^{m+1}(1), f^m(\alpha)]$. This contradicts that φ^{p^m} is monotone on $[f^{m+1}(1), f^m(\alpha)]$. Thus $\varphi(f^{m+1}(1)) < \lambda$. Similarly, we have $\varphi(f^{m+1}(\alpha)) < \lambda$. That is, (A.17) holds for $n = m+1$. Thereby, (A.17) holds by induction. This contradicts that $\varphi(0) = 1$. Thus $\varphi(\lambda) > \lambda$.

(iv) Firstly, suppose that q is a fixed point of φ; then by (A.5), we have $q \neq 1$. And by $\varphi(\alpha) = 0$, we have $q \neq \alpha$. If $q \in (\alpha, 1)$, then $q = \varphi(q) < \varphi(1)$. And by induction, for all $m \geq 0$, we have $q = \varphi^m(q) < \varphi^m(1)$. Specially,

$$q = \varphi^{p^n-1}(q) < \varphi^{p^n-1}(1) = \varphi^{p^n-1}(\varphi(0)) = \varphi^{p^n}(0). \tag{A.18}$$

This contradicts (A.5). Thereby, $q < \alpha$.

Secondly, there exists at least one fixed point $\beta \in (0, \alpha)$ by $\varphi(0) = 1$, $\varphi(\alpha) = 0$. We prove β is the unique fixed point as follows. We claim that

$$\beta \notin \left(f^{n+1}(1), f^n(1)\right], \quad \forall n \geq 1. \tag{A.19}$$

Suppose that there exists $m \geq 1$ such that $\beta \in [f^m(\alpha), f^m(1)]$; then by (A.3), we have

$$f^m\left(\varphi\left(f^{-m}(\beta)\right)\right) = \varphi^{p^m}\left(f^m\left(f^{-m}(\beta)\right)\right) = \varphi^{p^m}(\beta) = \beta; \tag{A.20}$$

that is, $\varphi(f^{-m}(\beta)) = f^{-m}(\beta)$. And from $f^{-m}(\beta) \in [\alpha, 1]$, this contradicts that φ does not have a fixed point on $[\alpha, 1]$. Thus, we have $\beta \notin [f^n(\alpha), f^n(1)]$, $\forall n \geq 1$. Suppose that there exists $m \geq 1$ such that $\beta \in (f^{m+1}(1), f^m(\alpha))$; then by (A.3), we have

$$\begin{aligned}
f^{m-1}\left(\varphi\left(f^{1-m}(\beta)\right)\right) &= \varphi^{p^m-1}\left(f^{m-1}\left(f^{1-m}(\beta)\right)\right)\\
&= \varphi^{p^m-1}(\beta) = \beta.
\end{aligned} \tag{A.21}$$

Thereby, $\varphi(f^{1-m}(\beta)) = f^{1-m}(\beta)$. That is, $f^{1-m}(\beta) \in (f^2(1), f(\alpha))$ is a fixed point of φ. Since φ has no fixed point on $[f(\alpha), f(1)]$ and $\varphi(\lambda) > \lambda$, we have that φ must be strictly increasing on $[f^2(1), f(\alpha)]$. And φ has the fixed point $f^{1-m}(\beta) \in (f^2(1), f(\alpha))$; thus φ^p is strictly increasing on $[f^2(1), f(\alpha)]$. But by (4) and the fact that φ is decreasing on $[\lambda, \alpha]$, we have that φ^p is strictly decreasing on $[f^2(1), f(\alpha)]$. This contradicts. Thus, we have $\beta \notin [f^{n+1}(1), f^n(\alpha)]$, $\forall n \geq 1$. Thereby, we prove (A.19). Thus, we have $\beta \in (\lambda, \alpha)$. And since φ is decreasing on $[\lambda, \alpha]$, we have that $\beta \in (\lambda, \alpha)$ is the unique fixed point.

(v) By (4), we have $0 = f(\varphi(\alpha)) = \varphi^p(f(\alpha))$. And since α is the unique minimum point of φ, it follows that $\varphi^{p-1}(f(\alpha)) = \alpha$. Thus the sufficiency is proved.

