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Research Article

New Hermite-Hadamard Type Inequalities for n-Times Differentiable and s-Logarithmically Preinvex Functions

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The concept of s-logarithmically preinvex function is introduced, and by creating an integral identity involving an n-times differentiable function, some new Hermite-Hadamard type inequalities for s-logarithmically preinvex functions are established.

1. Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{N}$ denote the set of all positive integers, I denote the interval in $\mathbb{R}$, and A denote the set in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Let us recall some definitions of various convex functions.

Definition 1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If inequality (1) reverses, then f is said to be concave on I.

Definition 2 (see [1]). A set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : A \times A \to \mathbb{R}^n$, if for every $x, y \in A$ and $t \in [0,1]$

$$y + t\eta(x, y) \in A. \tag{2}$$

The invex set A is also called a η-connected set.

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1], e.g.).

Definition 3 (see [1]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. For every $x, y \in A$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by

$$P_{xv} = \{z \mid z = x + t\eta(y, x), t \in [0,1]\}. \tag{3}$$

Definition 4 (see [2]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in A$ and $t \in [0,1]$

$$f(y + t\eta(x, y)) \le tf(x) + (1-t) f(y). \tag{4}$$

The function f is said to be preincave if and only if $-f$ is preinvex.

Every convex function is preinvex with respect to the map $\eta(x, y) = x - y$, but not conversely (see [2], e.g.).

Definition 5 (see [3]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set A is said to be logarithmically preinvex with respect to η, if for every $x, y \in A$ and $t \in [0,1]$

$$f(y + t\eta(x, y)) \le [f(x)]^t [f(y)]^{1-t}. \tag{5}$$

For properties and applications of preinvex and logarithmically preinvex functions, please refer to [1–8] and closely related references therein.

The most important inequality in the theory of convex functions, the well known Hermite-Hadamard's integral inequality, may be stated as follows. Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \to \mathbb{R}$ is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \qquad (6)$$

If f is concave on $[a, b]$, then inequality (6) is reversed.

Inequality (6) has been generalized by many mathematicians. Some of them may be recited as follows.

Theorem 6 (see [9, Theorem 2.2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I and $a, b \in I$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]}{8}. \qquad (7)$$

Theorem 7 (see[10, Theorem 1]). *Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \to \mathbb{R}$ is differentiable on $[a, b]$ such that $|f'(x)|^q$ is a convex function on $[a, b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}. \qquad (8)$$

Theorem 8 (see [11, Theorem 2.3]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I, $a, b \in I$ with $a < b$ and $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ &\quad\times\left\{\left[\left|f'(a)\right|^{p/(p-1)} + 3\left|f'(b)\right|^{p/(p-1)}\right]^{1-1/p}\right. \\ &\quad\left.+\left[3\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}\right]^{1-1/p}\right\}.\end{aligned} \qquad (9)$$

Theorem 9 (see [6]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. If $f : A \to \mathbb{R}_+$ is a preinvex function on A, then the following inequality holds:*

$$f\left(\frac{2b+\eta(a,b)}{2}\right) \leq \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \qquad (10)$$

Theorem 10 (see [4, Theorem 4.3]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is a twice differentiable function on A and $|f''(x)|$ is preinvex on A. If $q > 1$ and f'' is integrable on the η-path P_{bc} for $c = b + \eta(a, b)$, then*

$$\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right| \leq \frac{[\eta(a,b)]^2}{12}\left(\frac{1}{2}\right)^{1/q}\left[\left|f''(a)\right|^q + \left|f''(b)\right|^q\right]^{1/q}. \qquad (11)$$

Theorem 11 (see [12, Theorem 3.1]). *For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a, b)$. If $|f^{(n)}|^q$ is preinvex on A for $q \geq 1$, then*

$$\begin{aligned}&\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right. \\ &\quad+\sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]} \\ &\quad\left.\times\left[f^{(k)}(b) + (-1)^k f^{(k)}(b+\eta(a,b))\right]\right| \\ &\leq \frac{|\eta(a,b)|^n(n-1)^{1-1/q}}{4[(n+1)!](n+2)^{1/q}} \\ &\quad\times\left\{\left[n\left|f^{(n)}(a)\right|^q + (n^2-2)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right. \\ &\quad\left.+\left[(n^2-2)\left|f^{(n)}(a)\right|^q + n\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.\end{aligned} \qquad (12)$$

Theorem 12 (see [13, Theorem 5]). *For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is a function such that $f^{(n)}$ exits on A and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$. If $|f^{(n)}|^q$ is logarithmically preinvex on A for $q \geq 1$, then we have the inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right. \\ &\quad\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \qquad (13) \\ &\leq \frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left[E_1(n,q)\right]^{1/q},\end{aligned}$$

where

$$E_1(n,q) = \frac{(-1)^n n! \left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\}\left|f^{(n)}(a)\right|^q}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}} - \frac{2\left|f^{(n)}(b)\right|^q}{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]} - n!\left|f^{(n)}(b)\right|^q \times \sum_{k=1}^{n} \frac{(-1)^k\left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\}}{(n-k)!q^{k+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}. \quad (14)$$

Recently, some related inequalities for preinvex functions were also obtained in [14, 15].

