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Research Article

Boundary Stabilization of a Semilinear Wave Equation with Variable Coefficients under the Time-Varying and Nonlinear Feedback

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We study the boundary stabilization of a semilinear wave equation with variable coefficients under the time-varying and nonlinear feedback. By the Riemannian geometry methods, we obtain the stability results of the system under suitable assumptions of the bound of the time-varying term and the nonlinearity of the nonlinear term.

1. Introduction

Many results concerning the boundary stabilization of classical wave equations are available in literatures. See [1–6] for linear cases and [7–14] for nonlinear ones. The stability of a nondissipative system described by partial differential equations (PDEs) has attracted much attention. Reference [15] developed the exponential stability for an abstract nondissipative linear system, and in [16], the Riesz basis property was developed for a beam equation with nondissipativity.

In [17], the following semilinear wave equation was considered:

$$\begin{gathered} u_{tt} - \Delta_g u + h(\nabla u) + f(u) = 0 \qquad (x,t) \in \Omega \times (0,+\infty), \\ u(x,t)|_{\Gamma_2} = 0 \quad t \in (0,+\infty), \\ \frac{\partial u(x,t)}{\partial \mu} + l(u_t) = 0 \qquad (x,t) \in \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega \end{gathered} \tag{1}$$

and the well-posedness and uniform decay of the energy of the system (1) was also established with linearly bounded $l(u)$ in [17].

Based on [17], we study the system (1) with time-varying and nonlinear feedback:

$$\frac{\partial u(x,t)}{\partial \mu} + \phi(t)\, l(u) = 0 \quad (x,t) \in \Gamma_1 \times (0,+\infty). \tag{2}$$

The decay rate of the energy (when t goes to infinity) of the wave equation with time-varying feedback was established under the assumption ϕ is decreasing [18–20] or ϕ has an upper bound [21].

In this paper, we consider the decay rate of the energy under suitable assumptions of the bound of the time-varying term $\phi(t)$ and the nonlinearity of the nonlinear term $l(u)$.

2. Some Notation

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary Γ. It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$.

Let $A(x) = (a_{ij}(x))$ be symmetric, positively definite matrices for each $x \in \mathbb{R}^n$, and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$. As in [22], we define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \tag{3}$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold with an inner product:

$$\langle X, Y\rangle_g = \left\langle A^{-1}(x) X, Y\right\rangle, \qquad |X|_g^2 = \langle X, X\rangle_g \quad X, Y \in \mathbb{R}_x^n. \tag{4}$$

Denote by D, ∇_g, div_g, and Δ_g the Levi-Civita connection, the gradient operator, the divergence operator, and the Beltrami-Laplace operator in terms of the Riemannian metric g, respectively. It can be easily shown that, under the Euclidean coordinate,

$$\begin{gathered}\nabla_g f = \sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j} f\right)\frac{\partial}{\partial x_i} = A(x)\nabla f,\\ \left|\nabla_g u\right|_g^2 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \quad x \in \mathbb{R}^n,\\ \Delta_g f = \frac{1}{\sqrt{G}}\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sqrt{G}\sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j} f\right), \quad x \in \mathbb{R}^n,\end{gathered} \tag{5}$$

where ∇f is the gradient of f in the standard metric and $G = \det(g)$.

Let H be a vector field on $(\mathbb{R}_x^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DH of H determines a bilinear form on $\mathbb{R}_x^n$:

$$DH(X, Y) = \left\langle D_Y H, X\right\rangle_g \quad \forall X, Y \in \mathbb{R}_x^n, \tag{6}$$

where $D_Y H$ stands for the covariant derivative of the vector field H with respect to Y.

3. The Main Results

We consider the semilinear wave equation with variable coefficients under the time-varying and nonlinear boundary feedback:

$$\begin{gathered} u_{tt} - \Delta_g u + f(u) = 0 \qquad (x,t) \in \Omega \times (0, +\infty),\\ u(x,t)|_{\Gamma_2} = 0 \quad t \in (0, +\infty),\\ \frac{\partial u(x,t)}{\partial \mu} + \phi(t)\, l(u_t) = 0 \quad (x,t) \in \Gamma_1 \times (0, +\infty),\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \end{gathered} \tag{7}$$

