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Research Article

Construction of Biholomorphic Convex Mappings of Order α on B_p^n

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Some sufficient conditions for biholomorphic convex mappings of order α on the Reinhardt domain B_p^n in C^n are given; from that, criteria for biholomorphic convex mappings of order α with particular form become direct. As applications of these sufficient conditions, some concrete biholomorphic convex mappings of order α on B_p^n are provided.

1. Introduction and Preliminaries

The analytic functions of one complex variable, which map the unit disk $U = \{z \in C : |z| < 1\}$ onto starlike domains or convex domains, have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions. In the case of one complex variable, the following notions are well known.

Let $H(U)=\{f:U \mapsto \mathbb{C}$ be analytic in U with $f(0) = f'(0) - 1 = 0\}$. A function $f \in H(U)$ is said to be convex if $f(U)$ is convex, that is, given $w_1, w_2 \in f(U), tw_1 + (1-t)w_2 \in f(U)$ for all $t \in [0, 1]$. We let K denote the class of univalent convex functions in U. Suppose $\alpha \in [0, 1)$. If $f \in H(U)$ satisfies $f'(z) \neq 0$ for all $z \in U$ and the following inequality:

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha, \quad \forall z \in U, \tag{1}$$

then we call $f(z)$ a convex function of order α in U. We let $K(\alpha)$ denote the class of convex functions of order α in U. It is evident that $K \equiv K(0)$.

In higher dimensions, demanding that a mapping takes the unit ball to a convex domain turned out to be a very restrictive condition. It is rather hard to construct concrete biholomorphic convex mappings on some domains in $\mathbb{C}^n$, even on the Euclidean unit ball.

Suppose n is a fixed positive integer, $p > 1$. Let C^n be the space of n complex variables $z = (z_1, z_2, \ldots, z_n)$ with the usual inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, where $w = (w_1, w_2, \ldots, w_n) \in C^n$. We introduce the p-norm of $C^n : \|z\|_p = (\sum_{j=1}^n |z_j|^p)^{1/p}$, and let $B_p^n = \{z \in C^n : \|z\|_p < 1\}$; it is evident that B_p^n is a Reinhardt domain. For simplicity, let $\|z\| = \|z\|_2 = \sqrt{\langle z, z\rangle}$.

Let $H(B_p^n)$ be the class of holomorphic mappings $f(z) = (f_1(z), \ldots, f_n(z))$ in the Reinhardt domain B_p^n, where $z = (z_1, \ldots, z_n) \in C^n$. A mapping $f \in H(B_p^n)$ is said to be locally biholomorphic in B_p^n if f has a local inverse at each point $z \in B_p^n$ or, equivalently, if the first Fréchet derivative $Df(z) = (\partial f_j(z)/\partial z_k)_{1\le j,k\le n}$ is nonsingular at each point in B_p^n.

The second Fréchet derivative of a mapping $f \in H(B_p^n)$ is a symmetric bilinear operator $D^2 f(z)(\cdot,\cdot)$ on $C^n \times C^n$, and $D^2 f(z)(z,\cdot)$ is the linear operator obtained by restricting $D^2 f(z)$ to $\{z\} \times C^n$. The matrix representation of $D^2 f(z)(b,\cdot)$ is

$$D^2 f(z)(b,\cdot) = \left(\sum_{l=1}^n \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l\right)_{1\le j,k\le n}, \tag{2}$$

where $f(z) = (f_1(z), \ldots, f_n(z))$, $b = (b_1, \ldots, b_n) \in C^n$.

Let $N(B_p^n)$ denote the class of all locally biholomorphic mappings $f : B_p^n \to C^n$ such that $f(0) = 0$, $Df(0) = I$, where I is the unit matrix of $n \times n$. If $f \in N(B_p^n)$ is a biholomorphic mapping on B_p^n and $f(B_p^n)$ is a convex domain in C^n, then we call f a biholomorphic convex mapping on B_p^n. The class of all biholomorphic convex mappings on B_p^n is denoted by $K(B_p^n)$. Obviously, $K = K(B_p^1)$. The biholomorphic convex mapping of order α on B_p^n was introduced and investigated in [1–5]; the ε starlike and ε quasi-convex mappings were investigated in [4, 6].

Definition 1 (see [1–3, 5]). Suppose $0 \le \alpha < 1$, $p \ge 2$, $u(z) = \sum_{j=1}^n |z_j|^p$, and $f \in N(B_p^n)$. Assume that for any $z = (z_1, \ldots, z_n) \in B_p^n$ and $b = (b_1, \ldots, b_n) \in C^n$ with $\operatorname{Re}\langle b, \partial u/\partial \overline{z}\rangle = 0$, we have

$$\begin{aligned} J_f(z,b) = \operatorname{Re}\Bigg\{ & \frac{p^2}{4}\sum_{k=1}^{n}|z_k|^{p-2}|b_k|^2 + \frac{p}{2}\left(\frac{p}{2}-1\right) \\ & \times \sum_{k=1}^{n}\frac{|z_k|^p}{z_k^2}b_k^2 \\ & - \left\langle Df(z)^{-1}D^2f(z)(b,b), \frac{\partial u}{\partial \overline{z}}\right\rangle\Bigg\} \\ \ge \alpha \cdot & \frac{p}{2}\sum_{k=1}^{n}|z_k|^{p-2}|b_k|^2, \end{aligned} \tag{3}$$

where $\partial u/\partial \overline{z} = (\partial u/\partial \overline{z_1}, \ldots, \partial u/\partial \overline{z_n})$. Then, $f(z)$ is called a biholomorphic convex mapping of order α on B_p^n. The class of all biholomorphic convex mappings of order α on B_p^n is denoted by $K(B_p^n, \alpha)$. It is evident that $K(B_p^n) \equiv K(B_p^n, 0)$ and $K(B_p^1, \alpha) \equiv K(\alpha)$.

In 1995, Roper and Suffridge [7] proved that if $f \in K$ and $F(f)(z) = (f(z_1), \sqrt{f'(z_1)}z_0)$, where $z = (z_1, z_0) \in B^n$, $z_1 \in U$, $z_0 = (z_2, \ldots, z_n) \in C^{n-1}$, then $F(f)(z) \in K(B_2^n)$. $F(f)$ is popularly referred to as the Roper-Suffridge operator. Using this operator, we may construct a lot of concrete biholomorphic convex mappings on B_2^n. Roper and suffridge [8] also obtained some sufficient conditions for biholomorphic convex mappings on the Euclidean unit ball. Liu and Zhu [9] had given some sufficient conditions and concrete examples of biholomorphic convex mappings on the Reinhardt domain B_p^n. Liu [3] also gave some sufficient conditions for biholomorphic convex mappings of order α on B_p^n. A problem is naturally posed: can we give several direct criteria for biholomorphic convex mapping of order α on B_p^n? For example, can we get some sufficient conditions such that the mapping of the form

$$\begin{aligned} f(z) = (& p_1(z_1, z_2, \ldots, z_n), p_2(z_2, z_n), \ldots, \\ & p_{n-1}(z_{n-1}, z_n), p_n(z_n)) \end{aligned} \tag{4}$$

is a biholomorphic convex mapping of order α on B_p^n?

The aim of this paper is to give an answer to the above problem. From these, we may construct some concrete biholomorphic convex mappings of order α on B_p^n.

2. Main Results

Theorem 2. *Suppose that $n \ge 2$, $p \ge 2$, $0 \le \alpha < 1$, $q = p/(p-1)$. Let*

$$f(z) = (f_1(z_1, z_n), f_2(z_2, z_n) \ldots, f_{n-1}(z_{n-1}, z_n), p_n(z_n)), \tag{5}$$

where $z = (z_1, z_2, \ldots, z_n) \in B_p^n$, $p_n(\zeta) \in H(U)$ and $f_j(z_j, z_n) : B_p^2 \to C$ is holomorphic with $f_j(0,0) = 0$, $(\partial f_j/\partial z_j)(0,0) = 1$, $(\partial f_j/\partial z_n)(0,0) = 0$ $(j = 1, 2, \ldots, n-1)$. If f satisfies the following conditions:

$$\begin{aligned} &(1)\ \prod_{j=1}^{n-1}\frac{\partial f_j}{\partial z_j}\cdot p_n'(z_n) \ne 0, \\ &\quad \left|z_n p_n''(z_n)\right| \le (1-\alpha)\left|p_n'(z_n)\right|, \\ &(2)\ \left|z_j\frac{\partial^2 f_j}{\partial z_j^2}\right| + \left|z_j\frac{\partial^2 f_j}{\partial z_j \partial z_n}\right| \\ &\qquad \le (1-\alpha)\left|\frac{\partial f_j}{\partial z_j}\right| \quad (j = 1, 2, \ldots, n-1), \\ &(3)\ \left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \\ &\qquad + \left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \\ &\qquad + \left(1-|z_n|^p\right)^{1/q} \\ &\qquad \times \left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p} \\ &\qquad \le \left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)|z_n|^{p-2} \end{aligned} \tag{6}$$

for all $z = (z_1, \ldots, z_n) \in B_p^n$, then $f \in K(B_p^n, \alpha)$.