In the paper, the concept of s-logarithmically preinvex function is introduced, and by creating an integral identity involving an n-times differentiable function, some new Hermite-Hadamard type inequalities for s-logarithmically preinvex functions are established which generalize some known results.

2. New Definition and Lemmas

Now we introduce concepts of s-logarithmically preinvex functions.

Definition 13. Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set A is said to be s-logarithmically preinvex with respect to η, if for every $x, y \in A$, $t \in [0,1]$, and some $s \in (0,1]$

$$f(y + t\eta(x,y)) \le [f(x)]^{t^s}[f(y)]^{(1-t)^s}. \quad (15)$$

Clearly, when taking $s = 1$ in (15), then f becomes the standard logarithmically convex function on A.

In order to obtain our main results, we need the following lemmas.

Lemma 14. *For $n \in \mathbb{N}$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \ne b$. If $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$, then*

$$\frac{f(b) + f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx + \sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)} \times \left[(b-x)^{k-1}(2x - 2b - k\eta(a,b)) + (b+\eta(a,b)-x)^{k-1} \times (2b + 2\eta(a,b) - 2x - k\eta(a,b))\right] f^{(k-1)}(x)$$
$$= \frac{[\eta(a,b)]^n}{2n!} \times \left[\int_0^{(x-b)/(\eta(a,b))} (-t)^{n-1}(2t-n) f^{(n)}(b+t\eta(a,b))\,dt + \int_{(x-b)/(\eta(a,b))}^1 (1-t)^{n-1}(2t+n-2) \times f^{(n)}(b+t\eta(a,b))\,dt\right], \quad (16)$$

where $x \in [b, b+\eta(b,a)]$ and the above summation is zero for $n = 1$.

Proof. Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. When $n = 1$, by integrating by part in the right-hand side of (16), one gives

$$\frac{\eta(a,b)}{2}\left[\int_0^1 (2t-1) f'(b+t\eta(a,b))\,dt\right] = \frac{f(b) + f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx. \quad (17)$$

Hence, the identity (16) holds for $n = 1$.

When $n = m$ and $m \ge 1$, suppose that the identity (16) is valid.

When $n = m + 1$, by the hypothesis, we have

$$\frac{[\eta(a,b)]^{m+1}}{2(m+1)!} \times \left[\int_0^{(x-b)/(\eta(a,b))} (-t)^m (2t-m-1) \times f^{(m+1)}(b+t\eta(a,b))\,dt + \int_{(x-b)/(\eta(a,b))}^1 (1-t)^m (2t+m-1) \times f^{(m+1)}(b+t\eta(a,b))\,dt\right]$$
$$= \frac{[\eta(a,b)]^m}{2(m+1)!} \times \left[\left(\frac{b-x}{\eta(a,b)}\right)^m\left(\frac{2x-2b}{\eta(a,b)} - m - 1\right) f^{(m)}(x) - \int_0^{(x-b)/(\eta(a,b))} (-t)^{m-1}(m+1)(m-2t) \times f^{(m)}(b+t\eta(a,b))\,dt\right.$$

$$-\left(\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)^m\left(\frac{2x-2b}{\eta(a,b)}+m-1\right)f^{(m)}(x)$$
$$-\int_{(x-b)/(\eta(a,b))}^1(1-t)^{m-1}(m+1)$$
$$\times(2-2t-m)f^{(m)}(b+t\eta(a,b))\,dt\Bigg]$$
$$=\frac{1}{2(m+1)!\eta(a,b)}$$
$$\times\left\{(b-x)^m\left[2x-2b-(m+1)\eta(a,b)\right]\right.$$
$$\left.-(b+\eta(a,b)-x)^m\left[2x-2b+(m-1)\eta(a,b)\right]\right\}$$
$$\times f^{(m)}(x)+\frac{[\eta(a,b)]^m}{2m!}$$
$$\times\left[\int_0^{(x-b)/(\eta(a,b))}(-t)^{m-1}(2t-m)f^{(m)}(b+t\eta(a,b))\,dt\right.$$
$$+\int_{(x-b)/(\eta(a,b))}^1(1-t)^{m-1}(2t+m-2)$$
$$\left.\times f^{(m)}(b+t\eta(a,b))\,dt\right]$$
$$=\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx$$
$$+\sum_{k=2}^{m+1}\frac{1}{2k!\eta(a,b)}$$
$$\times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))\right.$$
$$\left.+(b+\eta(a,b)-x)^{k-1}(2b+2\eta(a,b)-2x-k\eta(a,b))\right]$$
$$\times f^{(k-1)}(x). \tag{18}$$