where l, f are continuous nonlinear functions and $\mu(x)$ is the outside unit normal vector of the Riemannian manifold (Ω, g) for each $x \in \Gamma$. Different from [18–21], in this paper, we consider a general ϕ; that is, $\phi \in C^1([0, +\infty))$ satisfies

$$\frac{1}{\Phi(t)} \le \phi \le \Phi(t) \quad \forall t \ge 0, \tag{8}$$

where $\Phi(t) \in C([0, +\infty))$ is a positive and nondecreasing function satisfying

$$\lim_{t\to+\infty}\frac{\Phi(t)}{t} = 0. \tag{9}$$

Let $\Phi'(t) \in C([0, +\infty))$ be a positive and nondecreasing function with 0 as the limit. Then $t\Phi'(t)$ satisfies (9). There are many examples of $\Phi'(t)$ such as $(1+t)^\alpha (\alpha < 0)$ and $e^{\beta t} (\beta < 0)$.

The main assumptions are listed as follows.

Assumption A. $f \in C^1(\mathbb{R})$, $f(0) = 0$ derives from a potential F:

$$F(s) = \int_0^s f(\tau)\, d\tau \ge 0 \quad \forall s \in \mathbb{R}, \tag{10}$$

and satisfies

$$\left|f'(s)\right| \le b_1|s|^\rho + b_2 \quad \forall s \in \mathbb{R}, \tag{11}$$

where b_1, b_2 are positive constants, and the parameter ρ satisfies

$$1 \le \rho \le \begin{cases} 2, & n = 2,\\ \dfrac{n}{n-2}, & n \ge 3. \end{cases} \tag{12}$$

Being different from [17], we assume the nonlinear term $l(u)$ has no growth restriction near zero as in [23, 24].

Assumption B. $l \in C^1(\mathbb{R})$ is a nondecreasing function satisfying

$$l(0) = 0, \qquad c_1|s|^2 \le s\, l(s) \le c_2|s|^2 \quad \forall |s| \ge 1. \tag{13}$$

Assumption C. There exists a vector field H on $\overline{\Omega}$ such that

$$DH(X, X) = c(x)|X|_g^2 \quad \text{for } X \in \mathbb{R}_x^n\ x \in \overline{\Omega}, \tag{14}$$

where $b = \min_{\overline{\Omega}} c(x)$ and $B = \max_{\overline{\Omega}} c(x)$

$$B < \min\left\{b + \frac{2b}{n}, rb\right\}, \tag{15}$$

where $r > 1$ is a constant. Moreover we assume that

$$\langle H, \mu\rangle_g \le 0 \quad x \in \Gamma_2, \qquad \langle H, \mu\rangle_g \ge 0 \quad x \in \Gamma_1. \tag{16}$$

Condition (14) as a checkable assumption is very useful to study the control and stabilization of the wave equation with variable coefficients and the quasilinear wave equation [22, 25]. For the examples of the condition, see [22, 26].

Based on condition (14), Assumption C was given by [17] to study the stabilization of the wave equation with variable coefficients and nonlinear boundary condition. Being different from [17], the lower bound of $\langle H, \mu\rangle_g$ was relaxed on Γ_1 from a positive constant to zero.

To facilitate the writing, we denote the volume element of (Ω, g) by dx and denote the volume element of (Γ, g) by $d\Gamma$. Define the energy of the system (7) by

$$E(t) = \int_\Omega \left(u_t^2 + \left|\nabla_g u\right|_g^2 + 2F(u)\right) dx. \tag{17}$$

As in [23, 24], we let $h \in C([0, +\infty))$ be a concave increasing function such that

$$h(0) = 0, \qquad s^2 + (g(s))^2 \le h(sg(s)) \quad \text{for } |s| \le 1. \tag{18}$$

With (18), the stabilization of the wave equation with variable coefficients and time dependent delay was studied by [27].

The main result of this paper is as follows.

Theorem 1. *Let Assumptions A–C hold true. Assume that*

$$2rF(s) \le sf(s) \quad \forall s \in \mathbb{R}, \tag{19}$$

where r is defined in (15).