We prove the necessity as follows. Suppose that $\varphi^i(x) = \alpha$ for some $x \in [0, \lambda]$ and $0 \le i \le p-1$. Firstly, we claim that

$$x \notin \left(f^{n+1}(1), f^n(\alpha)\right) \cup \left(f^n(\alpha), f^n(1)\right), \quad \forall n \ge 1. \tag{A.22}$$

Suppose that $x \in (f^n(\alpha), f^n(1))$; then φ^{i+1} is not monotone on $(f^n(\alpha), f^n(1))$. Thereby, φ^p is not monotone on $(f^n(\alpha), f^n(1))$. This contradicts that φ^p is monotone on $(f^n(\alpha), f^n(1))$ by (A.3). Thus $x \notin (f^n(\alpha), f^n(1))$. We have similarly that $x \notin (f^{n+1}(1), f^n(\alpha))$. Thereby we prove (A.22); that is, $x = f^n(1)$ or $x = f^n(\alpha)$, $\forall n \ge 1$.

Secondly, we prove that

$$x \ne f^n(1), \quad x \ne f^{n+1}(\alpha), \quad \forall n \ge 1. \tag{A.23}$$

Suppose that $x = f^n(1)$ for some $n \ge 1$; then $\varphi^i(f^n(1)) = \alpha$. And from (A.5), we have $\alpha = \varphi^i(f^n(1)) = \varphi^i(\varphi^{p^n}(0)) = \varphi^{i+p^n}(0)$. This contradicts that 0 is not a periodic point of φ. Thus we prove $x \ne f^n(1), \forall n \ge 1$. Suppose that $x = f^n(\alpha)$ for some $n \ge 2$; then $\varphi^i(f^n(\alpha)) = \alpha$. And from (A.3), we have

$$\begin{aligned} \varphi^i(0) &= \varphi^i\left(\varphi^{p^n}(f^n(\alpha))\right) = \varphi^{p^n}\left(\varphi^i(f^n(\alpha))\right) \\ &= \varphi^{p^n}(\alpha) = \varphi^{p^n-1}(0). \end{aligned} \tag{A.24}$$

This contradicts that 0 is a recurrent point of φ. Thus we prove $x \ne f^n(\alpha)$, $\forall n \ge 2$. Thereby (A.23) holds. That is, we get $x = f(\alpha)$ and $\varphi^i(f(\alpha)) = \alpha$.

Lastly, we prove $i = p-1$. Suppose that there exists $j \ne i$, $0 \le j \le p-1$, such that $\varphi^j(f(\alpha)) = \alpha$. Then we can suppose that $j < i$ and

$$0 = \varphi^{i+1}(f(\alpha)) = \varphi^{i-j}\left(\varphi^{j+1}(f(\alpha))\right) = \varphi^{i-j}(0). \tag{A.25}$$

This contradicts that 0 is not a periodic point of φ. Thus we prove $\varphi^j(f(\alpha)) \ne \alpha$, $\forall j \ne i$. Thereby we have $i = p-1$ and $x = f(\alpha)$. □

Proof of Lemma 4. (i) Firstly, we prove that, for all $i = 0, 1, \ldots, p-2$, we have $J_i \subset (\lambda, 1]$; that is, $J_i \cap J = \emptyset$. We claim that

$$\varphi^{i+1}(f(\alpha)) > \lambda, \quad \forall 0 \le i \le p-2. \tag{A.26}$$

Suppose that there exists $1 \le j \le p-1$ such that $\varphi^j(f(\alpha)) = x \le \lambda$. And from Lemma 3(v) we have $\alpha = \varphi^{p-1}(f(\alpha)) = \varphi^{p-1-j}(\varphi^j(f(\alpha))) = \varphi^{p-1-j}(x)$. This contradicts Lemma 3(v). Thus (A.26) holds. We claim that

$$\varphi^{i+1}(\lambda) > \lambda, \quad \forall 0 \le i \le p-2. \tag{A.27}$$

It is trivial that $\varphi^{i+1}(\lambda) \ne \lambda$ by 0 is a recurrent point of φ. Suppose that there exists $1 \le j \le p-1$ such that $\varphi^j(\lambda) < \lambda$. And from $\varphi^j(f(\alpha)) > \lambda$ we have that there exists $x \in (f(\alpha), f(1))$ such that $\varphi^j(x) = \lambda$. And from the fact that λ is an extreme point of φ, we get that φ^{j+1} is not monotone on $[f(\alpha), f(1)]$. This contradicts that φ^p is strictly monotone on $[f(\alpha), f(1)]$. Thus, (A.27) holds. We claim that

$$\varphi^{i+1}(x) > \lambda, \quad \forall 0 \le i \le p-2, \ x \in [0, \lambda]. \tag{A.28}$$