Therefore, when $n=m+1$, the identity (16) holds. By induction, the proof of Lemma 14 is complete. □

Remark 15. Under the conditions of Lemma 14, we have

$$\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx$$
$$+\sum_{k=2}^{n}\frac{[\eta(a,b)]^{k-1}(2-k)}{2k!}f^{(k-1)}(b)$$
$$=\frac{[\eta(a,b)]^n}{2n!}\int_0^1(1-t)^{n-1}(2t+n-2)$$
$$\times f^{(n)}(b+t\eta(a,b))\,dt, \tag{19}$$

$$\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx$$
$$+\sum_{k=2}^{n}\frac{[\eta(a,b)]^{k-1}(1-k)\left[1+(-1)^{k-1}\right]}{2^kk!}$$
$$\times f^{(k-1)}\left(b+\frac{\eta(a,b)}{2}\right)$$
$$=\frac{[\eta(a,b)]^n}{2n!}$$
$$\times\left[\int_0^{1/2}(-t)^{n-1}(2t-n)f^{(n)}(b+t\eta(a,b))\,dt\right.$$
$$\left.+\int_{1/2}^1(1-t)^{n-1}(2t+n-2)f^{(n)}(b+t\eta(a,b))\,dt\right], \tag{20}$$

$$\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx$$
$$+\sum_{k=2}^{n}\frac{[-\eta(a,b)]^{k-1}(2-k)}{2k!}f^{(k-1)}(b+\eta(a,b))$$
$$=\frac{[\eta(a,b)]^n}{2n!}\int_0^1(-t)^{n-1}(2t-n)f^{(n)}(b+t\eta(a,b))\,dt. \tag{21}$$

Proof. These are special cases of Lemma 14 for $x=b$, $b+\eta(a,b)/2$, $b+\eta(a,b)$, respectively. □

Remark 16. Adding the identities (19) and (21) and then dividing by 2 result in Lemma 14 from [12].

Lemma 17. *Let $\mu>0$ and $x\geq 0$. Then*

$$E(n;\mu,x)\triangleq\int_0^x t^n\mu^t\,dt$$
$$=\begin{cases}\dfrac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}}+n!\mu^x\displaystyle\sum_{k=0}^{n}\frac{(-1)^kx^{n-k}}{(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1,\\ \dfrac{x^{n+1}}{n+1}, & \mu=1\end{cases} \tag{22}$$

for $n\geq 0$, $n\in\mathbb{N}$.

Proof. When $\mu=1$, the proof is straightforward.

When $\mu\neq 1$, for $n=0$, we have

$$\int_0^x\mu^t\,dt=\frac{\mu^x-1}{\ln\mu}, \tag{23}$$

which coincides with the right-hand side of (22) for $n=0$.

For $n=1$, we get

$$\int_0^x t\mu^t\,dt=\frac{x\mu^x}{\ln\mu}-\frac{1}{\ln\mu}\int_0^x\mu^t\,dt$$
$$=\frac{1}{(\ln\mu)^2}+\mu^x\left[\frac{x}{\ln\mu}-\frac{1}{(\ln\mu)^2}\right], \tag{24}$$

which coincides with the right-hand side of (22) for $n=1$.

Suppose that (22) is true for $n = m$, $m \geq 0$, then, for $n = m+1$, it follows that

$$\begin{aligned}
\int_0^x t^{m+1}\mu^t dt &= \frac{x^{m+1}\mu^x}{\ln\mu} - \frac{m+1}{\ln\mu}\int_0^x t^m\mu^t dt \\
&= \frac{x^{m+1}\mu^x}{\ln\mu} - \frac{m+1}{\ln\mu}\left[\frac{(-1)^{m+1}m!}{(\ln\mu)^{m+1}}\right. \\
&\qquad \left. + m!\mu^x\sum_{k=0}^{m}\frac{(-1)^k x^{m-k}}{(m-k)!(\ln\mu)^{k+1}}\right] \\
&= \frac{(-1)^{m+2}(m+1)!}{(\ln\mu)^{m+2}} + \frac{x^{m+1}\mu^x}{\ln\mu} \\
&\quad + (m+1)!\mu^x\sum_{k=1}^{m+1}\frac{(-1)^k x^{m-k+1}}{(m-k+1)!(\ln\mu)^{k+1}} \\
&= \frac{(-1)^{m+2}(m+1)!}{(\ln\mu)^{m+2}} + (m+1)!\mu^x\sum_{k=0}^{m+1}\frac{(-1)^k x^{m-k+1}}{(m-k+1)!(\ln\mu)^{k+1}}.
\end{aligned} \tag{25}$$