(a) If the function l in (7) *satisfies*

$$c_1|s|^2 \le sl(s) \le c_2|s|^2 \quad \forall |s| < 1, \tag{20}$$

then there exist constants $C > 0$ such that

$$E(t) \le \frac{C\Phi(t)}{t} E(0) \quad t > 0. \tag{21}$$

(b) If the functions $\phi(t), l$ in (7) *satisfy*

$$\phi(t) \le \phi_0 \quad \forall t \ge 0, \qquad sl(s) \ge c_1|s|^2 \quad \forall |s| < 1, \tag{22}$$

where ϕ_0 is a positive constant, then there exist constants $C_1, C_2 > 0$ such that

$$C_1 h\left(\frac{C_2\Phi(T)}{T}E(0)\right) + \frac{C_1\Phi(T)}{T}E(0) \quad t > 0. \tag{23}$$

(c) If the function $\Phi(t)$ in (8) *is a constant function; that is,*

$$\Phi(t) = \Phi(0) \quad \forall t \ge 0, \tag{24}$$

then there exist constants $C_1, C_2 > 0$ such that

$$C_1 h\left(\frac{C_2 E(0)}{T}\right) + \frac{C_1}{T}E(0) \quad t > 0. \tag{25}$$

4. Well Posedness of the System

Define

$$H^1_{\Gamma_2}(\Omega) = \left\{u \in H^1 \mid (\Omega)\, u|_{\Gamma_2} = 0\right\}. \tag{26}$$

By a similar proof as Lemma 7.1 in [17], we have the following result.

Theorem 2. *Let Assumptions A-B hold true. For any initial data $(u_0, u_1) \in H^1_{\Gamma_2}(\Omega) \times L^2(\Omega)$, system* (7) *admits a unique weak solution u such that $u \in C([0,+\infty), H^1_{\Gamma_2}(\Omega)) \cap C^1([0,+\infty), L^2(\Omega))$.*

To prove Theorem 1, we still need several lemmas further. Define

$$E_0(t) = \int_\Omega \left(u_t^2 + \left|\nabla_g u\right|_g^2\right) dx. \tag{27}$$

Then, we have

$$E(t) = E_0(t) + 2\int_\Omega F(u)\, dx. \tag{28}$$

The following lemma shows the energy of the system (7) is decreasing.

Lemma 3. *Suppose that Assumptions A-B hold true. Let u be the solution of the system* (7). *Then*

$$E(0) - E(T) = 2\int_0^T \int_{\Gamma_1} \phi(t)\, u_t l(u_t)\, d\Gamma\, dt. \tag{29}$$

The assertion (29) *implies that $E(t)$ is decreasing.*

Proof. Differentiating (17), we obtain

$$\begin{aligned} E'(t) &= \int_\Omega \left(2u_t u_{tt} + 2\left\langle \nabla_g u, \nabla_g u_t\right\rangle_g + 2f(u)\right) dx \\ &= \int_{\Gamma_1} 2\phi(t)\, u_t l(u_t)\, d\Gamma. \end{aligned} \tag{30}$$

Then the inequality (29) follows directly from (30) integrating from 0 to T. □

5. Proofs of Theorem 1

Lemma 4. *Let $u(x,t)$ be the solution of the equation $u_{tt} + \Delta_g u + f(u) = 0, (x,t) \in \Omega \times (0,+\infty)$ and that $\mathscr{H}$ is a vector field defined on $\overline{\Omega}$. Then for $T \ge 0$*

$$\begin{aligned} &\int_0^T \int_\Gamma \frac{\partial u}{\partial \mu} \mathscr{H}(u)\, d\Gamma\, dt + \frac{1}{2}\int_0^T \int_\Gamma \left(u_t^2 - \left|\nabla_g u\right|_g^2 - 2F(u)\right) \\ &\qquad \times \langle \mathscr{H}, \mu\rangle_g d\Gamma\, dt \\ &\quad = (u_t, \mathscr{H}(u))|_0^T + \int_0^T \int_\Omega D\mathscr{H}\left(\nabla_g u, \nabla_g u\right) dx\, dt \\ &\qquad + \frac{1}{2}\int_0^T \int_\Omega \left(u_t^2 - \left|\nabla_g u\right|_g^2 - 2F(u)\right) \operatorname{div}_g \mathscr{H} dx\, dt. \end{aligned} \tag{31}$$

Moreover, assume that $P \in C^1(\overline{\Omega})$. Then

$$\begin{aligned} &\int_0^T \int_\Omega \left(u_t^2 - \left|\nabla_g u\right|_g^2 - uf(u)\right) P dx\, dt \\ &\quad = (u_t, uP)|_0^T + \frac{1}{2}\int_0^T \int_\Omega \nabla_g P\left(u^2\right) dx\, dt \\ &\qquad - \int_0^T \int_\Gamma P u \frac{\partial u}{\partial \mu} d\Gamma\, dt. \end{aligned} \tag{32}$$