Proof. By direct computation of the Fréchet derivatives of $f(z)$, we obtain

$$Df(z)=\begin{pmatrix}\frac{\partial f_1}{\partial z_1} & 0 & \cdots & 0 & \frac{\partial f_1}{\partial z_n}\\ 0 & \frac{\partial f_2}{\partial z_2} & \cdots & 0 & \frac{\partial f_2}{\partial z_n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \frac{\partial f_{n-1}}{\partial z_{n-1}} & \frac{\partial f_{n-1}}{\partial z_n}\\ 0 & 0 & \cdots & 0 & p_n'(z_n)\end{pmatrix},$$

$$Df(z)^{-1}=\begin{pmatrix}\frac{1}{\partial f_1/\partial z_1} & 0 & \cdots & 0 & -\frac{\partial f_1/\partial z_n}{(\partial f_1/\partial z_1)\,p_n'(z_n)}\\ 0 & \frac{1}{\partial f_2/\partial z_2} & \cdots & 0 & -\frac{\partial f_2/\partial z_n}{(\partial f_2/\partial z_2)\,p_n'(z_n)}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \frac{1}{\partial f_{n-1}/\partial z_{n-1}} & -\frac{\partial f_{n-1}/\partial z_n}{(\partial f_{n-1}/\partial z_{n-1})\,p_n'(z_n)}\\ 0 & 0 & \cdots & 0 & \frac{1}{p_n'(z_n)}\end{pmatrix}, \tag{7}$$

$$D^2f(z)(b,b)=\begin{pmatrix}B_1 & 0 & \cdots & 0 & A_1\\ 0 & B_2 & \cdots & 0 & A_2\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & B_{n-1} & A_{n-1}\\ 0 & 0 & \cdots & 0 & A_n\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ \cdots\\ b_{n-1}\\ b_n\end{pmatrix}=\begin{pmatrix}\frac{\partial^2 f_1}{\partial z_1^2}b_1^2+2\frac{\partial^2 f_1}{\partial z_1\partial z_n}b_1b_n+\frac{\partial^2 f_1}{\partial z_n^2}b_n^2\\ \frac{\partial^2 f_2}{\partial z_2^2}b_2^2+2\frac{\partial^2 f_2}{\partial z_2\partial z_n}b_2b_n+\frac{\partial^2 f_2}{\partial z_n^2}b_n^2\\ \vdots\\ \frac{\partial^2 f_{n-1}}{\partial z_{n-1}^2}b_{n-1}^2+2\frac{\partial^2 f_{n-1}}{\partial z_{n-1}\partial z_n}b_{n-1}b_n+\frac{\partial^2 f_{n-1}}{\partial z_n^2}b_n^2\\ p_n''(z_n)\,b_n^2\end{pmatrix},$$

where

$$A_j=\frac{\partial^2 f_j}{\partial z_n\partial z_j}b_j+\frac{\partial^2 f_j}{\partial z_n^2}b_n\quad(j=1,2,\ldots,n-1),$$
$$A_n=p_n''(z_n)\,b_n, \tag{8}$$
$$B_j=\frac{\partial^2 f_j}{\partial z_j^2}b_j+\frac{\partial^2 f_j}{\partial z_j\partial z_n}b_n\quad(j=1,2,\ldots,n-1).$$

Taking $z=(z_1,\ldots,z_n)\in B_p^n$, $b=(b_1,\ldots,b_n)\in C^n$ such that $\operatorname{Re}\langle b,(\partial u/\partial\bar z)\rangle=0$, by the hypothesis of Theorem 2, we have

$$\begin{aligned}&\frac{2}{p}J_f(z,b)-\alpha\sum_{j=1}^n|b_j|^2|z_j|^{p-2}\\ &\geq(1-\alpha)\sum_{j=1}^n|b_j|^2|z_j|^{p-2}\\ &\quad-\frac{2}{p}\operatorname{Re}\left\langle Df(z)^{-1}D^2f(z)(b,b),\frac{\partial u}{\partial\bar z}\right\rangle\\ &=(1-\alpha)\sum_{j=1}^n|b_j|^2|z_j|^{p-2}\\ &\quad-\operatorname{Re}\left\{\sum_{j=1}^{n-1}\frac{1}{\partial f_j/\partial z_j}\left[\frac{\partial^2 f_j}{\partial z_j^2}b_j^2+2\frac{\partial^2 f_j}{\partial z_j\partial z_n}b_jb_n+\frac{\partial^2 f_j}{\partial z_n}b_n^2-\frac{\partial f_j}{\partial z_n}\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\right]\right.\\ &\qquad\left.\times\frac{|z_j|^p}{z_j}+\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\frac{|z_n|^p}{z_n}\right\}\\ &\geq(1-\alpha)\sum_{j=1}^{n-1}|b_j|^2|z_j|^{p-2}\\ &\quad-\sum_{j=1}^n\left|\frac{1}{\partial f_j/\partial z_j}\right|\left[\left[\left|\frac{\partial^2 f_j}{\partial z_j^2}\right||b_j|^2+\left|\frac{\partial^2 f_j}{\partial z_j\partial z_n}\right||b_j|^2\right.\right.\\ &\qquad+\left|\frac{\partial^2 f_j}{\partial z_j\partial z_n}\right||b_n|^2+\left|\frac{\partial^2 f_j}{\partial z_n^2}\right||b_n|^2\end{aligned}$$

$$+\left|\frac{\partial f_j}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\left|b_n\right|^2\Bigg]\left|z_j\right|^{p-1}$$

$$-\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\left|b_n\right|^2\left|z_n\right|^{p-1}$$

$$=\sum_{j=1}^{n-1}\left|b_j\right|^2\left|z_j\right|^{p-2}$$

$$\times\left[1-\alpha-\frac{\left|\partial^2 f_j/\partial z_j^2\right|+\left|\partial^2 f_j/\partial z_j\partial z_n\right|}{\left|\partial f_j/\partial z_j\right|}\left|z_j\right|\right]$$

$$+\left|b_n\right|^2\Bigg[\left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)\left|z_n\right|^{p-2}$$

$$-\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|\left|z_j\right|^{p-1}$$

$$-\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_n^2}{\partial f_j/\partial z_j}\right|\left|z_j\right|^{p-1}$$

$$-\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\left|z_j\right|^{p-1}\Bigg]. \tag{9}$$

By Hölder's inequality, we have

$$\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|\left|z_j\right|^{p-1}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\left|z_j\right|^{(p-1)q}\right)^{1/q}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}\left(1-\left|z_n\right|^p\right)^{1/q},$$

$$\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_n^2}{\partial f_j/\partial z_j}\right|\left|z_j\right|^{p-1}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_n^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\left|z_j\right|^{(p-1)q}\right)^{1/q}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_n^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}\left(1-\left|z_n\right|^p\right)^{1/q},$$

$$\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\left|z_j\right|^{p-1}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p}\left(\sum_{j=1}^{n-1}\left|z_j\right|^{(p-1)q}\right)^{1/q}$$

$$\leq\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p}\left(1-\left|z_n\right|^p\right)^{1/q}. \tag{10}$$

Hence, we conclude from the above inequalities and the hypothesis of Theorem 2 that

$$\frac{2}{p}J_f(z,b)-\alpha\sum_{j=1}^{n}\left|b_j\right|^2\left|z_j\right|^{p-2}$$

$$\geq\left|b_n\right|^2\Bigg[\left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)\left|z_n\right|^{p-2}$$

$$-\left(1-\left|z_n\right|^p\right)^{1/q}$$

$$\times\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \tag{11}$$

$$-\left(1-\left|z_n\right|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}$$

$$-\left(1-\left|z_n\right|^p\right)^{1/q}$$

$$\times\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\cdot\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p}\Bigg]$$

$$\geq 0,$$

for all $z=(z_1,\ldots,z_n)\in B_p^n$, $b=(b_1,\ldots,b_n)$ such that $\mathrm{Re}\langle b,\partial u/\partial\overline{z}\rangle=0$. Thus, it follows from Definition 1 that $f\in K(B_p^n,\alpha)$. The proof is complete. □

Remark 3. Setting $\alpha=0$, $f_j(z_j,z_n)=p_j(z_j)+f_j(z_n)$, $(j=1,2,\ldots,n-1)$ in Theorem 2, we get Theorem 1 of [9].

Let us give two examples to illustrate the application of Theorem 2 in the following.

Example 4. Suppose that $p\geq 2$, $0\leq\alpha<1$, $0<|\lambda|\leq 1-\alpha$ and k is a positive integer such that $k<p\leq k+1$. Let

$$f(z)=\left(z_1+a_1z_1z_n^{k+1},z_2+a_2z_2z_n^{k+1},\ldots,z_{n-1}+a_{n-1}z_{n-1}z_n^{k+1},\frac{e^{\lambda z_n}-1}{\lambda}\right),$$
$$M(p)=\begin{cases}\dfrac{(1-\alpha-|\lambda|)^p}{(1+|\lambda|)^p(k+1)^p}, & p=k+1,\\[2ex] \dfrac{k^k(1-\alpha-|\lambda|)^p}{(1+|\lambda|)^p(k+1)^p(p-1)^{p-1}(k-p+1)^{k-p+1}}, & k<p<k+1.\end{cases} \tag{12}$$

If $\max\{|a_j|:j=1,2,\ldots,n-1\}\le(1-\alpha)/(k+2-\alpha)$ and

$$\sum_{j=1}^{n-1}\frac{|a_j|^p}{(1-|a_j|)^p}\le M(p), \tag{13}$$

then $f(z)\in K(B_p^n,\alpha)$.