Therefore, when $n = m+1$, the identity (22) holds. By induction, the proof of Lemma 17 is complete. □

Lemma 18 (see [16]). *Let $\gamma > 0$, $\mu > 0$, and $x \geq 0$. Then*

$$G(\gamma;\mu,x) \triangleq \int_0^x t^{\gamma-1}\mu^t dt = x^\gamma\mu^x\sum_{k=1}^{\infty}\frac{(-x\ln\mu)^{k-1}}{(\gamma)_k} < \infty, \tag{26}$$

where $(\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1)$.

By Lemmas 17 and 18, a straightforward computation gives the following lemmas.

Lemma 19. *Let $\mu > 0$ and $x \geq 0$. Then*

$$\begin{aligned}
F(n;\mu,x) &\triangleq nE(n-1;\mu,x) - 2E(n;\mu,x) \\
&= \begin{cases} \dfrac{2\mu^x x^n}{\ln\mu} + n!(\ln\mu+2) \\ \quad\times\left[\dfrac{(-1)^n}{(\ln\mu)^{n+1}} + \mu^x\displaystyle\sum_{k=1}^{n}\frac{(-1)^{k-1}x^{n-k}}{(n-k)!(\ln\mu)^{k+1}}\right], & \mu \neq 1, \\ \dfrac{x^n(n+1-2x)}{n+1}, & \mu = 1 \end{cases}
\end{aligned} \tag{27}$$

for $n \geq 0$, $n \in \mathbb{N}$.

Lemma 20. *Let $\gamma > 0$, $\mu > 0$, and $x \geq 0$. Then*

$$\begin{aligned}
H(\gamma;\mu,x) &\triangleq nG(\gamma;\mu,x) - 2G(\gamma+1;\mu,x) \\
&= x^\gamma\mu^x\sum_{k=1}^{\infty}\frac{(-x\ln\mu)^{k-1}[n(\gamma+k)-2x\gamma]}{(\gamma)_{k+1}},
\end{aligned} \tag{28}$$

where $(\gamma)_{k+1} = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k)$.

Lemma 21 (see [17]). *Let $0 < \phi \leq 1 \leq \psi$ and $0 < t, s \leq 1$. Then*

$$\begin{aligned}
\phi^{t^s} &\leq \phi^{st}, \\
\psi^{t^s} &\leq \psi^{st+1-s}.
\end{aligned} \tag{29}$$

3. Hermite-Hadamard Type Inequalities

Now we start out to establish some new Hermite-Hadamard type inequalities for n-times differentiable and s-logarithmically preinvex functions.

Theorem 22. *For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$. If $|f^{(n)}|^q$ is s-logarithmically preinvex on A for $q \geq 1$, then for $x \in [b, b+\eta(a,b)]$ and some $s \in (0,1]$*

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right. \\
&\quad + \sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)} \\
&\quad \times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))\right. \\
&\qquad \left.+(b+\eta(a,b)-x)^{k-1}(2b+2\eta(a,b)-2x-k\eta(a,b))\right] \\
&\quad \left.\times f^{(k-1)}(x)\right| \\
&\leq \frac{[\eta(a,b)]^{(n+1-q)/q}}{2n!(n+1)^{1-1/q}} \\
&\quad \times\left\{\left|f^{(n)}(a)\right|^\alpha\left|f^{(n)}(b)\right|^\beta\right. \\
&\qquad \times\left[(x-b)^n[(n+1)\eta(a,b)-2x+2b]\right]^{1-1/q} \\
&\qquad \times\left[F\left(n;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q} + \left|f^{(n)}(a)\right|^\delta\left|f^{(n)}(b)\right|^\theta \\
&\qquad \times\left[(b+\eta(a,b)-x)^n[(n-1)\eta(a,b)+2x-2b]\right]^{1-1/q} \\
&\qquad \left.\times\left[F\left(n;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\},
\end{aligned} \tag{30}$$

where $\xi = |f^{(n)}(a)/f^{(n)}(b)|$, $F(n;\mu,x)$ *is defined in Lemma 19, and*