Proof. Note that

$$\mathscr{H}(u) f(u) = \mathscr{H}(F(u)) = \operatorname{div}_g(F(u)\mathscr{H}) - F(u)\operatorname{div}_g \mathscr{H}. \tag{33}$$

The equality (31) and the equality (32) follow from Proposition 2.1 in [22]. □

Lemma 5. *Suppose that all assumptions in Theorem 1 hold true. Let u solve the system* (7). *Then there exist positive constants $\overline{T}, C$ for which*

$$E(T) \le \frac{C}{T}\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2\right) d\Gamma\, dt, \tag{34}$$

where $T \ge \overline{T}$.

Proof. From (15), we choose a positive constant θ satisfying

$$\theta < \frac{nb}{2}, \qquad b + \theta - \frac{nB}{2} > 0, \qquad 2r\theta > nB. \tag{35}$$

Set

$$\mathscr{H} = H, \qquad P = \theta. \tag{36}$$

We substitute the formula (32) into the formula (31), and we have

$$\begin{aligned}\Pi_\Gamma = &\left(u_t, H(u) + Pu\right)\big|_0^T \\ &+ \int_0^T \int_\Omega \left(DH\left(\nabla_g u, \nabla_g u\right) - b\left|\nabla_g u\right|_g^2\right) dx\,dt \\ &+ \int_0^T \int_\Omega \left(\left(\frac{1}{2}\operatorname{div} H - \theta\right) u_t^2 \right.\\ &\qquad \left. + \left(b + \theta - \frac{1}{2}\operatorname{div} H\right)\left|\nabla_g u\right|_g^2\right) dx\,dt \\ &+ \int_0^T \int_\Omega \left[\theta\left(uf(u) - 2rF(u)\right)\right.\\ &\qquad \left. + \left(2r\theta - \operatorname{div} H\right) F(u)\right] dx\,dt,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\Pi_\Gamma = &\int_0^T \int_\Gamma \frac{\partial u}{\partial \mu}\left(H(u) + uP\right) d\Gamma\,dt \\ &+ \frac{1}{2}\int_0^T \int_\Gamma \left(u_t^2 - \left|\nabla_g u\right|_g^2 - 2F(u)\right)\langle H, \mu\rangle_g\, d\Gamma\,dt.\end{aligned} \tag{38}$$

Decompose Π_Γ as

$$\Pi_\Gamma = \Pi_{\Gamma_1} + \Pi_{\Gamma_2}, \tag{39}$$

where $\Pi_{\Gamma_1}(\Pi_{\Gamma_2})$ stands by the value of the terms on the right side of (38) integrating on $\Gamma_1(\Gamma_2)$.

Similar to [5, 22], we deal with Π_{Γ_2} as follows.

Since $u|_{\Gamma_2} = 0$, we have $\nabla_\Gamma u|_{\Gamma_2} = 0$; that is,

$$\nabla_g u = \frac{\partial u}{\partial \mu}\mu \quad \text{for } x \in \Gamma_2. \tag{40}$$

Similarly, we obtain

$$H(u) = \left\langle H, \nabla_g u\right\rangle_g = \frac{\partial u}{\partial \mu}\langle H, \mu\rangle_g \quad \text{for } x \in \Gamma_2. \tag{41}$$

Using the equality (40) and (41) in the equality (38) on the portion Γ_2, with (16) we obtain

$$\Pi_{\Gamma_2} = \frac{1}{2}\int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \mu}\right)^2 \langle H, \mu\rangle_g d\Gamma\,dt \le 0. \tag{42}$$

Let H_1 be a vector field on $\overline{\Omega}$ such that

$$\begin{aligned} H_1 &= \mu \qquad x \in \Gamma_1, \\ H_1 &= 0 \qquad x \in \Gamma_2. \end{aligned} \tag{43}$$