Proof. Let

$$f_j(z_j,z_n)=z_j+a_jz_jz_n^{k+1}\quad(j=1,2,\ldots,n-1),$$
$$p_n(z_n)=\frac{e^{\lambda z_n}-1}{\lambda}. \tag{14}$$

Then,

$$\frac{\partial f_j}{\partial z_j}=1+a_jz_n^{k+1},\qquad\frac{\partial f_j}{\partial z_n}=(k+1)a_jz_jz_n^k,$$
$$\frac{\partial^2 f_j}{\partial z_j\partial z_n}=(k+1)a_jz_n^k,\qquad\frac{\partial^2 f_j}{\partial z_j^2}=0,$$
$$p_n'(z_n)=e^{\lambda z_n},\qquad p_n''(z_n)=\lambda p_n'(z_n). \tag{15}$$

So it follows from $|a_j|\le(1-\alpha)/(k+2-\alpha)<1$ that

$$\begin{aligned}&\left|z_j\frac{\partial^2 f_j}{\partial z_j^2}\right|+\left|z_j\frac{\partial^2 f_j}{\partial z_j\partial z_n}\right|\\&=\left|z_j(k+1)a_jz_n^k\right|\le(k+1)|a_j|\\&\le(1-\alpha)(1-|a_j|)\\&\le(1-\alpha)(1-|a_jz_n^{k+1}|)\\&\le(1-\alpha)\left|1+a_jz_n^{k+1}\right|\\&=(1-\alpha)\left|\frac{\partial f_j}{\partial z_j}\right|,\quad(j=1,2,\ldots,n-1),\end{aligned}$$
$$\left|\frac{z_np_n''(z_n)}{p_n'(z_n)}\right|=|\lambda||z_n|\le|\lambda|\le1-\alpha\Longrightarrow\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\le|\lambda|. \tag{16}$$

Since $\max\{|a_j|:j=1,2,\ldots,n-1\}\le(1-\alpha)/(k+2-\alpha)$, we have

$$\left|\frac{\partial f_j}{\partial z_j}\right|=\left|1+a_jz_n^{k+1}\right|\ge1-|a_j|>0. \tag{17}$$

By straightforward calculations, we obtain

$$\begin{aligned}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}&\le\left\{\sum_{j=1}^{n-1}\left|\frac{(k+1)a_jz_n^k}{1+a_jz_n^{k+1}}\right|^p\right\}^{1/p}\\&\le(k+1)\left\{\sum_{j=1}^{n-1}\frac{|a_j|^p}{(1-|a_j|)^p}\right\}^{1/p}\\&\quad\times|z_n|^{k-1},\end{aligned}$$
$$\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p}=0,$$
$$\begin{aligned}&\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p}\\&\le|\lambda|\left\{\sum_{j=1}^{n-1}\left|\frac{(k+1)a_jz_jz_n^k}{1+a_jz_n^{k+1}}\right|^p\right\}^{1/p}\\&\le(k+1)|\lambda|\\&\quad\times\left\{\sum_{j=1}^{n-1}\frac{|a_j|^p}{(1-|a_j|)^p}\right\}^{1/p}\\&\quad\times|z_n|^{k-1}.\end{aligned} \tag{18}$$

Set $q = p/(p-1)$. Then,

$$\begin{aligned} &\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \\ &\quad\le (k+1)\left\{\sum_{j=1}^{n-1}\frac{|a_j|^p}{\left(1-|a_j|\right)^p}\right\}^{1/p}\varphi\left(|z_n|\right)|z_n|^{p-2}, \\ &\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p} \\ &\quad\le (k+1)|\lambda|\left\{\sum_{j=1}^{n-1}\frac{|a_j|^p}{\left(1-|a_j|\right)^p}\right\}^{1/p}\varphi\left(|z_n|\right)|z_n|^{p-2}, \end{aligned} \tag{19}$$

where $\varphi(x) = (1-x^p)^{1/q}x^{k-p+1},\ x\in[0,1]$.

When $p = k+1$, we have $\max_{0\le x\le 1}\varphi(x) = 1$.

When $k < p < k+1$, we have $0 < k-p+1 < 1$ and

$$\varphi'(x) = \left(1-x^p\right)^{(1/q)-1}x^{k-p}\left[(k-p+1)-kx^p\right], \tag{20}$$

so

$$\begin{aligned} \max_{0\le x\le 1}\varphi(x) &= \varphi\left(\sqrt[p]{\frac{k-p+1}{k}}\right) \\ &= \left(\frac{p-1}{k}\right)^{1/q}\left(\frac{k-p+1}{k}\right)^{(k-p+1)/p}. \end{aligned} \tag{21}$$

Hence, when

$$\sum_{j=1}^{n-1}\frac{|a_j|^p}{\left(1-|a_j|\right)^p} \le M(p), \tag{22}$$

we have

$$\begin{aligned} &\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j\partial z_n}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \\ &\quad+\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial^2 f_j/\partial z_j^2}{\partial f_j/\partial z_j}\right|^p\right)^{1/p} \\ &\quad+\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=1}^{n-1}\left|\frac{\partial f_j/\partial z_n}{\partial f_j/\partial z_j}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p} \\ &\le \left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)|z_n|^{p-2}, \\ &\qquad z_j\in U,\ j=1,2,\ldots,n. \end{aligned} \tag{23}$$

By Theorem 2, we obtain that $f\in K(B_p^n,\alpha)$. The proof is complete. □

Example 5. Suppose that $p\ge 2$, $0\le\alpha<1$, $|a_n|\le(1-\alpha)/(4-2\alpha)$ and k is a positive integer such that $k<p\le k+1$. Let

$$\begin{aligned} f(z) &= \left(z_1+a_1z_1z_n^{k+1}, z_2+a_2z_2z_n^{k+1},\ldots,z_{n-1}+a_{n-1}z_{n-1}z_n^{k+1}, z_n+a_nz_n^2\right), \\ M'(p) &= \begin{cases} \dfrac{\left[1-\alpha-(4-2\alpha)|a_n|\right]^p}{(k+1)^p}, & p=k+1, \\ \dfrac{k^k\left[1-\alpha-(4-2\alpha)|a_n|\right]^p}{(k+1)^p(p-1)^{p-1}(k-p+1)^{k-p+1}}, & k<p<k+1. \end{cases} \end{aligned} \tag{24}$$

If $\max\{|a_j| : j=1,2,\ldots,n-1\}\le(1-\alpha)/(k+2-\alpha)<1$ and

$$\sum_{j=1}^{n-1}\frac{|a_j|^p}{\left(1-|a_j|\right)^p}\le M'(p), \tag{25}$$

then $f(z)\in K(B_p^n,\alpha)$.

By applying the same method of the proof for Theorem 2, we may prove the following result.

Theorem 6. *Suppose that $n\ge 2$, $0\le\alpha<1$, $p\ge 2$ and $f_j: U\to C$ are analytic on U, $f_j(0)=f_j'(0)=0$ $(j=1,2,\ldots,n-1)$, $p_j\in H(U)$, $P_j'(\xi)\neq 0$, $|\xi p_j''(\xi)|\le(1-\alpha)|p_j'(\xi)|$ $(\xi\in U, j=1,2,\ldots n)$. Let*

$$\begin{aligned} f(z) = \big(&p_1(z_1)+f_1(z_k), p_2(z_2)+f_2(z_k)\ldots, \\ &p_k(z_k),\ldots,p_{n-1}(z_{n-1})+f_{n-1}(z_n), p_n(z_n)\big), \end{aligned} \tag{26}$$

$(1\le k\le n$, when $k=j$, $f_j(z_k)=0)$.

If for any $z=(z_1,z_2,\ldots,z_n)\in B_p^n$, we have

$$\begin{aligned} &\left(1-|z_k|^p\right)^{1/q}\left(\sum_{j=1}^{k-1}\left|\frac{f_j''(z_k)}{p_j'(z_j)}\right|^p\right)^{1/p}+\left(1-|z_k|^p\right)^{1/q} \\ &\quad\times\left(\sum_{j=1}^{k-1}\left|\frac{f_j'(z_k)}{p_j'(z_j)}\right|^p\left|\frac{p_k''(z_k)}{p_k'(z_k)}\right|^p\right)^{1/p} \end{aligned}$$

$$\leq \left(1-\alpha-\left|\frac{z_k p_k''(z_k)}{p_k'(z_k)}\right|\right)|z_k|^{p-2},$$

$$\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=k+1}^{n-1}\left|\frac{f_j''(z_n)}{p_j'(z_j)}\right|^p\right)^{1/p}$$

$$+\left(1-|z_n|^p\right)^{1/q}\left(\sum_{j=k+1}^{n-1}\left|\frac{f_j'(z_n)}{p_j'(z_j)}\right|^p\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|^p\right)^{1/p}$$

$$\leq \left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)|z_n|^{p-2}, \tag{27}$$

then $f(z) \in K(B_p^n, \alpha)$.

Remark 7. Setting $k = n$, $\alpha = 0$ in Theorem 6, we get Theorem 1 in [9].