$$\{\alpha,\beta\} = \begin{cases} \{0,s\}, & \text{if } 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\le 1, \\ \{0,1\}, & \text{if } 0<\left|f^{(n)}(a)\right|<1<\left|f^{(n)}(b)\right|, \\ \{1-s,s\}, & \text{if } 0<\left|f^{(n)}(b)\right|<1<\left|f^{(n)}(a)\right|, \\ \{1-s,1\}, & \text{if } \left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|>1, \end{cases}$$
$$\{\delta,\theta\} = \begin{cases} \{s,0\}, & \text{if } 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\le 1, \\ \{s,1-s\}, & \text{if } 0<\left|f^{(n)}(a)\right|<1<\left|f^{(n)}(b)\right|, \\ \{1,0\}, & \text{if } 0<\left|f^{(n)}(b)\right|<1<\left|f^{(n)}(a)\right|, \\ \{1,1-s\}, & \text{if } \left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|>1. \end{cases} \tag{31}$$

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $b+t\eta(a,b)\in A$. Using Lemma 14, Hölder's inequality, and s-logarithmically preinvexity of $|f^{(n)}|^q$, it yields that

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx\right.\\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\
&\quad\times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))\right.\\
&\qquad\left.+(b+\eta(a,b)-x)^{k-1}(2b+2\eta(a,b)-2x-k\eta(a,b))\right]\\
&\quad\left.\times f^{(k-1)}(x)\right|\\
&\le\frac{[\eta(a,b)]^n}{2n!}\\
&\quad\times\left[\int_0^{(x-b)/(\eta(a,b))}t^{n-1}(n-2t)\left|f^{(n)}(b+t\eta(a,b))\right|dt\right.\\
&\qquad+\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\\
&\qquad\quad\left.\times\left|f^{(n)}(b+t\eta(a,b))\right|dt\right]\\
&\le\frac{[\eta(a,b)]^n}{2n!}\\
&\quad\times\left\{\left[\int_0^{(x-b)/(\eta(a,b))}t^{n-1}(n-2t)dt\right]^{1-1/q}\right.\\
&\qquad\times\left[\int_0^{(x-b)/(\eta(a,b))}t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\right.\\
&\qquad\quad\left.\times\left|f^{(n)}(a)\right|^{qt^s}dt\right]^{1/q}\\
&\qquad+\left[\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)dt\right]^{1-1/q}\\
&\qquad\times\left[\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\right.\\
&\qquad\quad\left.\left.\times\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt\right]^{1/q}\right\}.
\end{aligned}\tag{32}$$

By Lemmas 17 and 21, for $0<|f^{(n)}(a)|,|f^{(n)}(b)|\le 1$, we give

$$\begin{aligned}
&\int_0^{(x-b)/(\eta(a,b))}t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt\\
&\le\int_0^{(x-b)/(\eta(a,b))}t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{sq(1-t)}\left|f^{(n)}(a)\right|^{sqt}dt\\
&=\left|f^{(n)}(b)\right|^{sq}\left[nE\left(n-1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right.\\
&\qquad\left.-2E\left(n;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right],
\end{aligned}\tag{33}$$

$$\begin{aligned}
&\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{q(1-t)^s}\\
&\qquad\times\left|f^{(n)}(a)\right|^{qt^s}dt\\
&\le\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{sq(1-t)}\\
&\qquad\times\left|f^{(n)}(a)\right|^{sqt}dt\\
&=\int_0^{(b+\eta(a,b)-x)/(\eta(a,b))}t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{sqt}\left|f^{(n)}(a)\right|^{sq(1-t)}dt
\end{aligned}$$

$$= \left|f^{(n)}(a)\right|^{sq}\left[nE\left(n-1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]. \tag{34}$$

For $0<|f^{(n)}(a)|<1<|f^{(n)}(b)|$, we get

$$\int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q[s(1-t)+1-s]}\left|f^{(n)}(a)\right|^{sqt}dt$$
$$= \left|f^{(n)}(b)\right|^{q}\left[nE\left(n-1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right], \tag{35}$$

$$\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{q[s(1-t)+1-s]}$$
$$\times\left|f^{(n)}(a)\right|^{sqt}dt$$
$$= \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(st+1-s)}$$
$$\times\left|f^{(n)}(a)\right|^{sq(1-t)}dt$$
$$= \left|f^{(n)}(a)\right|^{sq}\left|f^{(n)}(b)\right|^{q(1-s)}\left[nE\left(n-1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]. \tag{36}$$