Set $\mathscr{H} = H_1$; it follows from (31) that

$$\begin{aligned}&\int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu}\right)^2 d\Gamma\,dt + \frac{1}{2}\int_0^T \int_{\Gamma_1} \left(u_t^2 - \left|\nabla_g u\right|_g^2\right) d\Gamma\,dt \\ &\quad = \left(u_t, H_1(u)\right)\big|_0^T + \int_0^T dt \int_\Omega DH_1\left(\nabla_g u, \nabla_g u\right) dx \\ &\qquad + \frac{1}{2}\int_0^T dt \int_\Omega \left(u_t^2 - \left|\nabla_g u\right|_g^2 - 2F(u)\right)\operatorname{div}_g H_1 dx.\end{aligned} \tag{44}$$

Then we obtain that

$$\begin{aligned}&\int_0^T \int_{\Gamma_1} \left|\nabla_g u\right|_g^2 d\Gamma\,dt \\ &\quad \le C\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2\right) d\Gamma\,dt + C\left(E_0(0) + E_0(T)\right) \\ &\qquad + C\int_0^T \int_\Omega \left(u_t^2 + \left|\nabla_g u\right|_g^2 + 2F(u)\right) dx\,dt.\end{aligned} \tag{45}$$

With (16) and (45), we have

$$\begin{aligned}\Pi_{\Gamma_1} = &\int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \mu}\left(H(u) + uP\right) d\Gamma\,dt \\ &+ \frac{1}{2}\int_0^T \int_{\Gamma_1} \left(u_t^2 - \left|\nabla_g u\right|_g^2 - F(u)\right)\langle H, \mu\rangle_g d\Gamma\,dt \\ \le\, &C_\varepsilon \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu}\right)^2 d\Gamma\,dt + \varepsilon\int_0^T \int_{\Gamma_1} \left(u^2 + \left|\nabla_g u\right|_g^2\right) d\Gamma\,dt \\ &+ C\int_0^T \int_{\Gamma_1} u_t^2 d\Gamma\,dt \\ \le\, &C\int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \mu}\right)^2 d\Gamma\,dt \\ &+ \varepsilon\left(E_0(0) + E_0(T) + \int_0^T E(t)\,dt\right) + C\int_0^T \int_{\Gamma_1} u_t^2 d\Gamma\,dt.\end{aligned} \tag{46}$$

Note that

$$nb \le \operatorname{div}_g H \le nB \quad \forall x \in \overline{\Omega}. \tag{47}$$

Substituting the formulas (42) and (46) into the formula (37), with (19) and (35), we obtain

$$\begin{aligned}\int_0^T E(t)\,dt \le\, &C\left(E_0(0) + E_0(T)\right) \\ &+ C\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2\right) d\Gamma\,dt.\end{aligned} \tag{48}$$

Since

$$E_0(0) = E_0(T) - \int_0^T \int_{\Gamma_1} u_t \frac{\partial u}{\partial \mu} d\Gamma\, dt \\ \le E_0(T) + \frac{1}{2}\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2 \right) d\Gamma\, dt, \tag{49}$$

from (48), we have

$$\int_0^T E(t)\, dt \le CE(T) + C\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2 \right) d\Gamma\, dt. \tag{50}$$

Since $E(t)$ is decreasing, we deduce that

$$\int_0^T E(t)\, dt \ge TE(T). \tag{51}$$

Substituting the formulas (51) into the formula (50), for sufficiently large T, we have

$$E(T) \le \frac{C}{T}\int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \mu}\right)^2 \right) d\Gamma\, dt. \tag{52}$$

The inequality (34) holds. □

Proof of Theorem 1. (a) From (8), (13), (20), (29), and (34), for $T \ge \overline{T}$ we deduce that

$$\begin{aligned} E(T) &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \left(\phi^2(t) + 1\right) u_t^2 d\Gamma\, dt \\ &\le \frac{C}{T}\Bigg(\sup\{\phi(t) \mid 0 \le t \le T\} \\ &\qquad + \sup\left\{\frac{1}{\phi(t)} \mid 0 \le t \le T\right\}\Bigg) \\ &\quad \times \int_0^T \int_{\Gamma_1} \phi(t) u_t^2 d\Gamma\, dt \le \frac{C\Phi(T)}{T} E(0). \end{aligned} \tag{53}$$

Note that $E(t)$ is decreasing, and the estimate (21) holds.