Example 8. Suppose that $p \geq 2$, $0 \leq \alpha < 1$, $0 < |\lambda| \leq 1-\alpha$ and k is a positive integer such that $k < p \leq k+1$. Let

$$f(z) = \left(z_1 + a_1' z_1^2 + a_1 z_2^{k+1} + b_1 z_2^{k+2}, \frac{e^{\lambda z_2 - 1}}{\lambda}, z_3 + a_3' z_3^2 + a_3 z_n^{k+1} + b_3 z_n^{k+2}, \ldots, z_{n-1} + a_{n-1}' z_{n-1}^2 + a_{n-1} z_n^{k+1} + b_{n-1} z_n^{k+2}, \frac{e^{\lambda z_n - 1}}{\lambda}\right),$$

$$M''(p) = \begin{cases} \dfrac{(1-2c)^p(1-\alpha-|\lambda|)^p}{(k+1+|\lambda|)^p}, & p = k+1, \\ \dfrac{(1-2c)^p k^k (1-\alpha-|\lambda|)^p}{(k+1+|\lambda|)^p (p-1)^{p-1} (k-p+1)^{k-p+1}}, & k < p < k+1, \end{cases} \tag{28}$$

where $c = \max\{|a_j'| : j = 1, 2, \ldots, n-1\}$. If $c \leq (1-\alpha)/4 < 1$ and

$$\left[(k+1)|a_1| + (k+2)|b_1|\right]^p \leq M''(p),$$
$$\sum_{j=3}^{n-1}\left[(k+1)|a_j| + (k+2)|b_j|\right]^p \leq M''(p), \tag{29}$$

then $f(z) \in K(B_p^n, \alpha)$.

Now, we give another sufficient condition for $K(B_p^n, \alpha)$, which gives an answer to the problem mentioned in the introduction.

Theorem 9. *Suppose that* $n \geq 2$, $0 \leq \alpha < 1$, $p \geq 2$ *and* k *is a positive integer such that* $k < p \leq k+1$. *Let*

$$f(z) = (p_1(z_1, z_2, \ldots, z_n), p_2(z_2, z_n), \ldots, p_{n-1}(z_{n-1}, z_n), p_n(z_n)), \tag{30}$$

where $z = (z_1, z_2, \ldots, z_n) \in B_p^n$, $p_n \in H(U)$, $p_j(z_j, z_n) : B_p^2 \to C$ *is holomorphic with* $p_j(0,0) = 0$, $(\partial p_j/\partial z_j)(0,0) = 1$, $(\partial p_j/\partial z_n)(0,0) = 0$ *and* $p_1(z_1, \ldots, z_n) : B_p^n \to C$ *is holomorphic with* $p_1(0,0,\ldots,0) = 0$, $(\partial p_1/\partial z_1)(0,0,\ldots,0) = 1$, $(\partial p_1/\partial z_l)(0,0,\ldots,0) = 0$ *for* $2 \leq l \leq n$. *If* $f(z)$ *satisfies the following conditions:*

(1) $\prod_{j=1}^{n-1}\dfrac{\partial p_j}{\partial z_j} p_n'(z_n) \neq 0,$

$$\left|z_n p_n''(z_n)\right| \leq (1-\alpha)\left|p_n'(z_n)\right|;$$

(2) $\sum_{l=1}^{n}\left|z_1\dfrac{\partial^2 p_1}{\partial z_1 \partial z_l}\right| \leq (1-\alpha)\left|\dfrac{\partial p_1}{\partial z_1}\right|,$

$$\left|z_j\frac{\partial^2 p_j}{\partial z_j^2}\right| + \left|z_j\frac{\partial^2 p_j}{\partial z_j \partial z_n}\right| \leq (1-\alpha)\left|\frac{\partial p_j}{\partial z_j}\right| \quad (j = 2, 3, \ldots, n-1);$$

(3) $\dfrac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|}\left(\dfrac{|\partial p_1/\partial z_j|\left(|\partial^2 p_j/\partial z_j^2| + |\partial^2 p_j/\partial z_j \partial z_n|\right)}{|\partial p_j/\partial z_j|} + \sum_{l=1}^{n}\left|\dfrac{\partial^2 p_1}{\partial z_j \partial z_l}\right|\right)$

$$\leq |z_j|^{p-2}\left(1-\alpha-\frac{|z_j \partial^2 p_j/\partial z_j^2| + |z_j \partial^2 p_j/\partial z_j \partial z_n|}{|\partial p_j/\partial z_j|}\right) \quad (j = 2, 3, \ldots, n-1);$$

(4) $\sum_{j=2}^{n-1}\left|\dfrac{\partial^2 p_j/\partial z_j \partial z_n}{\partial p_j/\partial z_j}\right||z_j|^{p-1} + \sum_{j=2}^{n-1}\left|\dfrac{\partial^2 p_j/\partial z_n^2}{\partial p_j/\partial z_j}\right||z_j|^{p-1} + \sum_{j=2}^{n-1}\left|\dfrac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\dfrac{p_n''(z_n)}{p_n'(z_n)}\right||z_j|^{p-1}$

$$+\frac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|} \times \left(\sum_{l=1}^{n}\left|\frac{\partial^2 p_1}{\partial z_n \partial z_l}\right| + \sum_{j=2}^{n-1}\left(\left|\frac{\partial p_1}{\partial z_j}\right|\left(\left|\frac{\partial^2 p_j}{\partial z_j \partial z_n}\right| + \left|\frac{\partial^2 p_j}{\partial z_n^2}\right|\right) \times \left|\frac{\partial p_j}{\partial z_j}\right|^{-1}\right) + \left|\frac{\partial p_1}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right| + \sum_{j=2}^{n-1}\left|\frac{\partial p_1}{\partial z_j}\right|\left|\frac{(\partial p_j/\partial z_n)}{(\partial p_j/\partial z_j)}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\right)$$

$$\leq |z_n|^{p-2}\left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right); \tag{31}$$

then $f \in K(B_p^n, \alpha)$.

Proof. By calculating the Fréchet derivatives of $f(z)$ straightforwardly, we obtain

$$Df(z) = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} & \frac{\partial p_1}{\partial z_2} & \cdots & \frac{\partial p_1}{\partial z_{n-1}} & \frac{\partial p_1}{\partial z_n} \\ 0 & \frac{\partial p_2}{\partial z_2} & \cdots & 0 & \frac{\partial p_2}{\partial z_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{\partial p_{n-1}}{\partial z_{n-1}} & \frac{\partial p_{n-1}}{\partial z_n} \\ 0 & 0 & \cdots & 0 & p_n'(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} \frac{1}{\partial p_1/\partial z_1} & -\frac{\partial p_1/\partial z_2}{(\partial p_1/\partial z_1)(\partial p_2/\partial z_2)} & \cdots & -\frac{\partial p_1/\partial z_{n-1}}{(\partial p_1/\partial z_1)(\partial p_{n-1}/\partial z_{n-1})} & -\frac{\partial p_1/\partial z_n}{(\partial p_1/\partial z_1)\, p_n'(z_n)} + \sum_{j=2}^{n-1}\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\frac{\partial p_1/\partial z_j}{(\partial p_1/\partial z_1)\, p_n'(z_n)} \\ 0 & \frac{1}{\partial p_2/\partial z_2} & \cdots & 0 & -\frac{\partial p_2/\partial z_n}{(\partial p_2/\partial z_2)\, p_n'(z_n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\partial p_{n-1}/\partial z_{n-1}} & -\frac{\partial p_{n-1}/\partial z_n}{(\partial p_{n-1}/\partial z_{n-1})\, p_n'(z_n)} \\ 0 & 0 & \cdots & 0 & \frac{1}{p_n'(z_n)} \end{pmatrix} \tag{32}$$

$$D^2 f(z)(b,b) = \begin{pmatrix} \sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_1\partial z_l}b_l & \sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_2\partial z_l}b_l & \cdots & \sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_{n-1}\partial z_l}b_l & C_1 \\ 0 & D_2 & \cdots & 0 & C_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_{n-1} & C_{n-1} \\ 0 & 0 & \cdots & 0 & C_n \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_j\partial z_l}b_l b_j \\ \frac{\partial^2 p_2}{\partial z_2^2}b_2^2 + 2\frac{\partial^2 p_2}{\partial z_2\partial z_n}b_2 b_n + \frac{\partial^2 p_2}{\partial z_n^2}b_n^2 \\ \vdots \\ \frac{\partial^2 p_{n-1}}{\partial z_{n-1}^2}b_{n-1}^2 + 2\frac{\partial^2 p_{n-1}}{\partial z_{n-1}\partial z_n}b_{n-1}b_n + \frac{\partial^2 p_{n-1}}{\partial z_n^2}b_n^2 \\ p_n''(z_n)\, b_n^2 \end{pmatrix},$$

where

$$\begin{aligned} C_1 &= \sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_n\partial z_l}b_l, \\ C_j &= \frac{\partial^2 p_j}{\partial z_n\partial z_j}b_j + \frac{\partial^2 p_j}{\partial z_n^2}b_n \quad (j = 2,3,\ldots,n-1), \\ C_n &= p_n''(z_n)\, b_n, \\ D_j &= \frac{\partial^2 p_j}{\partial z_j^2}b_j + \frac{\partial^2 p_j}{\partial z_j\partial z_n}b_n \quad (j = 2,3,\ldots,n-1). \end{aligned} \tag{33}$$