For $0<|f^{(n)}(b)|<1<|f^{(n)}(a)|$, we obtain

$$\int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{qs(1-t)}\left|f^{(n)}(a)\right|^{q(st+1-s)}dt$$
$$= \left|f^{(n)}(b)\right|^{qs}\left|f^{(n)}(a)\right|^{q(1-s)}\left[nE\left(n-1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right],$$

$$\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{qs(1-t)}$$
$$\times\left|f^{(n)}(a)\right|^{q(st+1-s)}dt$$
$$= \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{qst}$$
$$\times\left|f^{(n)}(a)\right|^{q[s(1-t)+1-s]}dt$$
$$= \left|f^{(n)}(a)\right|^{q}\left[nE\left(n-1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]. \tag{37}$$

For $|f^{(n)}(a)|,|f^{(n)}(b)|>1$, we have

$$\int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_0^{(x-b)/(\eta(a,b))} t^{n-1}(n-2t)\left|f^{(n)}(b)\right|^{q[s(1-t)+1-s]}$$
$$\times\left|f^{(n)}(a)\right|^{q(st+1-s)}dt$$
$$= \left|f^{(n)}(b)\right|^{q}\left|f^{(n)}(a)\right|^{q(1-s)}\left[nE\left(n-1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right], \tag{38}$$

$$\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt$$
$$\leq \int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{n-1}(2t+n-2)$$
$$\times\left|f^{(n)}(b)\right|^{q[s(1-t)+1-s]}\left|f^{(n)}(a)\right|^{q(st+1-s)}dt$$
$$= \int_0^{(b+\eta(a,b)-x)/(\eta(a,b))} t^{n-1}(n-2t)$$
$$\times\left|f^{(n)}(b)\right|^{q(st+1-s)}\left|f^{(n)}(a)\right|^{q[s(1-t)+1-s]}dt$$
$$= \left|f^{(n)}(a)\right|^{q}\left|f^{(n)}(b)\right|^{q(1-s)}\left[nE\left(n-1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right.$$
$$\left.-2E\left(n;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]. \tag{39}$$

Using Lemma 19 and substituting (33) to (39) into (32), we get inequality (30).

Theorem 22 is thus proved. □

Corollary 23. *Under the assumptions of Theorem 22,*

(1) if $q = 1$, then

$$\begin{aligned}&\left|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right.\\&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\&\quad\times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\right.\\&\qquad\left.\left.\times(2b+2\eta(a,b)-2x-k\eta(a,b))\right]f^{(k-1)}(x)\right|\\&\le\frac{[\eta(a,b)]^n}{2n!}\left\{\left|f^{(n)}(a)\right|^{\alpha}\left|f^{(n)}(b)\right|^{\beta}\left[F\left(n;\xi^{s},\frac{x-b}{\eta(a,b)}\right)\right]\right.\\&\qquad+\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\\&\qquad\left.\times\left[F\left(n;\xi^{-s},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]\right\};\end{aligned}\tag{40}$$

(2) if $s = 1$, then

$$\begin{aligned}&\left|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right.\\&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\&\quad\times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\right.\\&\qquad\left.\left.\times(2b+2\eta(a,b)-2x-k\eta(a,b))\right]f^{(k-1)}(x)\right|\\&\le\frac{[\eta(a,b)]^{(n+1-q)/q}}{2n!(n+1)^{1-1/q}}\\&\quad\times\left\{\left|f^{(n)}(b)\right|\left[(x-b)^n\left[(n+1)\eta(a,b)-2x+2b\right]\right]^{1-1/q}\right.\\&\qquad\times\left[F\left(n;\xi^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}+\left|f^{(n)}(a)\right|\\&\qquad\times\left[(b+\eta(a,b)-x)^n\left[(n-1)\eta(a,b)+2x-2b\right]\right]^{1-1/q}\\&\qquad\left.\times\left[F\left(n;\xi^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}.\end{aligned}\tag{41}$$

Theorem 24. *For $n \in \mathbb{N}$ and $n \ge 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \ne b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b+\eta(a,b)$. If $|f^{(n)}|^q$ is s-logarithmically preinvex on A for $q > 1$, then for $x \in [b, b+\eta(a,b)]$ and some $s \in (0,1]$*

$$\begin{aligned}&\left|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,dx\right.\\&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\&\quad\times\left[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\right.\\&\qquad\left.\left.\times(2b+2\eta(a,b)-2x-k\eta(a,b))\right]f^{(k-1)}(x)\right|\\&\le\frac{[\eta(a,b)]^{n-2+1/q}}{2^{2-1/q}n!}\left(\frac{1-q}{2q-1}\right)^{1-1/q}\\&\quad\times\left\{\left[(n\eta(a,b)-2x+2b)^{(2q-1)/(q-1)}\right.\right.\\&\qquad\left.-(n\eta(a,b))^{(2q-1)/(q-1)}\right]^{1-1/q}\left|f^{(n)}(a)\right|^{\alpha}\left|f^{(n)}(b)\right|^{\beta}\\&\quad\times\left[G\left(nq-q+1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\\&\quad+\left[((n-2)\eta(a,b)+2x-2b)^{(2q-1)/(q-1)}\right.\\&\qquad\left.-(n\eta(a,b))^{(2q-1)/(q-1)}\right]^{1-1/q}\\&\quad\times\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\\&\quad\left.\times\left[G\left(nq-q+1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\},\end{aligned}\tag{42}$$

where ξ, $\{\alpha,\beta\}$ and $\{\delta,\theta\}$ are given in Theorem 22 and $G(\gamma;\mu,x)$ is defined by (26).