Suppose that there exists $x \in [0, \lambda)$ and $1 \le j \le p-1$ such that $\varphi^j(x) = \lambda$. And from (4) we get $\varphi^{p-j}(\lambda) = \varphi^p(x) = f(\varphi(f^{-1}(x))) \le f(1) = \lambda$. This contradicts (A.27). Thus $\varphi^{i+1}(x) \ne \lambda$, $\forall 0 \le i \le p-2$, $x \in [0, \lambda]$. Suppose that there exists $x \in [0, \lambda)$ and $1 \le j \le p-1$ such that $\varphi^j(x) < \lambda$. And from (A.27) we have that there exists $z \in (x, \lambda)$ such that $\varphi^j(z) = \lambda$. And from (4) we get $\varphi^{p-j}(\lambda) = \varphi^p(z) = f(\varphi(f^{-1}(z))) \le f(1) = \lambda$. This contradicts (A.27). Thus (A.28) holds. That is, $J_i \cap J = \emptyset$.

Secondly, we prove that J_i, $\forall 0 \le i \le p-2$, are pairwise disjoint. Suppose that there exists $0 \le i < j \le p-2$, such that $J_i \cap J_j = J_{ij} \ne \emptyset$. Let $y \in J_{ij}$; then there exist $x_i \in [0, \lambda], x_j \in [0, \lambda]$, such that $\varphi^{i+1}(x_i) = y = \varphi^{j+1}(x_j)$. Thereby we have $\varphi^{p-j+i}(x_i) = \varphi^{p-1-j}(\varphi^{i+1}(x_i)) = \varphi^{p-1-j}(\varphi^{j+1}(x_j)) = \varphi^p(x_j) = f(\varphi(f^{-1}(x_j))) \le f(1) = \lambda$. This contradicts $J_i \subset (\lambda, 1]$. Thus we proved that $J_0, J_1, \ldots, J_{p-2}$ are pairwise disjoint.

(ii) For all $i = 0, 1, \ldots, p-2$, then $\varphi^{i+1} : J \mapsto J_i$ is a homeomorphism by Lemmas 3(i) and 3(v). Thereby $\varphi^i : J_0 \mapsto J_i$ is also a homeomorphism. □

Proof of Lemma 5. λ is a maximum point of φ on $[f(\alpha), \alpha]$ by Lemma 3(iii). Suppose that $f^m(1)$ is the maximum point of φ on $[f^m(\alpha), \alpha]$, where m is some certain integer. Firstly, we prove $f^m(\alpha)$ is an extreme point of φ. If otherwise, φ is strictly increasing on $[f^{m+1}(1), f^m(\alpha)]$. Thus $\varphi(f^{m+1}(1)) < \varphi(f^m(\alpha))$ and φ is strictly increasing on Δ_m. We claim that φ is strictly increasing on Δ_k for all $k \ge m$. It holds obviously when $k = m$. Suppose that φ is strictly increasing on Δ_k for all $m \le k \le l$, where l is some certain integer. φ^p is strictly increasing on Δ_{l+1} by (4). Thereby φ is strictly monotone on Δ_{l+1}. If φ is strictly decreasing on Δ_{l+1}, then we claim that

$$\varphi(f^s(1)) < \varphi\left(f^l(\alpha)\right), \quad \forall s \ge l+1. \tag{A.29}$$

Its proof is similar to (A.7) and we omit it here. Thereby $\varphi(f^s(1)) < \varphi(f^l(\alpha)) < 1$, $\forall s \ge l+1$. This contradicts $\varphi(0) = 1$. Thus φ is strictly increasing on Δ_{l+1}. Thereby φ is strictly increasing on $[0, f^m(1)]$ by induction. Furthermore $\varphi(0) < \varphi(f^m(1)) < 1$. This contradicts $\varphi(0) = 1$. Thus $f^m(\alpha)$ is an extreme point of φ.