Taking $z = (z_1,\ldots,z_n) \in B_p^n$, $b = (b_1,\ldots,b_n) \in C^n$ such that $\mathrm{Re}\langle b, \partial u/\partial \bar{z}\rangle = 0$, by Definition 1 and the hypothesis of Theorem 9, we have

$$\begin{aligned} &\frac{2}{p}J_f(z,b) - \alpha\sum_{j=1}^{n}|b_j|^2|z_j|^{p-2} \\ &\quad\geq (1-\alpha)\sum_{j=1}^{n}|b_j|^2|z_j|^{p-2} \\ &\qquad - \frac{2}{p}\mathrm{Re}\left\langle Df(z)^{-1}D^2 f(z)(b,b), \frac{\partial u}{\partial \bar{z}}\right\rangle \\ &\quad= (1-\alpha)\sum_{j=1}^{n}|b_j|^2|\, z_j|^{p-2} \end{aligned}$$

$$
\begin{aligned}
&-\operatorname{Re}\left\{\frac{1}{\partial p_1/\partial z_1}\right.\\
&\quad\times\left[\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 p_1}{\partial z_j\partial z_l}b_l b_j\right.\\
&\qquad-\sum_{j=2}^{n-1}\frac{\partial p_1/\partial z_j}{\partial p_j/\partial z_j}\\
&\qquad\times\left(\frac{\partial^2 p_j}{\partial z_j^2}b_j^2+2\frac{\partial^2 p_j}{\partial z_j\partial z_n}b_j b_n+\frac{\partial^2 p_j}{\partial z_n^2}b_n^2\right)\\
&\qquad-\frac{\partial p_1}{\partial z_n}\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\\
&\qquad\left.+\sum_{j=2}^{n-1}\frac{(\partial p_j/\partial z_n)(\partial p_1/\partial z_j)}{\partial p_j/\partial z_j}\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\right]\frac{|z_1^p|}{z_1}\\
&\qquad+\sum_{j=2}^{n-1}\frac{1}{\partial p_j/\partial z_j}\left(\frac{\partial^2 p_j}{\partial z_j^2}b_j^2+2\frac{\partial^2 p_j}{\partial z_j\partial z_n}b_j b_n\right.\\
&\qquad\qquad\left.+\frac{\partial^2 p_j}{\partial z_n^2}b_n^2-\frac{\partial p_j}{\partial z_n}\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\right)\\
&\qquad\left.\times\frac{|z_j|^p}{z_j}+\frac{p_n''(z_n)}{p_n'(z_n)}b_n^2\frac{|z_n|^p}{z_n}\right\}\\
&\geq(1-\alpha)\sum_{j=1}^{n}|b_j|^2|z_j|^{p-2}-\frac{1}{|\partial p_1/\partial z_1|}\\
&\quad\times\left[\sum_{j=1}^{n}\sum_{l=1}^{n}\left|\frac{\partial^2 p_1}{\partial z_j\partial z_l}\right||b_j|^2+\sum_{j=2}^{n-1}\left|\frac{\partial p_1/\partial z_j}{\partial p_j/\partial z_j}\right|\right.\\
&\quad\times\left(\left|\frac{\partial^2 p_j}{\partial z_j^2}\right||b_j|^2+\left|\frac{\partial^2 p_j}{\partial z_j\partial z_n}\right||b_j|^2\right.\\
&\qquad\left.+\left|\frac{\partial^2 p_j}{\partial z_j\partial z_n}\right||b_n|^2+\left|\frac{\partial^2 p_j}{\partial z_n}\right||b_n^2|\right)+\left|\frac{\partial p_1}{\partial z_n}\right|\\
&\quad\times\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||b_n|^2\\
&\qquad\left.+\sum_{j=2}^{n-1}\left|\frac{(\partial p_j/\partial z_n)(\partial p_1/\partial z_j)}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||b_n|^2\right]|z_1|^{p-1}\\
&\quad-\sum_{j=2}^{n-1}\frac{1}{|\partial p_j/\partial z_j|}\left(\left|\frac{\partial^2 p_j}{\partial z_j^2}\right||b_j|^2+\left|\frac{\partial^2 p_j}{\partial z_j\partial z_n}\right||b_j|^2\right.\\
&\qquad+\left|\frac{\partial p_j}{\partial z_j\partial z_n}\right||b_n|^2+\left|\frac{\partial^2 p_j}{\partial z_n^2}\right||b_n|^2\\
&\qquad\left.+\left|\frac{\partial p_j}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||b_n|^2\right)|z_j|^{p-1}\\
&\quad-\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||b_n|^2|z_n|^{p-1}\\
&=|b_1|\,|z_1|^{p-2}\\
&\quad\times\left[1-\alpha-\frac{\sum_{l=1}^{n}|z_1\partial^2 p_1/\partial z_1\partial z_l|}{|\partial p_1/\partial z_1|}\right]\\
&\quad+\sum_{j=2}^{n-1}|b_j|^2\left[|z_j|^{p-2}\right.\\
&\qquad\times\left(1-\alpha\right.\\
&\qquad\qquad\left.-\frac{|z_j\partial^2 p_j/\partial z_j^2|+|z_j\partial^2 p_j/\partial z_j\partial z_n|}{|\partial p_j/\partial z_j|}\right)\\
&\qquad-\frac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|}\\
&\qquad\times\left(\sum_{l=1}^{n}\left|\frac{\partial^2 p_1}{\partial z_j\partial z_l}\right|+\left|\frac{\partial p_1}{\partial z_j}\right|\right.\\
&\qquad\qquad\left.\left.\times\frac{|\partial^2 p_j/\partial z_j^2|+|\partial^2 p_j/\partial z_j\partial z_n|}{|\partial p_j/\partial z_j|}\right)\right]\\
&\quad+|b_n|^2\left[|z_n|^{p-2}\left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right)\right.\\
&\qquad-\frac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|}\\
&\qquad\times\left(\sum_{l=1}^{n}\left|\frac{\partial^2 p_1}{\partial z_n\partial z_l}\right|\right.\\
&\qquad\qquad+\sum_{j=2}^{n-1}\left|\frac{\partial p_1}{\partial z_j}\right|\\
&\qquad\qquad\times\frac{|\partial^2 p_j/\partial z_n^2|+|\partial^2 p_j/\partial z_j\partial z_n|}{|\partial p_j/\partial z_j|}\\
&\qquad\qquad+\left|\frac{\partial p_1}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|+\sum_{j=2}^{n-1}\left|\frac{\partial p_1}{\partial z_j}\right|\\
&\qquad\qquad\left.\times\left|\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=2}^{n-1} \left| \frac{\partial^2 p_j / \partial z_j \partial z_n}{\partial p_j / \partial z_j} \right| \left| z_j \right|^{p-1} \\
&- \sum_{j=2}^{n-1} \left| \frac{\partial^2 p_j / \partial z_n^2}{\partial p_j / \partial z_j} \right| \left| z_j \right|^{p-1} \\
&- \sum_{j=2}^{n-1} \left| \frac{\partial p_j / \partial z_n}{\partial p_j / \partial z_j} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| \left| z_j \right|^{p-1} \Bigg] \\
&\geq 0
\end{aligned} \tag{34}
$$

for all $z = (z_1, \dots, z_n) \in B_p^n$, $b = (b_1, \dots, b_n)$ such that $\operatorname{Re}\langle b, \partial u / \partial \bar{z} \rangle = 0$. Thus, it follows from Definition 1 that $f \in K(B_p^n, \alpha)$. The proof is complete. □

Corollary 10. *Suppose that $0 \leq \alpha < 1$, $n \geq 2$, $p \geq 2$ and k is a positive integer such that $k < p \leq k+1$. Let*

$$
\begin{aligned}
f(z) = (&p_1(z_1, z_2, \dots, z_n), p_2(z_2) + f_2(z_n), \dots, \\
&p_{n-1}(z_{n-1}) + f_{n-1}(z_n), p_n(z_n)),
\end{aligned} \tag{35}
$$

where $z = (z_1, z_2, \dots, z_n) \in B_p^n$, $f_j : U \to C$ is holomorphic with $f_j(0) = 0$, $f_j'(0) = 0$ $(j = 2, 3, \dots, n-1)$, $p_j \in H(U)$ $(j = 2, 3, \dots, n)$ and $p_1(z_1, \dots, z_n) : B_p^n \to C$ is holomorphic with $p_1(0, 0, \dots, 0) = 0$, $(\partial p_1/\partial z_1)(0, 0, \dots, 0) = 1$, $(\partial p_1 / \partial z_l)(0, 0, \dots, 0) = 0$ $(l = 2, 3, \dots, n)$. If f satisfies the following conditions:

(1) $$\frac{\partial p_1}{\partial z_1} \cdot \prod_{j=2}^{n} p_j'(z_j) \neq 0,$$

$$\left| z_j p_j''(z_j) \right| \leq (1-\alpha) \left| p_j'(z_j) \right|, \quad j = 2, \dots, n;$$

(2) $$\sum_{l=1}^{n} \left| z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l} \right| \leq (1-\alpha) \left| \frac{\partial p_1}{\partial z_1} \right|;$$

(3) $$
\begin{aligned}
&|z_1|^{p-1} \left| \frac{(\partial p_1/\partial z_j) \cdot (p_j''(z_j)/p_j'(z_j))}{\partial p_1/\partial z_1} \right| + |z_1|^{p-1} \\
&\times \sum_{l=1}^{n} \left| \frac{\partial^2 p_1 / \partial z_j \partial z_l}{\partial p_1 / \partial z_1} \right| \\
&\leq \left(1 - \alpha - \left| \frac{z_j p_j''(z_j)}{p_j'(z_j)} \right| \right) |z_j|^{p-2}, \\
&\qquad j = 2, \dots, n-1;
\end{aligned}
$$