Proof. Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$.

Using Lemma 14, Hölder's inequality, and s-logarithmically preinvexity of $|f^{(n)}|^q$, it follows that

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx \\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)} \\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1} \\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg| \\
&\leq\frac{[\eta(a,b)]^n}{2n!} \\
&\quad\times\Bigg\{\left[\int_0^{(x-b)/(\eta(a,b))}(n-2t)^{q/(q-1)}dt\right]^{1-1/q} \\
&\qquad\times\left[\int_0^{(x-b)/(\eta(a,b))}t^{(n-1)q}\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt\right]^{1/q} \\
&\qquad+\left[\int_{(x-b)/(\eta(a,b))}^{1}(2t+n-2)^{q/(q-1)}dt\right]^{1-1/q} \\
&\qquad\times\left[\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{(n-1)q}\left|f^{(n)}(b)\right|^{q(1-t)^s}\right. \\
&\qquad\quad\left.\times\left|f^{(n)}(a)\right|^{qt^s}dt\right]^{1/q}\Bigg\}.
\end{aligned} \tag{43}$$

The rest is the same as the proof of Theorem 22. □

Corollary 25. *Under the assumptions of Theorem 24, if $s=1$, then*

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx \\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)} \\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1} \\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg| \\
&\leq\frac{[\eta(a,b)]^{n-2+1/q}}{2^{2-1/q}n!}\left(\frac{1-q}{2q-1}\right)^{1-1/q} \\
&\quad\times\Bigg\{\Big[(n\eta(a,b)-2x+2b)^{(2q-1)/(q-1)} \\
&\qquad-(n\eta(a,b))^{(2q-1)/(q-1)}\Big]^{1-1/q}\left|f^{(n)}(b)\right| \\
&\qquad\times\left[G\left(nq-q+1;\xi^q,\frac{x-b}{\eta(a,b)}\right)\right]^{1/q} \\
&\qquad+\Big[((n-2)\eta(a,b)+2x-2b)^{(2q-1)/(q-1)} \\
&\qquad-(n\eta(a,b))^{(2q-1)/(q-1)}\Big]^{1-1/q}\left|f^{(n)}(a)\right| \\
&\qquad\times\left[G\left(nq-q+1;\xi^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\}.
\end{aligned} \tag{44}$$

Theorem 26. *For $n\in\mathbb{N}$ and $n\geq 2$, let $A\subseteq\mathbb{R}$ be an open invex set with respect to $\eta: A\times A\to\mathbb{R}$ and $a,b\in A$ with $\eta(a,b)>0$ for all $a\neq b$. Suppose that $f:A\to\mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c=b+\eta(a,b)$. If $|f^{(n)}|^q$ is s-logarithmically preinvex on A for $q>1$, then for $x\in[b,b+\eta(a,b)]$, $0\leq r\leq(n-1)q$, and some $s\in(0,1]$*

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx \\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)} \\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1} \\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg| \\
&\leq\frac{[\eta(a,b)]^{(r+2-q)/q}}{2n!}\left[\frac{q-1}{(nq-r-1)(nq+q-r-2)}\right]^{1-1/q} \\
&\quad\times\Bigg\{\left|f^{(n)}(a)\right|^{\alpha}\left|f^{(n)}(b)\right|^{\beta}(x-b)^{(nq-r-1)/q} \\
&\qquad\times[n(nq+q-r-2)\eta(a,b) \\
&\qquad-2(nq-r-1)(x-b)]^{1-1/q} \\
&\qquad\times\left[H\left(r+1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}+\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta} \\
&\qquad\times[n(nq+q-r-2)\eta(a,b) \\
&\qquad-2(nq-r-1)(b+\eta(a,b)-x)]^{1-1/q} \\
&\qquad\times(b+\eta(a,b)-x)^{(nq-r-1)/q} \\
&\qquad\times\left[H\left(r+1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\},
\end{aligned} \tag{45}$$

where ξ, $\{\alpha,\beta\}$ and $\{\delta,\theta\}$ are given in Theorem 22 and $H(\gamma;\mu,x)$ is defined by (28).