(b) From (8), (13), (22), (29), and (34), for $T \ge \overline{T}$ we deduce that

$$\begin{aligned} E(T) &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \left(\phi^2(t) g^2(u_t) + u_t^2\right) d\Gamma\, dt \\ &\le \frac{C}{T}\Bigg\{ \int_0^T \int_{\Gamma_1} \phi(t) g^2(u_t)\, d\Gamma\, dt \\ &\qquad + \Phi(T)\int_0^T \int_{\Gamma_1} \phi(t) u_t^2 d\Gamma\, dt \Bigg\} \\ &\le \frac{C}{T}\Bigg\{ \int_0^T \int_{\{x\in\Gamma_1, |u_t|\le 1\}} \phi(t) g^2(u_t)\, d\Gamma\, dt \\ &\qquad + \Phi(T)\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t)\, d\Gamma\, dt \Bigg\} \\ &\le \frac{C}{T}\int_0^T \int_{\{x\in\Gamma_1, |u_t|\le 1\}} \phi(t) h(u_t g(u_t))\, d\Gamma\, dt \\ &\qquad + \frac{C\Phi(T)}{T} E(0) \\ &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \phi(t) h(u_t g(u_t))\, d\Gamma\, dt + \frac{C\Phi(T)}{T} E(0) \\ &\le \frac{C\int_0^T \phi(t)\, dt \cdot \mathrm{meas}(\Gamma_1)}{T} h \\ &\quad \times \left(\frac{\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t)\, d\Gamma\, dt}{\int_0^T \phi(t)\, dt \cdot \mathrm{meas}(\Gamma_1)} \right) + \frac{C\Phi(T)}{T} E(0) \\ &\le C_1 h\left(\frac{C_2\Phi(T)}{T} E(0) \right) + \frac{C_1\Phi(T)}{T} E(0). \end{aligned} \tag{54}$$

Note that $E(t)$ is decreasing, and the estimate (23) holds.

(c) From (8), (13), (24), (29), and (34), for $T \ge \overline{T}$ we deduce that

$$\begin{aligned} E(T) &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \left(\phi^2(t) g^2(u_t) + u_t^2\right) d\Gamma\, dt \\ &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \phi(t) \left(g^2(u_t) + u_t^2\right) d\Gamma\, dt \\ &\le \frac{C}{T}\int_0^T \int_{\{x\in\Gamma_1, |u_t|\le 1\}} \phi(t) h(u_t g(u_t))\, d\Gamma\, dt \\ &\qquad + \frac{C}{T}\int_0^T \int_{\{x\in\Gamma_1, |u_t|> 1\}} \phi(t) u_t^2 d\Gamma\, dt \\ &\le \frac{C}{T}\int_0^T \int_{\Gamma_1} \phi(t) h(u_t g(u_t))\, d\Gamma\, dt \\ &\qquad + \frac{C}{T}\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t)\, d\Gamma\, dt \\ &\le \frac{C\int_0^T \phi(t)\, dt \cdot \mathrm{meas}(\Gamma_1)}{T} h \\ &\quad \times \left(\frac{\int_0^T \int_{\Gamma_1} \phi(t) u_t g(u_t)\, d\Gamma\, dt}{\int_0^T \phi(t)\, dt \cdot \mathrm{meas}(\Gamma_1)} \right) + \frac{C}{T} E(0) \\ &\le C_1 h\left(\frac{C_2 E(0)}{T} \right) + \frac{C_1}{T} E(0). \end{aligned} \tag{55}$$

Note that $E(t)$ is decreasing, and the estimate (25) holds. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. E. Lagnese, "Note on boundary stabilization of wave equations," *SIAM Journal on Control and Optimization*, vol. 26, no. 5, pp. 1250–1256, 1988.

[2] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions," *SIAM Review*, vol. 20, pp. 639–739, 1978.

[3] R. Triggiani, "Wave equation on a bounded domain with boundary dissipation: an operator approach," *Journal of Mathematical Analysis and Applications*, vol. 137, no. 2, pp. 438–461, 1989.

[4] Y. You, "Energy decay and exact controllability for the Petrovsky equation in a bounded domain," *Advances in Applied Mathematics*, vol. 11, no. 3, pp. 372–388, 1990.

[5] Z.-H. Ning and Q.-X. Yan, "Stabilization of the wave equation with variable coefficients and a delay in dissipative boundary feedback," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 1, pp. 167–173, 2010.

[6] Z. H. Ning, C. X. Shen, and X. P. Zhao, "Stabilization of the wave equation with variable coeffcients and a delay in dissipative internal feedback," *Journal of Mathematical Analysis and Applications*, vol. 405, no. 1, pp. 148–155, 2013.