(4) $$
\begin{aligned}
&\sum_{j=2}^{n-1} \left| \frac{f_j''(z_n)}{p_j'(z_j)} \right| |z_j|^{p-1} \\
&+ \sum_{j=2}^{n-1} \left| \frac{f_j'(z_n)}{p_j'(z_j)} \right| \left| \frac{p_n''(z_n)}{p_n'(z_n)} \right| |z_j|^{p-1} \\
&+ |z_1|^{p-1} \sum_{j=2}^{n-1} \left| \frac{(f_j''(z_n)/p_j'(z_j))(\partial p_1 / \partial z_j)}{\partial p_1/\partial z_1} \right| \\
&+ |z_1|^{p-1} \\
&\times \sum_{j=2}^{n-1} \left| \frac{f_j'(z_n)}{p_j'(z_j)} \frac{p_n''(z_n)}{p_n'(z_n)} \frac{\partial p_1}{\partial z_j} \left(\frac{\partial p_1}{\partial z_1} \right)^{-1} \right| \\
&+ |z_1|^{p-1} \sum_{l=1}^{n} \frac{\partial^2 p_1 / \partial z_l \partial z_n}{\partial p_1 / \partial z_1} \\
&+ |z_1|^{p-1} \left| \frac{(p_n''(z_n)/p_n'(z_n))(\partial p_1/\partial z_n)}{\partial p_1/\partial z_1} \right| \\
&\leq \left(1 - \alpha - \left| \frac{z_n p_n''(z_n)}{p_n'(z_n)} \right| \right) |z_n|^{p-2},
\end{aligned} \tag{36}
$$

for all $z = (z_1, \dots, z_n) \in B_p^n$, then $f \in K(B_p^n, \alpha)$.

Remark 11. Setting $p_j(z_j, z_n) = f_j(z_j)$, $2 \leq j \leq n-1$, $\alpha = 0$ in Theorem 9 or $f_j(z_n) = 0$, $2 \leq j \leq n$, $\alpha = 0$ in Corollary 10, we get Theorem 2 in [9].

Example 12. Suppose that $p \geq 2$, $0 \leq \alpha < 1$, $0 < |\lambda| \leq 1 - \alpha$ and k is a positive integer such that $k < p \leq k+1$. Let

$$
\begin{aligned}
f(z) &= \left(z_1 + \sum_{j=2}^{n-1} a_j z_j^{k+1} + a_n z_1 z_n^{k+1}, z_2 + b_2 z_2 z_n^{k+1}, \dots, z_{n-1} + b_{n-1} z_{n-1} z_n^{k+1}, \frac{e^{\lambda z_n} - 1}{\lambda} \right), \\
N(p) &= \begin{cases} \left(\displaystyle\sum_{j=2}^{n-1} \frac{|b_j|^p}{1 - |b_j|^p} \right)^{1/p} + \dfrac{a}{1-a} \left(1 + (k+1) \displaystyle\sum_{j=2}^{n-1} \frac{|b_j|}{1 - |b_j|} \right), & p = k+1, \\ \left(\displaystyle\sum_{j=2}^{n-1} \frac{|b_j|^p}{1 - |b_j|^p} \right)^{1/p} \left(\dfrac{p-1}{k} \right)^{1/q} \left(\dfrac{k-p+1}{k} \right)^{(k-p+1)/p} + \dfrac{a}{1-a} \left(1 + (k+1) \displaystyle\sum_{j=2}^{n-1} \frac{|b_j|}{1 - |b_j|} \right), & k < p < k+1, \end{cases}
\end{aligned} \tag{37}
$$

where $a = \max\{|a_j| : j = 2, \dots, n\}$, $b = \max\{|b_j| : j = 2, \dots, n-1\}$. If

$$a \le \frac{1 - |\lambda| - \alpha}{(k+1)^2 (k + 1 + |\lambda|) + 1 - |\lambda| - \alpha} < 1,$$
$$b \le \frac{(k + |\lambda|)(1-\alpha) + |\lambda|}{(k + 2 - \alpha)(k + 1 + |\lambda|)} < 1, \quad (38)$$
$$N(p) \le \frac{1 - \alpha - |\lambda|}{(k+1)(k+1+|\lambda|)},$$

for all $z = (z_1, \dots, z_n) \in B_p^n$, then $f(z) \in K(B_p^n, \alpha)$.

Proof. Put

$$p_1(z_1, z_2, \dots, z_n) = z_1 + \sum_{j=2}^{n-1} a_j z_j^{k+1} + a_n z_1 z_n^{k+1},$$
$$p_n(z_n) = \frac{e^{\lambda z_n} - 1}{\lambda}, \quad (39)$$
$$p_j(z_j, z_n) = z_j + b_j z_j z_n^{k+1} \quad (j = 2, 3, \dots, n-1).$$

Then,

$$\frac{\partial p_1}{\partial z_1} = 1 + a_n z_n^{k+1},$$
$$\frac{\partial p_1}{\partial z_j} = (k+1) a_j z_j^k \quad (j = 2, 3, \dots, n-1),$$
$$\frac{\partial p_1}{\partial z_n} = (k+1) a_n z_1 z_n^k,$$
$$\frac{\partial^2 p_1}{\partial z_n^2} = k(k+1) a_n z_1 z_n^{k-1},$$
$$\frac{\partial^2 p_1}{\partial z_j^2} = k(k+1) a_j z_j^{k-1},$$
$$\frac{\partial^2 p_1}{\partial z_1 \partial z_n} = \frac{\partial^2 p_1}{\partial z_n \partial z_1} = (k+1) a_n z_n^k, \quad (40)$$
$$\frac{\partial^2 p_1}{\partial z_1 \partial z_j} = 0 \quad (j = 2, 3, \dots, n-1),$$
$$\frac{\partial p_j}{\partial z_j} = 1 + b_j z_n^{k+1}, \qquad \frac{\partial p_j}{\partial z_n} = (k+1) b_j z_j z_n^k,$$
$$\frac{\partial^2 p_j}{\partial z_j \partial z_n} = \frac{\partial^2 p_j}{\partial z_n \partial z_j} = (k+1) b_j z_n^k,$$
$$\frac{\partial^2 p_j}{\partial z_n^2} = k(k+1) b_j z_j z_n^{k-1},$$
$$p_n'(z_n) = e^{\lambda z_n}, \qquad p_n''(z_n) = \lambda p_n'(z_n),$$

so it follows from $a = \max\{|a_j| : j = 2, \dots, n\} \le (1 - |\lambda| - \alpha)/((k+1)^2(k+1+|\lambda|) + 1 - |\lambda| - \alpha) < (1-\alpha)/(k+2-\alpha) < 1$ and $b = \max\{|b_j| : j = 2, \dots, n-1\} \le ((k + |\lambda|)(1-\alpha) + |\lambda|)/(k+2-\alpha)(k+1+|\lambda|) \le (1-\alpha)/(k+2-\alpha) < 1$ that

$$\left|\frac{\partial p_1}{\partial z_1}\right| = \left|1 + a_n z_n^{k+1}\right| \ge 1 - |a_n| \ge 1 - a > 0,$$
$$\left|\frac{\partial p_j}{\partial z_j}\right| = \left|1 + b_j z_n^{k+1}\right| \ge 1 - |b_j| \quad (41)$$
$$\ge 1 - b > 0 \quad (j = 2, 3, \dots, n-1),$$
$$p_n'(z_n) = e^{\lambda z_n} \ne 0.$$

By calculating straightforwardly, we obtain

$$(1-\alpha)\left|\frac{\partial p_1}{\partial z_1}\right| - \sum_{l=1}^{n}\left|z_1 \frac{\partial^2 p_1}{\partial z_1 \partial z_l}\right|$$
$$= (1-\alpha)\left|1 + a_n z_n^{k+1}\right| - \left|(k+1) a_n z_1 z_n^k\right|$$
$$\ge (1-\alpha)(1 - |a_n|) - (k+1)|a_n|$$
$$= 1 - \alpha - (k+2-\alpha)|a_n|$$
$$> 1 - \alpha - (k+2-\alpha)\frac{1-\alpha}{k+2-\alpha} = 0,$$
$$(1-\alpha)\left|\frac{\partial p_j}{\partial z_j}\right| - \left|z_j \frac{\partial^2 p_j}{\partial z_j^2}\right| - \left|z_j \frac{\partial^2 p_j}{\partial z_j \partial z_n}\right| \quad (42)$$
$$= (1-\alpha)\left|1 + b_j z_n^{k+1}\right| - (k+1)\left|b_j z_j z_n^k\right|$$
$$\ge (1-\alpha)(1 - |b_j|) - (k+1)|b_j|$$
$$= 1 - \alpha - (k+2-\alpha)|b_j|$$
$$\ge 1 - \alpha - (k+2-\alpha)\frac{1-\alpha}{k+2-\alpha} = 0,$$
$$(j = 2, \dots, n-1),$$
$$\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right| = |\lambda||z_n| \le |\lambda| \le 1 - \alpha.$$