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $b+t\eta(a,b)\in A$.

Using Lemma 14, Hölder's inequality, and s-logarithmically preinvexity of $|f^{(n)}|^q$, it turns out that

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx\\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg|\\
&\leq\frac{[\eta(a,b)]^n}{2n!}\\
&\quad\times\Bigg\{\left[\int_0^{(x-b)/(\eta(a,b))}t^{(nq-q-r)/(q-1)}(n-2t)dt\right]^{1-1/q}\\
&\qquad\times\Bigg[\int_0^{(x-b)/(\eta(a,b))}t^{r}(n-2t)\left|f^{(n)}(b)\right|^{q(1-t)^s}\\
&\qquad\quad\times\left|f^{(n)}(a)\right|^{qt^s}dt\Bigg]^{1/q}\\
&\qquad+\left[\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{(nq-q-r)/(q-1)}(2t+n-2)dt\right]^{1-1/q}\\
&\qquad\times\Bigg[\int_{(x-b)/(\eta(a,b))}^{1}(1-t)^{r}(2t+n-2)\\
&\qquad\quad\times\left|f^{(n)}(b)\right|^{q(1-t)^s}\left|f^{(n)}(a)\right|^{qt^s}dt\Bigg]^{1/q}\Bigg\}.
\end{aligned}\tag{46}$$

The rest is also similar to the proof of Theorem 22. □

Corollary 27. *Under the assumptions of Theorem 26,*

(1) if $r=0$, then

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx\\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg|\\
&\leq\frac{[\eta(a,b)]^{(2-q)/q}}{2n!}\left[\frac{q-1}{(nq-1)(nq+q-2)}\right]^{1-1/q}\\
&\quad\times\Bigg\{\left|f^{(n)}(a)\right|^{\alpha}\left|f^{(n)}(b)\right|^{\beta}(x-b)^{(nq-1)/q}\\
&\quad\times[n(nq+q-2)\eta(a,b)-2(nq-1)(x-b)]^{1-1/q}\\
&\quad\times\left[H\left(1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}+\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\\
&\quad\times[(n-1)(nq-2)\eta(a,b)-2(nq-1)(b-x)]^{1-1/q}\\
&\quad\times(b+\eta(a,b)-x)^{nq-1/q}\\
&\quad\times\left[H\left(1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\};
\end{aligned}\tag{47}$$

(2) if $r=(n-1)q$, then

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx\\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg|\\
&\leq\frac{[\eta(a,b)]^{n-2+2/q}}{2n!}\\
&\quad\times\Bigg\{\left|f^{(n)}(a)\right|^{\alpha}\left|f^{(n)}(b)\right|^{\beta}[(x-b)(n\eta(a,b)-x+b)]^{1-1/q}\\
&\quad\times\left[H\left((n-1)q+1;\xi^{sq},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}+\left|f^{(n)}(a)\right|^{\delta}\\
&\quad\times\left|f^{(n)}(b)\right|^{\theta}[(b+\eta(a,b)-x)\\
&\qquad\times((n-1)\eta(a,b)+x-b)]^{1-1/q}\\
&\quad\times\left[H\left((n-1)q+1;\xi^{-sq},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\Bigg\};
\end{aligned}\tag{48}$$

(3) if $s=1$, then

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,dx\\
&\quad+\sum_{k=2}^{n}\frac{1}{2k!\eta(a,b)}\\
&\quad\times\Big[(b-x)^{k-1}(2x-2b-k\eta(a,b))+(b+\eta(a,b)-x)^{k-1}\\
&\qquad\times(2b+2\eta(a,b)-2x-k\eta(a,b))\Big]f^{(k-1)}(x)\Bigg|
\end{aligned}$$

$$\leq \frac{[\eta(a,b)]^{(r+2-q)/q}}{2n!}\left[\frac{q-1}{(nq-r-1)(nq+q-r-2)}\right]^{1-1/q}$$
$$\times \left\{ \left|f^{(n)}(b)\right|(x-b)^{(nq-r-1)/q} \right.$$
$$\times [n(nq+q-r-2)\eta(a,b)$$
$$-2(nq-r-1)(x-b)]^{1-1/q}$$
$$\times \left[H\left(r+1;\xi^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}+\left|f^{(n)}(a)\right|$$
$$\times [n(nq+q-r-2)\eta(a,b)$$
$$-2(nq-r-1)(b+\eta(a,b)-x)]^{1-1/q}$$
$$\times (b+\eta(a,b)-x)^{(nq-r-1)/q}$$
$$\left.\times \left[H\left(r+1;\xi^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}. \quad (49)$$

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved the final paper.

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