[7] M. Aassila, M. M. Cavalcanti, and V. N. Domingos Cavalcanti, "Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term," *Calculus of Variations and Partial Differential Equations*, vol. 15, no. 2, pp. 155–180, 2002.

[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, "Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term," *Journal of Differential Equations*, vol. 203, no. 1, pp. 119–158, 2004.

[9] F. Conrad and B. Rao, "Decay of solutions of the wave equation in a star-shaped domain with nonlinear boundary feedback," *Asymptotic Analysis*, vol. 7, no. 3, pp. 159–177, 1993.

[10] V. Komornik, "On the nonlinear boundary stabilization of the wave equation," *Chinese Annals of Mathematics B*, vol. 14, no. 2, pp. 153–164, 1993.

[11] V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method*, John Wiley and Sons. Ltd, Chichester, UK, 1994.

[12] V. Komornik and E. Zuazua, "A direct method for the boundary stabilization of wave equation," *Journal de Mathématiques Pures et Appliquées*, vol. 69, pp. 33–54, 1990.

[13] I. Lasiecka and D. Tataru, "Uniform boundary stabilization of semilinear wave equation with nonlinear boundary condition," *Differential Integral Equations*, vol. 6, pp. 507–533, 1993.

[14] E. Zuazua, "Uniform stabilization of the wave equation by nonlinear boundary feedback," *SIAM Journal on Control and Optimization*, vol. 28, no. 2, pp. 466–477, 1990.

[15] K. S. Liu, Z. Y. Liu, and B. Rao, "Exponential stability of an abstract nondissipative linear system," *SIAM Journal on Control and Optimization*, vol. 40, no. 1, pp. 149–165, 2002.

[16] B. Z. Guo, J. M. Wang, and S. P. Yung, "On the C0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam," *Systems & Control Letters*, vol. 18, no. 6, pp. 1013–1038, 2005.

[17] B.-Z. Guo and Z.-C. Shao, "Exponential stability of a semilinear wave equation with variable coeffcients under the nonlinear boundary feedback," *Nonlinear Analysis*, vol. 71, pp. 5961–5978, 2009.

[18] M. Bellassoued, "Decay of solutions of the wave equation with arbitrary localized nonlinear damping," *Journal of Differential Equations*, vol. 211, no. 2, pp. 303–332, 2005.

[19] A. Benaissa and A. Guesmia, "Energy decay for wave equations of φ-Laplacian type with weakly nonlinear dissipation," *Electronic Journal of Differential Equations*, vol. 109, pp. 1–22, 2008.

[20] A. Benaissa, A. Benaissa, and S. A. Messaoudi, "Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks," *Journal of Mathematical Physics*, vol. 53, no. 12, Article ID 123514, 2012.

[21] H. Li, C. S. Lin, S. P. Wang, and Y. M. Zhang, "Stabilization of the wave equation with boundary time-varying delay," *Advances in Mathematical Physics*, vol. 2014, Article ID 735341, 6 pages, 2014.

[22] P.-F. Yao, "On the observability inequalities for exact controllability of wave equations with variable coefficients," *SIAM Journal on Control and Optimization*, vol. 37, no. 5, pp. 1568–1599, 1999.

[23] I. Lasiecka and D. Tataru, "Uniform boundary stabilization of semilinear wave equation with nonlinear boundary dissipation," *Differential Integral Equations*, vol. 6, pp. 507–533, 1993.

[24] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and I. Lasiecka, "Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction," *Journal of Differential Equations*, vol. 236, no. 2, pp. 407–459, 2007.

[25] P. F. Yao, "Boundary controllability for the quasilinear wave equation," *Applied Mathematics and Optimization*, vol. 61, no. 2, pp. 191–233, 2010.

[26] P. F. Yao, *Modeling and Control in Vibrational and Structural Dynamics. A Differential Geometric Approach*, Applied Mathematics and Nonlinear Science Series, Chapman and Hall/CRC CRC Press, Boca Raton, Fla, USA, 2011.

[27] Z. H. Ning, C. X. Shen, X. P. Zhao, H. Li, C. S. Lin, and Y. M. Zhang, "Nonlinear Boundary Stabilization of the Wave Equations with Variable coeffcients and time dependent delay," *Applied Mathematics and Computation*, vol. 232, pp. 511–520, 2014.