By calculating straightforwardly, we also obtain

$$|z_j|^{p-2}\left(1 - \alpha - \frac{\left|z_j \partial^2 p_j / \partial z_j^2\right| + \left|\left(\partial^2 p_j / \partial z_j \partial z_n\right)\right|}{\left(\partial p_j / \partial z_j\right)}\right)$$
$$- \frac{|z_1|^{p-1}}{|\partial p_1 / \partial z_1|}$$
$$\times \left(\frac{|\partial p_1 / \partial z_j|\left(\left|\partial^2 p_j / \partial z_j^2\right| + \left|\partial^2 p_j / \partial z_j \partial z_n\right|\right)}{\left|\partial p_j / \partial z_j\right|}\right.$$

$$
\begin{aligned}
&\quad + \sum_{l=1}^{n} \left| \frac{\partial^2 p_1}{\partial z_j \partial z_l} \right| \Bigg) \\
&= \left|z_j\right|^{p-2} \left(1 - \alpha - \left| \frac{(k+1) z_j b_j z_n^k}{1 + b_j z_n^{k+1}} \right| \right) \\
&\quad - \frac{\left|z_1\right|^{p-1}}{\left|1 + a_n z_n^{k+1}\right|} \\
&\quad \times \left(\left| \frac{(k+1) a_j z_j^k (k+1) b_j z_n^k}{b_j z_n^{k+1}} \right| \right. \\
&\quad \left. + \left| k (k+1) a_j z_j^{k-1} \right| \right) \\
&\geq \left|z_j\right|^{p-2} \left(1 - \alpha - \frac{(k+1) \left|b_j\right|}{1 - \left|b_j\right|} \right) \\
&\quad - \frac{1}{1 - \left|a_n\right|} \left(\frac{(k+1)^2 \left|a_j\right| \left|b_j\right|}{1 - \left|b_j\right|} + k (k+1) \left|a_j\right| \right) \\
&\quad \times \left|z_j\right|^{k-1} \\
&\geq \left|z_j\right|^{p-2} \left(1 - \alpha - \frac{(k+1) b}{1 - b} \right) \\
&\quad - \frac{1}{1-a} \left(\frac{(k+1)^2 ab}{1 - b} + (k+1)^2 a \right) \left|z_j\right|^{p-2} \\
&\geq \left|z_j\right|^{p-2} \left(1 - \alpha - \frac{(k+1) b}{1 - b} - \frac{(k+1) 2a}{1 - a} \frac{1}{1 - b} \right) \\
&= \frac{\left|z_j\right|^{p-2}}{1 - b} \left((1 - \alpha)(1 - b) - (k+1) b - \frac{(k+1)^2}{1 - a} a \right) \\
&= \frac{\left|z_j\right|^{p-2}}{1 - b} \left(1 - \alpha - (k + 2 - \alpha) b - (k+1)^2 \frac{a}{1 - a} \right) \\
&\geq \frac{\left|z_j\right|^{p-2}}{1 - b} \Bigg(1 - \alpha - (k + 2 - \alpha) \\
&\quad \times \frac{(k + |\lambda|)(1 - \alpha) + |\lambda|}{(k + 2 - \alpha)(k + 1 + |\lambda|)} \\
&\quad - \frac{(k+1)^2 (1 - |\lambda| - \alpha)}{(k+1)^2 (k + 1 + |\lambda|) + 1 - |\lambda| - \alpha} \\
&\quad \times \frac{(k+1)^2 (k + 1 + |\lambda|) + 1 - |\lambda| - \alpha}{(k+1)^2 (k + 1 + |\lambda|)} \Bigg) \\
&= \frac{\left|z_j\right|^{p-2}}{1 - b} \left(1 - \alpha - \frac{(k + |\lambda|)(1 - \alpha) + |\lambda|}{k + 1 + |\lambda|} - \frac{1 - |\lambda| - \alpha}{k + 1 + |\lambda|} \right) \\
&= 0.
\end{aligned} \tag{43}
$$

Set $q = p/(p-1)$, then by Hölder's inequality, we have

$$
\begin{aligned}
&\sum_{j=2}^{n-1} \left| \frac{\partial^2 p_j / \partial z_j \partial z_n}{\partial p_j / \partial z_j} \right| \left|z_j\right|^{p-1} \\
&\quad = \sum_{j=2}^{n-1} \left| \frac{(k+1) b_j z_n^k}{1 + b_j z_n^{k+1}} \right| \left|z_j\right|^{p-1} \\
&\quad \leq (k+1) \sum_{j=2}^{n-1} \frac{\left|b_j\right| \left|z_n\right|^{k-1}}{1 - \left|b_j\right|} \left|z_j\right|^{p-1} \\
&\quad \leq (k+1) \left[\sum_{j=2}^{n-1} \frac{\left|b_j\right|^p}{\left(1 - \left|b_j\right|\right)^p} \right]^{1/p} \\
&\quad \quad \times \left[\sum_{j=2}^{n-1} \left|z_j\right|^{(p-1)q} \right]^{1/q} \left|z_n\right|^{k-1} \\
&\quad \leq (k+1) \left[\sum_{j=2}^{n-1} \frac{\left|b_j\right|^p}{\left(1 - \left|b_j\right|\right)^p} \right]^{1/p} \\
&\quad \quad \times \left(1 - \left|z_1\right|^p - \left|z_n\right|^p\right)^{1/q} \left|z_n\right|^{k-1} \\
&\quad \leq (k+1) \left[\sum_{j=2}^{n-1} \frac{\left|b_j\right|^p}{\left(1 - \left|b_j\right|\right)^p} \right]^{1/p} \varphi\left(\left|z_n\right|, \left|z_1\right|\right) \left|z_n\right|^{p-2}, \\
&\sum_{j=2}^{n-1} \left| \frac{\partial^2 p_j / \partial z_n^2}{\partial p_j / \partial z_j} \right| \left|z_j\right|^{p-1} \\
&\quad = \sum_{j=2}^{n-1} \left| \frac{k (k+1) b_j z_j z_n^{k+1}}{1 + b_j z_n^{k+1}} \right| \left|z_j\right|^{p-1} \\
&\quad \leq k (k+1) \sum_{j=2}^{n-1} \frac{\left|b_j\right| \left|z_n\right|^{k-1}}{1 - \left|b_j\right|} \left|z_j\right|^{p-1} \\
&\quad \leq k (k+1) \left[\sum_{j=2}^{n-1} \frac{\left|b_j\right|^p}{\left(1 - \left|b_j\right|\right)^p} \right]^{1/p} \\
&\quad \quad \times \left[\sum_{j=2}^{n-1} \left|z_j\right|^{(p-1)q} \right]^{1/q} \left|z_n\right|^{k-1}, \\
&\quad \leq k (k+1) \left[\sum_{j=2}^{n-1} \frac{\left|b_j\right|^p}{\left(1 - \left|b_j\right|\right)^p} \right]^{1/p} \\
&\quad \quad \times \left(1 - \left|z_1\right|^p - \left|z_n\right|^p\right)^{1/q} \left|z_n\right|^{k-1}
\end{aligned}
$$

$$\leq k(k+1)\left[\sum_{j=2}^{n-1}\frac{|b_j|^p}{(1-|b_j|)^p}\right]^{1/p} \times \varphi(|z_n|,|z_1|)|z_n|^{p-2},$$

$$\begin{aligned}\sum_{j=2}^{n-1}\left|\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||z_j|^{p-1} &= \sum_{j=2}^{n-1}\left|\frac{(k+1)b_j z_j z_n^k}{1+b_j z_n^{k+1}}\right||\lambda||z_j|^{p-1} \\ &\leq (k+1)|\lambda|\sum_{j=2}^{n-1}\frac{|b_j||z_n|^{k-1}}{1-|b_j|}|z_j|^{p-1} \\ &\leq (k+1)|\lambda|\left[\sum_{j=2}^{n-1}\frac{|b_j|^p}{(1-|b_j|)^p}\right]^{1/p} \times\left[\sum_{j=2}^{n-1}|z_j|^{(p-1)q}\right]^{1/q}|z_n|^{k-1} \\ &\leq (k+1)|\lambda|\left[\sum_{j=2}^{n-1}\frac{|b_j|^p}{(1-|b_j|)^p}\right]^{1/p} \times\left(1-|z_1|^p-|z_n|^p\right)^{1/q}|z_n|^{k-1} \\ &\leq (k+1)|\lambda|\left[\sum_{j=2}^{n-1}\frac{|b_j|^p}{(1-|b_j|)^p}\right]^{1/p} \times\varphi(|z_n|,|z_1|)|z_n|^{p-2},\end{aligned} \quad (44)$$

where $\psi(x)=(1-x^p-y^p)^{1/q}x^{k-p+1}$, $x,y\in[0,1]$.

When $p=k+1$, we have $\max_{0\leq x,y\leq 1}\psi(x,y)=1$.