Secondly, we prove $f^m(\alpha)$ is the minimum point of φ on $[0, f^m(1)]$. Suppose that there exists $l \ge m+1$ such that $\varphi(f^l(\alpha)) < \varphi(f^m(\alpha))$ or $\varphi(f^l(1)) < \varphi(f^m(\alpha))$; then we claim that

$$\varphi(f^s(\alpha)) < \varphi(f^m(\alpha)), \quad \forall s \ge l, \tag{A.30}$$

or

$$\varphi(f^s(1)) < \varphi(f^m(\alpha)), \quad \forall s \ge l. \tag{A.31}$$

The proof is similar to (A.7) and we omit it here. Thereby $\varphi(0) < \varphi(f^m(\alpha)) < 1$. This contradicts $\varphi(0) = 1$. Thus $f^m(\alpha)$ is the minimum point of φ on $[0, f^m(1)]$.

Thirdly, we prove $f^{m+1}(1)$ is an extreme point of φ. φ^{p-1} is strictly increasing on $[\varphi(f^m(\alpha)), \varphi(f^{m+1}(1))]$ by (4) and φ is strictly decreasing on $[f^{m+1}(1), f^m(\alpha)]$. And from Lemma 4(ii) we have that φ^{p-1} is strictly increasing on $J_0 = [\varphi(f(\alpha)), 1]$. Suppose that $f^{m+1}(1)$ is not an extreme point of φ; then φ is strictly decreasing on $[f^{m+1}(\alpha), f^{m+1}(1)]$. And from (4) and the fact that φ is strictly increasing on $[f^m(\alpha), f^m(1)]$, we get that φ^p is strictly increasing on $[f^{m+1}(\alpha), f^{m+1}(1)]$. Thereby φ^{p-1} is strictly decreasing on $[\varphi(f^{m+1}(1)), \varphi(f^{m+1}(\alpha))]$. This contradicts that φ^{p-1} is strictly increasing on $J_0 = [\varphi(f(\alpha)), 1]$. Thus $f^{m+1}(1)$ is the extreme point of φ.

Lastly, we have that $f^{m+1}(1)$ is the maximum point of φ on $[f^{m+1}(\alpha), \alpha]$. The proof is similar to $f^m(\alpha)$ and we omit it here. Thereby $f^n(1)$ is the maximum point of φ on $[f^n(\alpha), \alpha]$ and $f^n(\alpha)$ is the minimum point of φ on $[0, f^n(1)]$ for all $n \geq 1$ by induction. □

Proof of Lemma 6. It is trivial that $x = 1$ is a solution of the equation $\varphi^{p-1}(x) = f(x)$ by (4). Suppose that $x = x_0 \in (\varphi(f(\alpha)), 1]$ is an arbitrary solution of this equation; that is, $\varphi^{p-1}(x_0) = f(x_0)$. Since $\varphi([0, f(\alpha)]) = [\varphi(f(\alpha)), 1]$, we have that there exists $y \in [0, f(\alpha)]$, such that $\varphi(y) = x_0$. We claim that

$$\varphi\left(f^n\left(y\right)\right) = x_0, \quad \forall n \geq 0. \tag{A.32}$$

Obviously, (A.32) holds for $n = 0$. Suppose that (A.32) holds for $n = k$, where $k \geq 0$ is a certain integer. Therefore, by (4) we have that

$$\varphi^{p-1}\left(\varphi\left(f^{k+1}\left(y\right)\right)\right) = f\left(\varphi\left(f^k\left(y\right)\right)\right) = f\left(x_0\right) = \varphi^{p-1}\left(x_0\right). \tag{A.33}$$