When $k<p<k+1$, we have $0<k-p+1<1$ and

$$\begin{aligned}\psi_x(x,y) &= (1-x^p-y^p)^{(1/q)-1}x^{k-p} \times[(k-p+1)(1-y^p)-kx^p], \\ \psi_y(x,y) &= (1-p)y^{p-1}(1-x^p-y^p)^{(1/q)-1}x^{k-p+1},\end{aligned} \quad (45)$$

so

$$\begin{aligned}\max_{0\leq x,y\leq 1}\psi(x,y) &= \psi\left(\sqrt[p]{\frac{k-p+1}{k}},0\right) \\ &= \left(\frac{p-1}{k}\right)^{1/q}\left(\frac{k-p+1}{k}\right)^{(k-p+1)/p},\end{aligned}$$

$$\begin{aligned}&\frac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|} \times\left(\sum_{l=1}^{n}\left|\frac{\partial^2 p_1}{\partial z_n\partial z_l}\right| + \sum_{j=2}^{n-1}\left(\left|\frac{\partial p_1}{\partial z_j}\right| \times\left(\left|\frac{\partial^2 p_j}{\partial z_j\partial z_n}\right|+\left|\frac{\partial^2 p_j}{\partial z_n^2}\right|\right) \times\left(\left|\frac{\partial p_j}{\partial z_j}\right|\right)^{-1}\right)\right. \\ &\quad\left. +\left|\frac{\partial p_1}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right| +\sum_{j=2}^{n-1}\left|\frac{\partial p_1}{\partial z_j}\right|\left|\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\right) \\ &\leq \frac{1}{1-|a_n|}\left[(k+1)|a_n|+k(k+1)|a_n| +\sum_{j=2}^{n-1}(k+1)|a_j|\frac{(k+1)^2|b_j|}{1-|b_j|} +(k+1)|a_n||\lambda| +\sum_{j=2}^{n-1}\frac{(k+1)^2|\lambda||a_j||b_j|}{1-|b_j|}\right] \times|z_n|^{p-2} \\ &\leq \frac{(k+1)(k+1+\lambda)a}{1-a} \times\left[1+(k+1)\sum_{j=2}^{n-1}\frac{|b_j|}{1-|b_j|}\right]|z_n|^{p-2}.\end{aligned} \quad (46)$$

Hence, we have

$$\sum_{j=2}^{n-1}\left|\frac{\partial^2 p_j/\partial z_j\partial z_n}{\partial p_j/\partial z_j}\right||z_j|^{p-1}+\sum_{j=2}^{n-1}\left|\frac{\partial^2 p_j/\partial z_n^2}{\partial p_j/\partial z_j}\right||z_j|^{p-1} +\sum_{j=2}^{n-1}\left|\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right||z_j|^{p-1}+\frac{|z_1|^{p-1}}{|\partial p_1/\partial z_1|} \times\left(\sum_{j=2}^{n-1}\left|\frac{\partial^2 p_1}{\partial z_n\partial z_j}\right|\right.$$

$$
\begin{aligned}
&+\sum_{j=2}^{n-1}\frac{\left|\partial p_1/\partial z_j\right|\left(\left|\partial^2 p_j/\partial z_j\partial z_n\right|+\left|\partial^2 p_j/\partial z_n^2\right|\right)}{\left|\partial p_j/\partial z_j\right|}\\
&+\left|\frac{\partial p_1}{\partial z_n}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\\
&+\sum_{j=2}^{n-1}\left|\frac{\partial p_1}{\partial z_j}\right|\left|\frac{\partial p_j/\partial z_n}{\partial p_j/\partial z_j}\right|\left|\frac{p_n''(z_n)}{p_n'(z_n)}\right|\Bigg)\\
&\le (k+1)(k+1+\lambda)N(p)\left|z_n\right|^{p-2}\\
&\le (k+1)(k+1+\lambda)\frac{1-\alpha-|\lambda|}{(k+1)(k+1+|\lambda|)}\left|z_n\right|^{p-2}\\
&\le (1-\alpha-|\lambda|)\left|z_n\right|^{p-2}\\
&\le \left(1-\alpha-\left|\lambda z_n\right|\right)\left|z_n\right|^{p-2}\\
&=\left|z_n\right|^{p-2}\left(1-\alpha-\left|\frac{z_n p_n''(z_n)}{p_n'(z_n)}\right|\right).
\end{aligned}
\tag{47}
$$

By Theorem 6, we obtain that $f \in K(B_p^n,\alpha)$. The proof is complete. □

By applying the same method of the proof for Example 12, we only need to let $2|b_n|/(1-2|b_n|)$ instead of $|\lambda|$, we may prove the following result.

Example 13. Suppose that $p \ge 2,\ 0 \le \alpha < 1,\ 0 \le |b_n| \le (1-\alpha)/(4-2\alpha)$ and k is a positive integer such that $k < p \le k+1$. Let

$$
\begin{aligned}
f(z)=\Bigg(&z_1+\sum_{j=2}^{n-1}a_j z_j^{k+1}+a_n z_1 z_n^{k+1}, z_2+b_2 z_2 z_n^{k+1},\dots,\\
&z_{n-1}+b_{n-1}z_{n-1}z_n^{k+1}, z_n+b_n z_n^2\Bigg),
\end{aligned}
\tag{48}
$$

where $a=\max\{|a_j| : j=2,\dots,n\}$ and $b=\max\{|b_j| : j=2,\dots,n-1\}$. If

$$
a \le \frac{1-(2|b_n|/(1-2|b_n|))-\alpha}{(k+1)^2(k+1+2|b_n|/(1-2|b_n|))+1-(2|b_n|/(1-2|b_n|))-\alpha} < 1,
\tag{49}
$$

$b \le ((k+(2|b_n|/(1-2|b_n|)))(1-\alpha)+(2|b_n|/(1-2|b_n|)))/(k+2-\alpha)(k+1+(2|b_n|/(1-2|b_n|))) < 1$, and

$$
\begin{aligned}
N(p) &\le \frac{1-\alpha-(2|b_n|/(1-2|b_n|))}{(k+1)(k+1+(2|b_n|/(1-2|b_n|)))}\\
&=\frac{1-(4-2\alpha)|b_n|-\alpha}{(k+1)(k+1-2k|b_n|)},
\end{aligned}
\tag{50}
$$

where $N(p)$ is defined in Example 12, then $f(z) \in K(B_p^n,\alpha)$.

By applying the same method of the proof for Theorem 2, we may get the following result.

Theorem 14. *Suppose that $0 \le \alpha < 1,\ n \ge 2,\ p \ge 2$ and l is a positive integer such that $l < p \le l+1$. Let*

$$
\begin{aligned}
f(z)=\big(&p_1(z_1,z_2,\dots,z_n), p_2(z_2)+f_2(z_k),\dots,\\
&p_k(z_k),\dots p_n(z_n)+f_n(z_k)\big) \quad (2 \le k \le n),
\end{aligned}
\tag{51}
$$

where $z=(z_1,z_2,\dots,z_n) \in B_p^n$, $f_k(z_k)=0$, $f_j : U \to C$ is holomorphic with $f_j(0)=0$, $f_j'(0)=0$ $(j=2,3,\dots,n-1)$, $p_j \in H(U)$ $(j=2,3,\dots,n)$ and $p_1(z_1,\dots,z_n) : B_p^n \to C$ is holomorphic with $p_1(0,0,\dots,0)=0$, $(\partial p_1/\partial z_1)(0,0,\dots,0)=1$, $(\partial p_1/\partial z_l)(0,0,\dots,0)=0$ $(l=2,3,\dots,n)$. If f satisfies the following conditions:

(1) $$\frac{\partial p_1}{\partial z_1}\cdot\prod_{j=2}^{n}p_j'(z_j) \ne 0,$$

$$\left|z_j p_j''(z_j)\right| \le (1-\alpha)\left|p_j'(z_j)\right|, \quad j=2,\dots,n;$$

(2) $$\sum_{l=1}^{n}\left|z_1\frac{\partial^2 p_1}{\partial z_1\partial z_l}\right| \le (1-\alpha)\left|\frac{\partial p_1}{\partial z_1}\right|;$$

(3) $$
\begin{aligned}
&\left|z_1\right|^{p-1}\left|\frac{(\partial p_1/\partial z_j)\cdot\left(p_j''(z_j)/p_j'(z_j)\right)}{\partial p_1/\partial z_1}\right|\\
&+\left|z_1\right|^{p-1}\sum_{l=1}^{n}\left|\frac{\partial^2 p_1/\partial z_j\partial z_l}{\partial p_1/\partial z_1}\right|\\
&\le\left(1-\alpha-\left|\frac{z_j p_j''(z_j)}{p_j'(z_j)}\right|\right)\left|z_j\right|^{p-2},
\end{aligned}
$$

$$(j=2,\dots,k-1,k+1,\dots,n-1);$$

$$(4)\quad \sum_{j=2,j\neq k}^{n}\left|\frac{f_j''(z_k)}{p_j'(z_j)}\right||z_j|^{p-1}$$
$$+\sum_{j=2,j\neq k}^{n}\left|\frac{f_j'(z_k)}{p_j'(z_j)}\right|\left|\frac{p_k''(z_k)}{p_k'(z_k)}\right||z_j|^{p-1}$$
$$+\sum_{j=2,j\neq k}^{n}\left|\frac{f_j''(z_k)}{p_j'(z_j)}\frac{\partial p_1}{\partial z_j}\left(\frac{\partial p_1}{\partial z_1}\right)^{-1}\right||z_1|^{p-1}$$
$$+\sum_{j=2,j\neq k}^{n}\left|\frac{f_j'(z_k)}{p_j'(z_j)}\frac{p_k''(z_k)}{p_k'(z_k)}\times\frac{\partial p_1}{\partial z_j}\left(\frac{\partial p_1}{\partial z_1}\right)^{-1}\right|\times|z_1|^{p-1}$$
$$+\sum_{l=1}^{n}\left|\frac{(\partial^2 p_1/\partial z_l\partial z_k)}{(\partial p_1/\partial z_1)}\right||z_1|^{p-1}$$
$$+\left|\frac{(p_k''(z_k)/p_k'(z_k))(\partial p_1/\partial z_k)}{(\partial p_1/\partial z_1)}\right||z_1|^{p-1}$$
$$\leq\left(1-\alpha-\left|\frac{z_k p_k''(z_k)}{p_k'(z_k)}\right|\right)|z_k|^{p-2}, \quad (52)$$

for all $z=(z_1,\ldots,z_n)\in B_p^n$, *then* $f\in K(B_p^n,\alpha)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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