Since $f^{k+1}(y) \in [0, f(\alpha))$, we have $\varphi(f^{k+1}(y)) \in (\varphi(f(\alpha)), 1]$. And since φ^{p-1} is strictly increasing on $[\varphi(f(\alpha)), 1]$ by $\varphi(f(\alpha)) > \alpha$, we have $\varphi(f^{k+1}(y)) = x_0$. That is, (A.32) holds for $n = k + 1$. Thereby, (A.32) is proved by induction. By the fact that $\{f^n(y)\}$ is strictly decreasing and $\lim_{n\to\infty} f^n(y) = 0$, we have $x_0 = \lim_{n\to+\infty}\varphi(f^n(y)) = \varphi(0) = 1$. □

Proof of Lemma 7. There exist $\alpha \in (\lambda, 1)$, $\beta \in (\lambda, 1)$ such that $\varphi_i(\alpha) = 0$, $\varphi_i(\beta) = \beta\,(i = 1, 2)$ by (8). Denote $\varphi_0(x) = \varphi_1(x) = \varphi_2(x)\,(x \in [\lambda, 1])$ and $\alpha_1 = \varphi_1(f(\alpha)) < \varphi_1(\lambda) = \varphi_0(\lambda)$, $\alpha_2 = \varphi_2(f(\alpha)) < \varphi_2(\lambda) = \varphi_0(\lambda)$. We prove that $\alpha_1 = \alpha_2$ as follows. It is trivial that $\alpha_1 > \lambda$, $\alpha_2 > \lambda$, and

$$\begin{aligned} \varphi_0^{p-1}\left(\alpha_1\right) &= \varphi_1^{p-1}\left(\alpha_1\right) = \varphi_1^{p}\left(f\left(\alpha\right)\right) = 0, \\ \varphi_0^{p-1}\left(\alpha_2\right) &= \varphi_2^{p-1}\left(\alpha_2\right) = \varphi_2^{p}\left(f\left(\alpha\right)\right) = 0, \end{aligned} \tag{A.34}$$

by Lemmas 3(v) and 4. Since φ_0^{p-1} is strictly monotone on $[\min\{\alpha_1, \alpha_2\}, 1]$, we have that $\alpha_1 = \alpha_2$. Let $\alpha_0 = \alpha_1 = \alpha_2$; then $\alpha_0 < \varphi_0(\lambda)$ and $\varphi_0^{p-1}(\alpha_0) = 0$. Define $\psi_+ = \varphi_0^{p-1}|_{[\alpha_0,1]}$. By Lemmas 3(v) and 4, we have that ψ_+ is a homeomorphism. And by $\psi_+(\alpha_0) = 0 \leq \psi_+(1)$, we know that ψ_+ is strictly increasing.

We prove $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \geq 0$ by induction as follows.

Obviously, $\varphi_1(x) = \varphi_2(x)$ holds on Δ_0. Suppose that $\varphi_1(x) = \varphi_2(x)$ holds on Δ_k for all $k \leq m$, where $m \geq 0$ is a certain integer. Let

$$\varphi\left(x\right) = \varphi_1\left(x\right) = \varphi_2\left(x\right), \quad x \in \left[f^{m+1}\left(1\right), 1\right]. \tag{A.35}$$

Since $\varphi_i(x) \geq \varphi_i(f(\alpha)) = \alpha_0 > \lambda$ for $x \leq \lambda$ and from (4) we have

$$\begin{aligned} f\left(\varphi\left(f^{-1}\left(x\right)\right)\right) &= f\left(\varphi_i\left(f^{-1}\left(x\right)\right)\right) = \varphi_i^{p-1}\left(\varphi_i\left(x\right)\right) \\ &= \psi_+\left(\varphi_i\left(x\right)\right), \quad \left(i = 1, 2, x \in \Delta_{m+1}\right). \end{aligned} \tag{A.36}$$

Thereby

$$\varphi_i\left(x\right) = \psi_+^{-1}\left(f\left(\varphi\left(f^{-1}\left(x\right)\right)\right)\right), \quad \left(i = 1, 2, x \in \Delta_{m+1}\right). \tag{A.37}$$

Thus we have $\varphi_1(x) = \varphi_2(x)$ on Δ_{m+1}. Thus we get $\varphi_1(x) = \varphi_2(x)$ on Δ_k for all $k \geq 0$ by induction. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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